Cálculo A

Função Exponencial e logarítmica [1]

1. Expresse as quantidades abaixo na forma de um único logaritmo

   a) $\log_5 a + \log_5 b - \log_5 c$

   b) $\log_2 x + 5\log_2 (x+1) + \frac{1}{2}\log_2 (x-1)$

   c) $\frac{1}{3}\ln x - 4\ln(2x+3)$

   d) $\ln x + a\ln y - b\ln z$

2. Resolva as seguintes equações

   $a)\ \log_2 x = 3 \quad b)\ 2 = \log_5 (x-1) \quad c)\ 3^{x+2} = m\ (m > 0) \quad d)\ \ln x = 2$

   $e)\ \ln x = \ln 2 + \ln 8 \quad f)\ \ln(e^{2x-1}) = 5 \quad g)\ m = \ln(\ln x)$

3. Determine o domínio das funções

   (a) $f(x) = \frac{1}{16^{x^2} - 2^x}$

   (b) $f(x) = \sqrt{2^x - 3^x}$

   (c) $f(x) = \log_2 x^2$
   $f(x) = 2\log_2 x$

   (d) $f(x) = \log_x 5$

   (e) $f(x) = \frac{1}{\log(100-x)}$

   (f) $f(x) = \ln x + \ln(x-1)$

   (g) $f(x) = \ln x(x-1)$

   (h) $f(x) = \log_{3+x}(x^2-1)$

   (i) $f(x) = \log_3(\log_{\frac{1}{2}} x)$

   (j) $f(x) = \log(x^2+1)$

   (k) $f(x) = \log\left(\frac{3x-x^2}{x-1}\right)$

   (l) $f(x) = \sqrt{\log_3 \frac{2x-3}{x-1}}$

   (m) $f(x) = \frac{\sqrt{x+5}}{\log(9-5x)}$

   (n) $f(x) = \frac{\sqrt{x^2-4}}{\log_2(x^2+2x-3)}$

   (o) $f(x) = \log_{x+1}(x^2-3x+2)$

---

[1] (i) Ao escrevermos $\log x$, sem especificar a base do sistema de logaritmos que estamos empregando, assume-se que a base é qualquer número real positivo.
(ii) Por $\ln x$ entendemos $\log_e x$, onde $e = 2.71...$

(p) $f(x) = \log_x \log_{\frac{1}{2}}(\frac{4}{3} - 2^{x-1})$

4. Determine a imagem das funções
   (a) $f(x) = 10^{-x^2}$
   (b) $f(x) = \frac{1}{1-2^{-x}}$
   (c) $f(x) = 4^x - 2^x + 1$
   (d) $f(x) = \log(x^2 + 10)$
   (e) $f(x) = \log_2(4 - x^4)$
   (f) $f(x) = \log_3 x + \log_x 3$

5. Determine x solução de $\log_{\frac{1}{4}}(x+1) = \log_4(x-1)$

6. Seja $f : A \to \mathbb{R}$ definida por $f(x) = |\ln(x^2 - x + 1)|$. Determine A de modo a termos f injetiva.

7. Seja $f(x) = \ln(x^2 + x + 1), x \in \mathbb{R}$. Determine funções $h, g : \mathbb{R} \to \mathbb{R}$ tais que $f(x) = g(x) + h(x), \forall x \in \mathbb{R}$, sendo h uma função par e g uma função ímpar.

8. Seja $a^2 + b^2 = 7ab$. Mostre que $\log \frac{a+b}{3} = \frac{1}{2}(\log a + \log b)$

9. Mostre
   a) $\log_a b \log_b c = \log_a c$
   b) $\log_a b = \frac{1}{\log_b a}$

10. Mostre que $\log_2 5$ é irracional. Isto é, mostre que ele **não** pode ser escrito na forma $\frac{p}{q}$ com $p, q \in Z$.

11. Suponha que b, c, p, q são positivos e que $b/c = p/q$. Mostre que $\ln b - \ln c = \ln p - \ln q$.

12. Seja

$$f(x) = \frac{e^x}{e^{2x} + 1}$$

Mostre que f é função par.

13. Seja $f(x) = \frac{1}{2}(a^x + a^{-x}), (a > 0)$. Mostrar que

$$f(x+y) + f(x-y) = 2f(x)f(y)$$

14. Mostre que $\frac{\log_a n}{\log_{am} n} = 1 + \log_a m$

15. Sejam x, y, z tal que se tenha

$$\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\log y} = \frac{z(x+y-z)}{\log z}$$

Mostre que $x^y y^x = z^y y^z = x^z z^x$

16. Simplifique a expressão

$$a^{\frac{\log(\log a)}{\log a}}$$

17. Sejam $y = 10^{\frac{1}{1-\log_{10} x}}$, $z = 10^{\frac{1}{1-\log_{10} y}}$. Mostre que $x = 10^{\frac{1}{1-\log_{10} z}}$

18. Sejam a, b, c números reais positivos satisfazendo $a^2 + b^2 = c^2$. Mostre que

$$\log_{b+c} a + \log_{c-b} a = 2 \log_{c+b} a \, \log_{c-b} a$$

19. Sejam $a > 0, c > 0, b = \sqrt{ac}$, $a \neq 1, c \neq 1, ac \neq 1$ e $N > 0$. Mostre que

$$\frac{\log_a N}{\log_c N} = \frac{\log_a N - \log_b N}{\log_b N - \log_c N}$$

20. Mostre que

$$\log_{a_1 a_2 \ldots a_n} x = \frac{1}{\frac{1}{\log_{a_1} x} + \frac{1}{\log_{a_2} x} + \ldots + \frac{1}{\log_{a_n} x}}$$

21. Sejam dadas
$a, a_1, a_2, ..., a_n, ...$: progressão geométrica de razão $q > 0$
$b, b_1, b_2, ..., b_n, ...$: progressão aritmética com diferença $r > 0$.
Encontre a base β de um sistema de logarítmos onde se tem

$$\log_\beta a_n - b_n = \log_\beta a - b, \ \forall n \in \mathbb{N}$$

Respostas

1. (a) $\log_5 \frac{ab}{c}$
   (b) $\log_2 \frac{x(x+1)^5}{\sqrt{x-1}}$
   (c) $\ln \frac{\sqrt[3]{x}}{(2x+3)^4}$
   (d) $\ln \frac{xy^a}{z^b}$

2. (a) $x = 8$
   (b) $x = 26$

- (c) $x = -2 + \log_3 m$
- (d) $x = e^2$
- (e) $x = 16$
- (f) $x = 3$
- (g) $x = e^{e^m}$

3. (a) $\mathbb{R} - \{0, \frac{1}{4}\}$
   (b) $(-\infty, 0]$
   (c) $\mathbb{R} - \{0\}$
   $(0, \infty)$
   (d) $(0, 1) \cup (1, \infty)$
   (e) $(-\infty, 99) \cup (99, 100)$
   (f) $(1, \infty)$
   (g) $(-\infty, 0) \cup (1, \infty)$
   (h) $(-3, -2) \cup (-2, -1) \cup (1, \infty)$
   (i) $(0, 1)$
   (j) $\mathbb{R}$
   (k) $(-\infty, 0) \cup (1, 3)$
   (l) $(-\infty, 1) \cup [2, \infty)$
   (m) $[-5, \frac{8}{5}) \cup (\frac{8}{5}, \frac{9}{5})$
   (n) $(-\infty, -1 - \sqrt{5}) \cup (-1 - \sqrt{5}, -3) \cup [2, \infty)$
   (o) $(-1, 0) \cup (0, 1) \cup (2, \infty)$
   (p) $(0, 1) \cup (1, 1 + \log_2 \frac{4}{3})$

4. (a) $(0, 1]$
   (b) $(-\infty, 0) \cup (1, \infty)$
   (c) $[\frac{3}{4}, \infty)$
   (d) $[1, \infty)$
   (e) $(-\infty, 2]$
   (f) $(-\infty, -2] \cup [2, \infty)$

5. $\sqrt{2}$

6. $A = [0, \frac{1}{2}]$ ou $A = (-\infty, 0]$, ou $A = [\frac{1}{2}, 1]$ ou $A = [1, \infty)$
7. $g(x) = \frac{1}{2} \ln\left(\frac{x^2+x+1}{x^2-x+1}\right)$
   $h(x) = \frac{1}{2} \ln(x^4 + x^2 + 1)$
8.
9.
10.
11.
12.
13.
14.
15.
16. $\log a$
17.
18.
19.
20.
21. $\beta = q^{\frac{1}{r}}$

1.

a) $\underbrace{\log_5 a + \log_5 b}_{} - \log_5 c =$

$= \log_5 ab - \log_5 c$

$= \log_5 \frac{ab}{c}$

b) $\log_2 x + 5 \log_2 (x+1) + \frac{1}{2} \log_2 (x-1) =$

$= \log_2 x + \log_2 (x+1)^5 + \log_2 \sqrt{x-1}$

$= \log \frac{x(x+1)^5}{\sqrt{x-1}}$

c) $\frac{1}{3} \ln x - 4 \ln (2x+3) =$

$= \ln x^{1/3} - \ln (2x+3)^4$

$= \ln \frac{\sqrt[3]{x}}{(2x+3)^4}$

d) $\ln x + a \ln y - b \ln z =$

$= \ln x + \ln y^a - \ln z^b$

$= \ln \frac{xy^a}{z^b}$

2.

a) $\log_2 x = 3$

$\therefore \quad 2^3 = x$

$\therefore \quad \underline{\underline{x = 8}}$

b) $2 = \log_5 (x-1)$

$\therefore \quad 5^2 = x - 1$

$\therefore \quad 25 = x - 1 \quad \Rightarrow \quad \underline{\underline{x = 26}}$

c) $3^{x+2} = m \qquad (m > 0)$

$\therefore \quad \log 3^{x+2} = \log m$

$(x+2) \log 3 = \log m$

$(x+2) = \dfrac{\log m}{\log 3} = \log_3 m$

$// \; x = -2 + \log_3 m \; //$

d) $\ln x = 2$

$\therefore \quad // \, e^2 = x \, //$

e) $\ln x = \ln 2 + \ln 8$

$\therefore \ln x = \ln 16$

$\underline{x = 16}$

f) $\ln(e^{2x-1}) = 5$

$\therefore e^5 = e^{2x-1}$

$2x-1=5$

$2x = 6$

$// x = 3 //$

g) $m = \ln(\ln x)$

$\therefore e^m = \ln x$

$\therefore // x = e^{e^m} //$

3

(K. pg. 137)

a) $f(x) = \dfrac{1}{16^{x^2} - 2^x}$

$x \notin \text{Dom } f \implies 16^{x^2} - 2^x = 0$

$16^{x^2} = 2^x$

$\therefore \log_2 (16^{x^2}) = \log_2 (2^x)$

$x^2 \log_2 16 = x$

$x^2 \cdot 4 = x$

$4x^2 - x = 0$

$x(4x - 1) = 0$

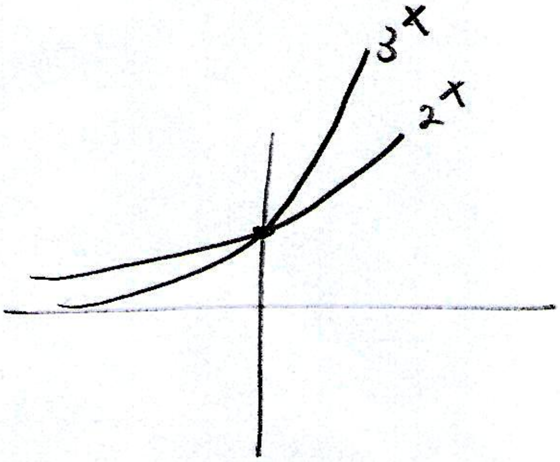
$x = 0 \quad \therefore \quad x = \frac{1}{4}$

$\therefore \quad // \text{ Dom } f : \mathbb{R} - \{0, \frac{1}{4}\} //$

b) $f(x) = \sqrt{2^x - 3^x}$

$2^x - 3^x \geq 0$

$2^x \geq 3^x > 0$



se $x \leq 0$: $2^x \geq 3^x$

$\therefore$ // Dom $f = (-\infty, 0]$ //

---

c) $f(x) = \log_2 x^2$

$x^2 > 0$ $\therefore$ // $x \in \mathbb{R} - \{0\}$ //

$f(x) = 2 \log_2 x$ $\therefore$ // $x > 0$ //

---

d) $f(x) = \log_x 5$ $\Rightarrow$ // Dom $f = (0,1) \cup (1,+\infty)$ //

e) $f(x) = \dfrac{1}{\lg(100-x)}$

$$\begin{cases} 100 - x > 0 \quad \therefore \quad 100 > x \\ \lg(100-x) \neq 0 \Rightarrow 100 - x \neq 1 \\ \qquad\qquad\qquad\qquad x \neq 99 \end{cases}$$

$\therefore$ // $\text{Dom } f = (-\infty, 99) \cup (99, 100)$ //

---

f) $f(x) = \ln x + \ln(x-1)$

$\underline{x > 0}$ e $x - 1 > 0$ $\Big\}$ $\therefore$ $x > 1$

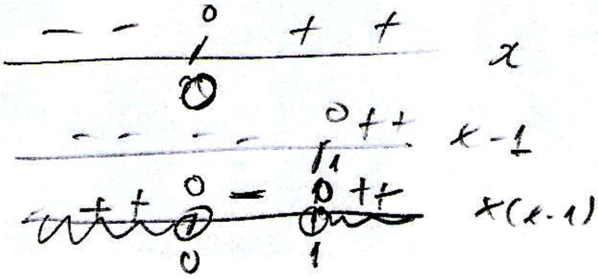
$\therefore \underline{x > 1}$

// $\text{Dom } f = (1, +\infty)$ //

---

g) $f(x) = \ln x(x-1)$

$\therefore \quad x(x-1) > 0$



// $\text{Dom } f: (-\infty, 0) \cup (1, +\infty)$ //

h) $\log_{3+x}(x^2-1)$

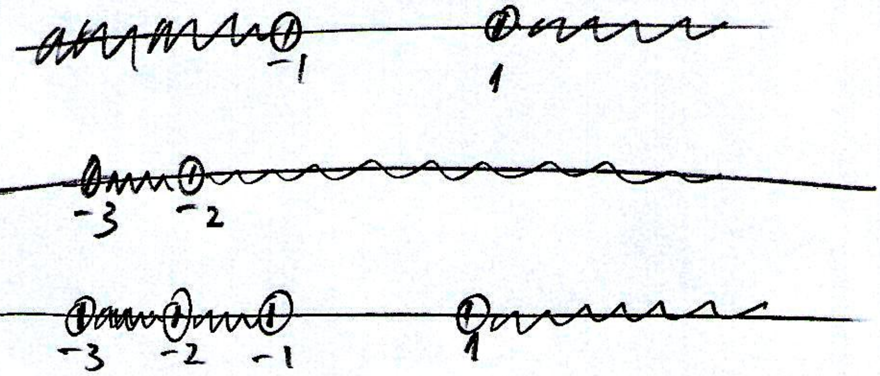
e $\begin{cases} x^2-1>0 & (*) \\ 3+x \in (0,1)\cup(1,+\infty) & (**) \end{cases}$

De (*) : $x<-1$ ou $x>1$ (3*)

De (**) : $0<3+x<1$ ou $3+x>1$

$-3<x<-2$ ou $x>-2$ (4*)

De (3*) e (4*) :

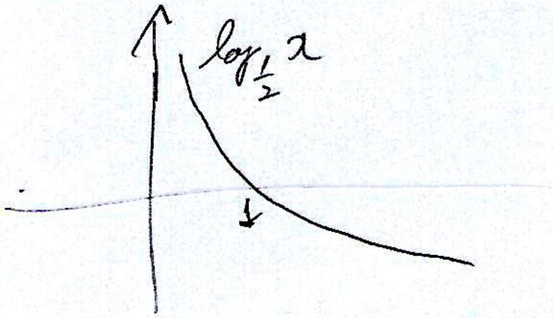


// Dom f $= (-3,-2)\cup(-2,-1)\cup(1,+\infty)$ //

i) $f = \log_3(\log_{\frac{1}{2}} x)$

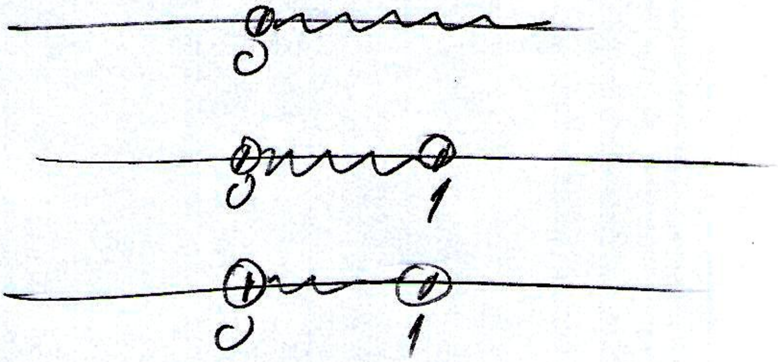
$\log_{\frac{1}{2}} x \Rightarrow x > 0$ (1*)

$\log_3(\log_{\frac{1}{2}} x) \Rightarrow \log_{\frac{1}{2}} x > 0$ (2*)



$\therefore 0 < x < 1$ (3*)

De (1*) e (3*) :



$\| \text{Dom} f = (0,1) \|$

j) Pg. 135 $f(x) = \log(x^2+1)$

$x^2+1 > 0 \Rightarrow x \in \mathbb{R}$

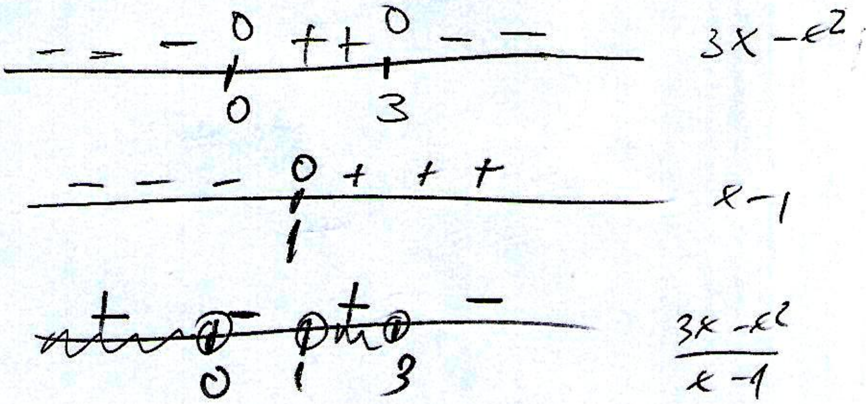
// $\text{Dom } f = \mathbb{R}$ //

---

k) $f(x) = \log\left(\frac{3x-x^2}{x-1}\right)$

$$\frac{3x-x^2}{x-1} > 0$$

$3x - x^2 = x(3-x)$



// $\text{Dom } f = (-\infty, 0) \cup (1, 3)$ //

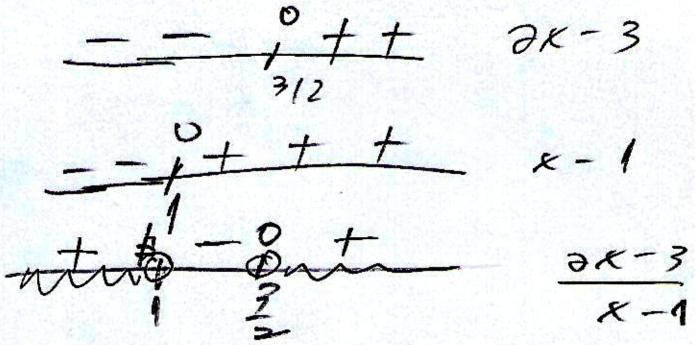
e) $f(x) = \sqrt{\log_3 \frac{2x-3}{x-1}}$

$\frac{2x-3}{x-1} > 0$ ⓧ

și

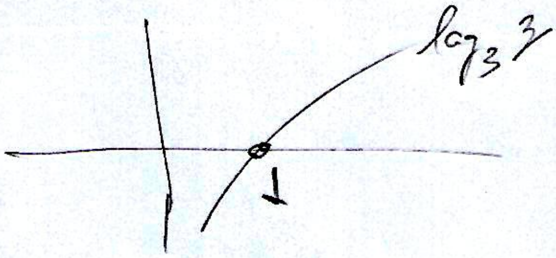
$\log_3 \frac{2x-3}{x-1} \geq 0$ (2*)

De ⓧ :



$\frac{2x-3}{x-1} > 0 \Rightarrow x \in (-\infty, 1) \cup (\frac{3}{2}, +\infty)$ (3*)

De (2*) :



$\log_3 \frac{2x-3}{x-1} \geq 0 \Rightarrow \frac{2x-3}{x-1} \geq 1$

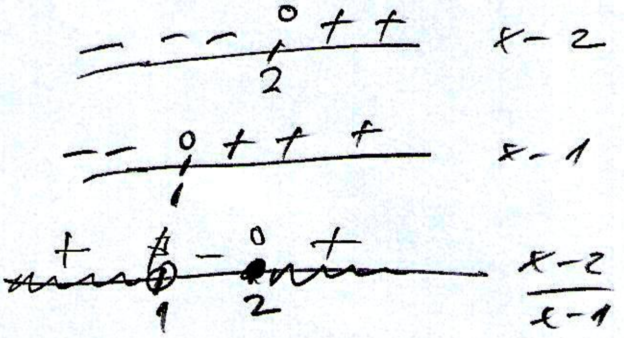
$\frac{2x-3}{x-1} - 1 \geq 0$

$\frac{2x-3-(x-1)}{x-1} \geq 0$

→

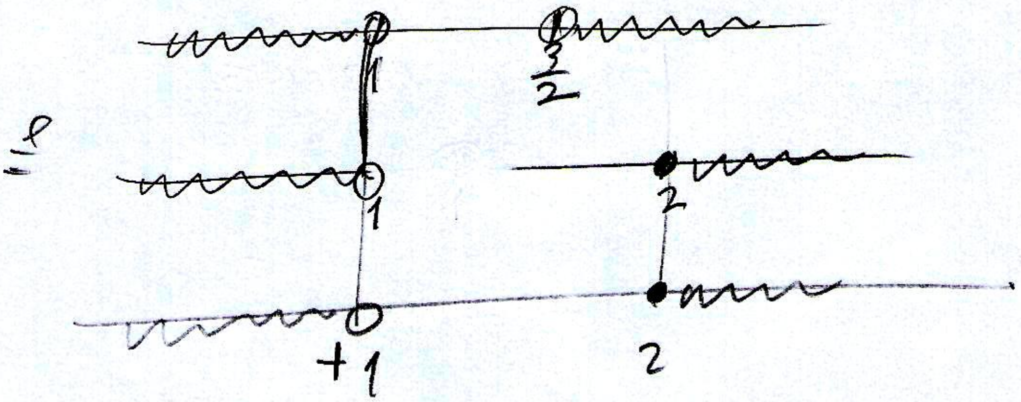
$$\frac{2x-3-x+1}{x-1} \geq 0$$

$$\frac{x-2}{x-1} \geq 0$$



$\frac{x-2}{x-1} \geq 0 \Longrightarrow x \in (-\infty, 1) \cup [2, +\infty)$ ④*

De ③* e ④* :



$$\text{Dom} f = (-\infty, +1) \cup [2, +\infty)$$

m) $f(x) = \dfrac{\sqrt{x+5}}{\log(9-5x)}$

$x+5 \geq 0 \Longrightarrow x \geq -5$ $\circledast$

<u>e</u>

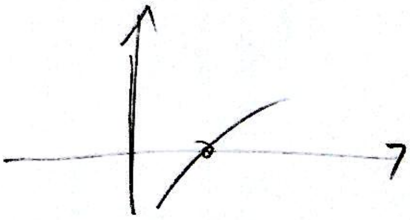
$9-5x > 0 \Longrightarrow \frac{9}{5} > x$ $\textcircled{2*}$

<u>e</u>

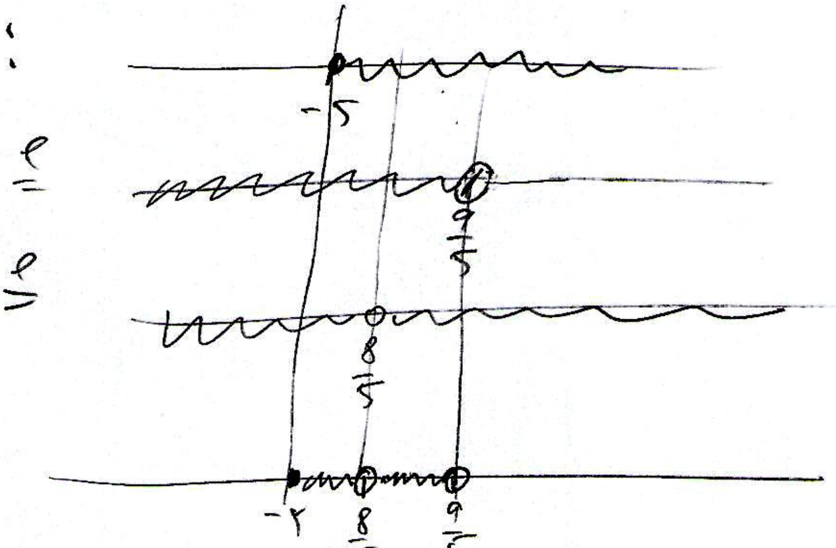
$\log(9-5x) \neq 0 \Longrightarrow 9-5x \neq 1$

$\therefore -5x \neq -8$

$x \neq \frac{8}{5}$ $\textcircled{3*}$



De $\circledast$ <u>e</u> $\textcircled{2*}$ e $\textcircled{3*}$:



$$//\ \text{Dom } f = \left[-5, \frac{8}{5}\right) \cup \left(\frac{8}{5}, \frac{9}{5}\right)\ //$$

n) $f(x) = \dfrac{\sqrt{x^2-4}}{\log_2(x^2+2x-3)}$

$x^2-4 \geq 0 \implies \underline{x \leq -2 \text{ ou } x \geq 2}$

$x^2+2x-3 > 0 \implies$ $\overset{+}{} \underset{-3}{\overset{0}{\circ}} \overset{-}{} \underset{1}{\overset{0}{\circ}} \overset{+}{}$, $\underline{x<-3 \text{ ou } x>1}$

$\log_2(x^2+2x-3) \neq 0 \implies x^2+2x-3 \neq 1$

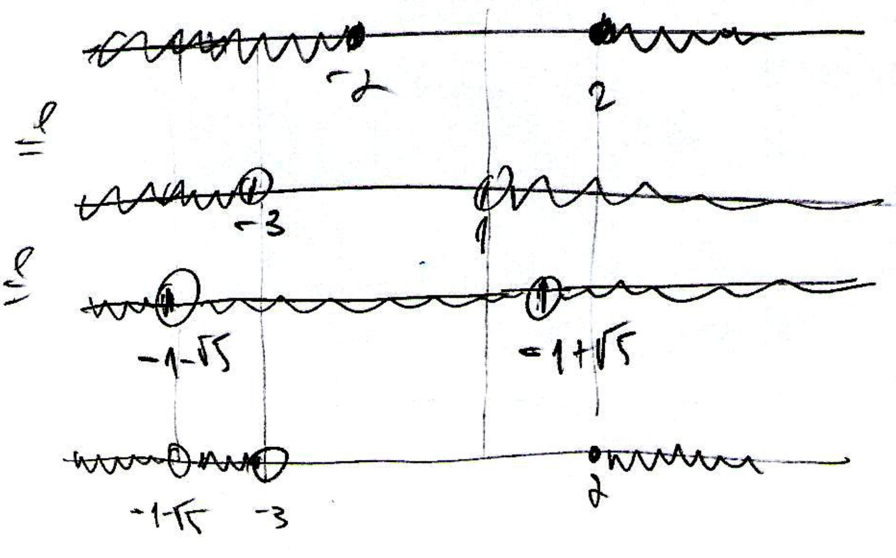
$x^2+2x-4 \neq 0$

$x = \dfrac{-2 \pm \sqrt{4+16}}{2}$

$= \dfrac{-2 \pm 2\sqrt{5}}{2}$

$= -1 \pm \sqrt{5}$

$\therefore \quad \underline{x \neq -1 \pm \sqrt{5}}$



$\boxed{\text{Dom} f = (-\infty, -1-\sqrt{5}) \cup (-1-\sqrt{5},}$
$\cup\ [2, +\infty)$

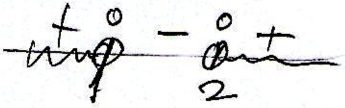
a) $y = \log_{x+1}(x^2-3x+2)$

$x^2-3x+2 > 0$ ⊛

e

$x+1 \in (0,1) \cup (1,+\infty)$ ⊛⊛

De ⊛ :



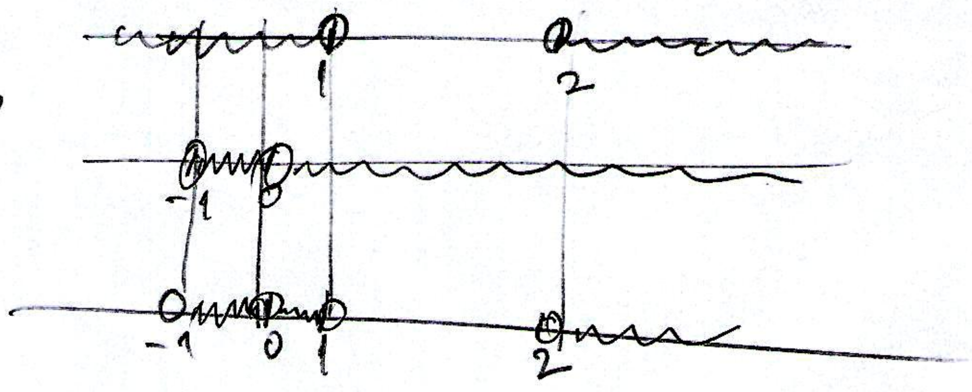
∴ $x < 1$ ou $x > 2$

De ⊛⊛ : $0 < x+1 < 1$ ou $1 < x+1$

$-1 < x < 0$ ou $0 < x$

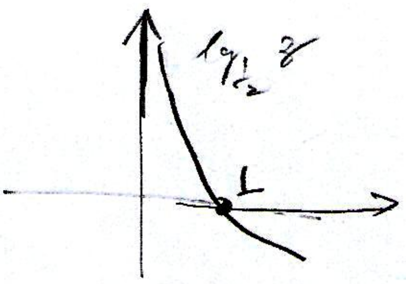
∴

e



$$\text{Dom } f = (-1,0) \cup (0,1) \cup (2,+\infty)$$

φ) $f(x) = \log_x \lg_{\frac{1}{2}}\left(\frac{4}{3} - 2^{x-1}\right)$



$\rightarrow \frac{4}{3} - 2^{x-1} > 0$ (*)

$\left(\Leftarrow \lg_{\frac{1}{2}}\left(\frac{4}{3} - 2^{x-1}\right)\right)$

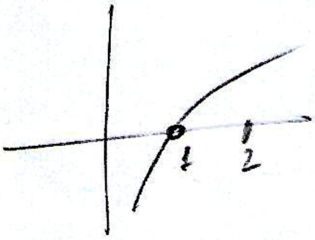
$\rightarrow \lg_{\frac{1}{2}}\left(\frac{4}{3} - 2^{x-1}\right) > 0$ (**) $\left(\Leftarrow \log_x \lg_{\frac{1}{2}}\left(\frac{4}{3} - 2^{x-1}\right)\right)$

$\rightarrow x \in (0,1) \cup (1,+\infty)$ (3*)

De (*) : $\frac{4}{3} > 2^{x-1}$ (ln é crescente)

$$\ln \frac{4}{3} > \ln 2^{x-1}$$

$$\ln \frac{4}{3} > (x-1) \underbrace{\ln 2}_{>0}$$

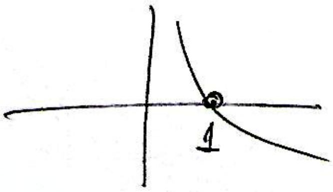


$$\underbrace{\frac{\ln \frac{4}{3}}{\ln 2}} > x - 1$$

$\log_2 \frac{4}{3} > x - 1$ ∴ $\underline{x < 1 + \log_2 \frac{4}{3}}$ (4*)

De (**) :

$$\log_{\frac{1}{2}}\left(\frac{4}{3} - 2^{x-1}\right) > 0$$



∴ $\frac{4}{3} - 2^{x-1} < 1$

$$\underbrace{\frac{4}{3} - 1}_{} < 2^{(x-1)}$$

$$\frac{1}{3} < 2^{(x-1)}$$

ln é crescente

$$\ln\frac{1}{3} < (x-1)\underbrace{\ln 2}_{>0}$$

$$\frac{0.48}{0.3}$$

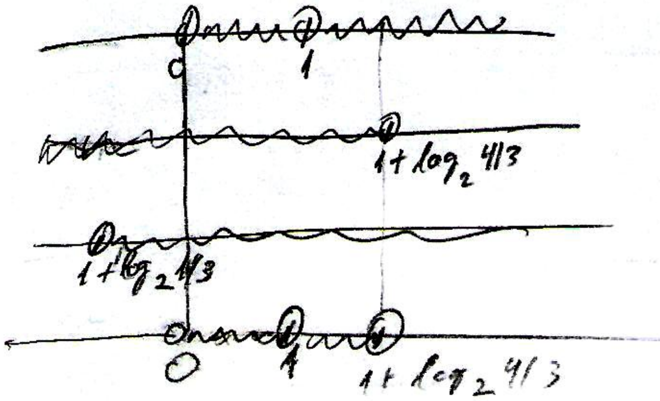
∴ $\frac{\ln\frac{1}{3}}{\ln 2} < x-1$

$$\log_2 3 = \frac{\lg 3}{\lg 2}$$

$$\log_2\frac{1}{3} < x-1$$

∴ $\underline{\overbrace{\log_2\frac{1}{3}}^{\sim \ominus 1.6} + 1 < x}$ ⑤*

De ③*, ④* e ⑤* :



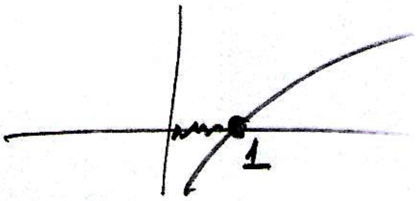
$$\boxed{\text{Dom} f = (0,1)\cup\left(1, 1+\log_2\frac{4}{3}\right)}$$

4.

a) $f(x) = 10^{-x^2}$

$\therefore \log_{10} y = -x^2 \log_{10} 10 = -x^2 \leq 0$

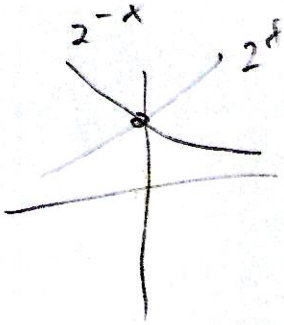
$\therefore \log_{10} y \leq 0$



$\therefore 0 < y \leq 1$

$\therefore$ // $\text{Im} f = [0,1]$ //

b) $f(x) = \dfrac{1}{1-2^{-x}}$; $x \neq 0$

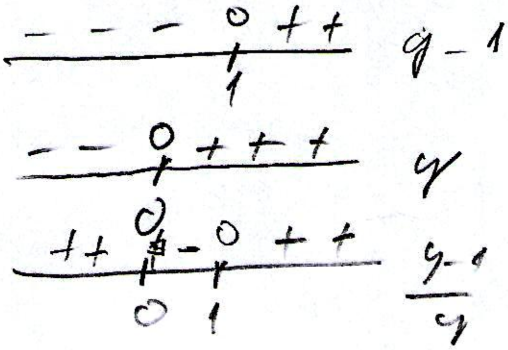


$y = \dfrac{1}{1-2^{-x}}$ $\therefore$ $y \neq 0$

$1 - 2^{-x} = \dfrac{1}{y}$

$1 - \dfrac{1}{y} = 2^{-x} > 0$

$\therefore \dfrac{y-1}{y} > 0$



$\dfrac{y-1}{y} > 0 \Rightarrow y < 0$ ou $y > 1$

$\therefore$ // $\text{Im} f = (-\infty, 0) \cup (1, +\infty)$ //

e)

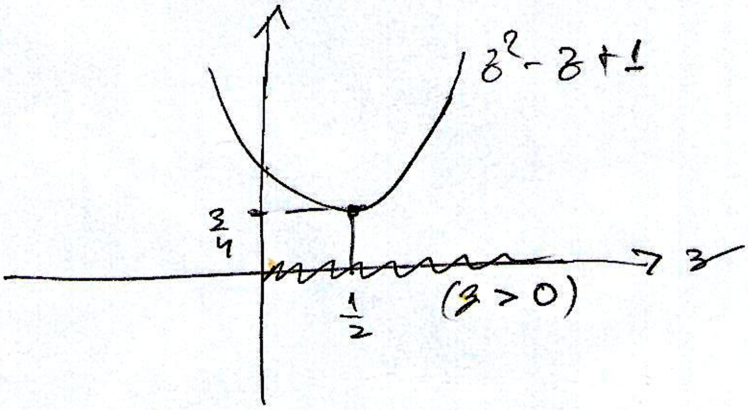
$f(x) = 4^x - 2^x + 1$

$y = 4^x - 2^x + 1$

$= (2^x)^2 - 2^x + 1$

Seja $z = 2^x > 0$

$\therefore$ $y = z^2 - z + 1$



$V = \left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$

$= \left(\frac{1}{2}, -\frac{-3}{4}\right)$

$= \left(\frac{1}{2}, \frac{3}{4}\right)$

Quando $z > 0$ temos $\underbrace{z^2 - z + 1} \geq \frac{3}{4}$

$y \geq \frac{3}{4}$

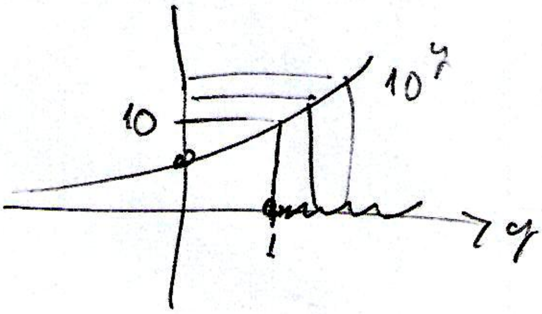
$\therefore$ $\| \text{Im} f = \left[\frac{3}{4}, +\infty\right) \|$

d) $f(x) = \log_{10}(x^2+10)$; $x \in \mathbb{R}$

$y = \log_{10}(x^2+10)$,

$\therefore \; 10^y = x^2+10 \geq 10^1$

$\therefore$



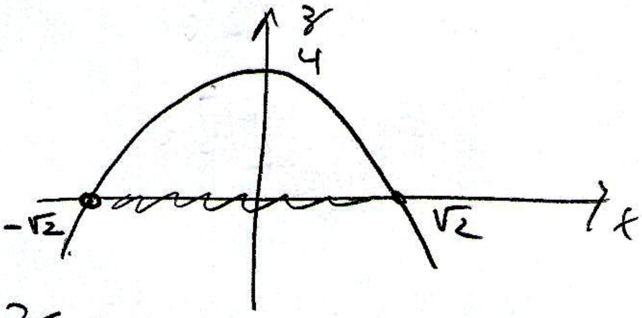
$y \geq 1$

$\therefore \; /\!/ \; \text{Im} f = [1, +\infty) \; /\!/$

---

e) $f(x) = \log_2(4-x^4)$; $4-x^4 > 0$

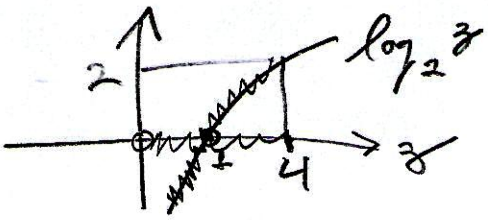
Seja $z = 4-x^4$



$y = \log_2(4-x^4) = \log_2 z$

Sendo $z = 4-x^4$ temos que $z > 0 \Rightarrow$

$0 < z \leq 4$



$y = \log_2 z$

$\therefore \; 2^y = z$, daí $z = 4 \Rightarrow y = 2$

Assim, vemos do gráfico $y = \log_2 z$ que quando $0 < z \leq 4$ temos $y \leq 2$

$\therefore$ // $\operatorname{Im} f = (-\infty, 2]$ //

---

f: $y = \log_3 x + \log_x 3$; $x > 0$
$x \neq 1$ (para definirmos $\log_x 3$)

$\therefore$ $y = \log_3 x + \frac{1}{\log_3 x}$

Seja $z = \log_3 x$

$\therefore$ $x \neq 1 \Longrightarrow 3^z \neq 1 \rightarrow \underline{\underline{z \neq 0}}$

Daí $y = \log_3 x + \frac{1}{\log_3 x} = z + \frac{1}{z}$

$\therefore$ $y = \frac{z^2+1}{z}$

$\therefore$ $yz = z^2 + 1$

$\therefore$ $z^2 - yz + 1 = 0$

$z = (y \pm \sqrt{y^2-4})/2$

$\rightarrow$ $z \in \mathbb{R}, z \neq 0$
tem-se basicamente que impor que
$y^2 - 4 \geq 0$
$\therefore$ $y \leq -2$ ou $y \geq 2$

// $\operatorname{Im} f = (-\infty, -2] \cup$
$\cup [2, +\infty)$ //

5. $\log_{\frac{1}{4}}(x+1) = \lg_4(x-1)$ (*) $\begin{cases} x+1>0 \\ x-1>0 \end{cases}$

Seja $y = \lg_{\frac{1}{4}}(x+1) \Longrightarrow \left(\frac{1}{4}\right)^y = x+1$

$4^{-y} = x+1$

Seja $z = \lg_4(x-1) \Longrightarrow 4^z = x-1$ (*1)

De (*) : $y = z$

$\therefore \quad 4^{-y} = x+1$

$\therefore \quad 4^{-z} = x+1$

$\therefore \quad \frac{1}{4^z} = x+1$

De (*1) : $\frac{1}{x-1} = x+1$

$\therefore \quad 1 = (x+1)(x-1)$

$1 = x^2 - 1$

$\therefore \quad x^2 - 2 = 0 \quad \therefore \quad x = \pm\sqrt{2}$ (**)

Mas $\left.\begin{matrix} x+1>0 \\ x-1>0 \end{matrix}\right\} \Longrightarrow \left.\begin{matrix} x>-1 \\ x>1 \end{matrix}\right\} \therefore \; x>1$

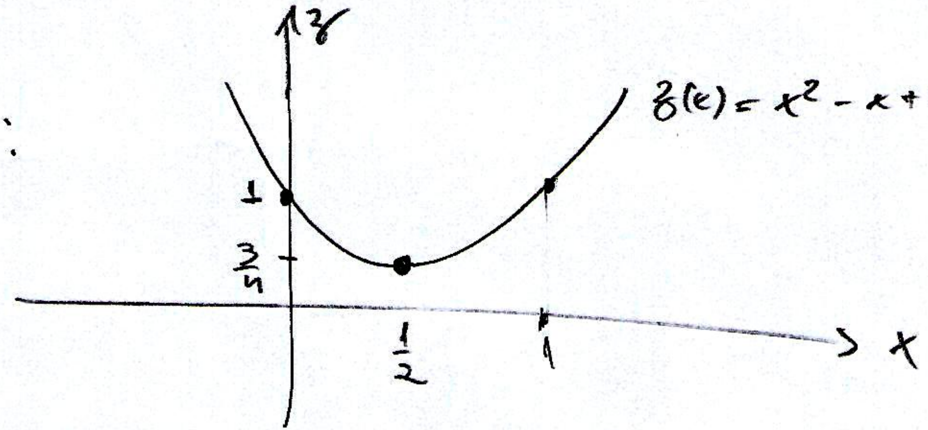
Daí devemos ter em (**) : $//x = \sqrt{2}//$.

6. $$\begin{cases} f : A \to \mathbb{R} \\ x \to f(x) = |\ln(x^2 - x + 1)| \end{cases}$$

Seja $z = x^2 - x + 1$.

$\therefore \quad f(x) = |\ln z|$

→ Para $f(x)$ ser injetiva devemos inicialmente determinar os intervalos onde $z(x)$ é injetiva.



Temos:

$$\begin{cases} \Delta = b^2 - 4ac \\ \quad = -3 \\ V = \left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right) = \left(\frac{1}{2}, \frac{3}{4}\right) \end{cases}$$

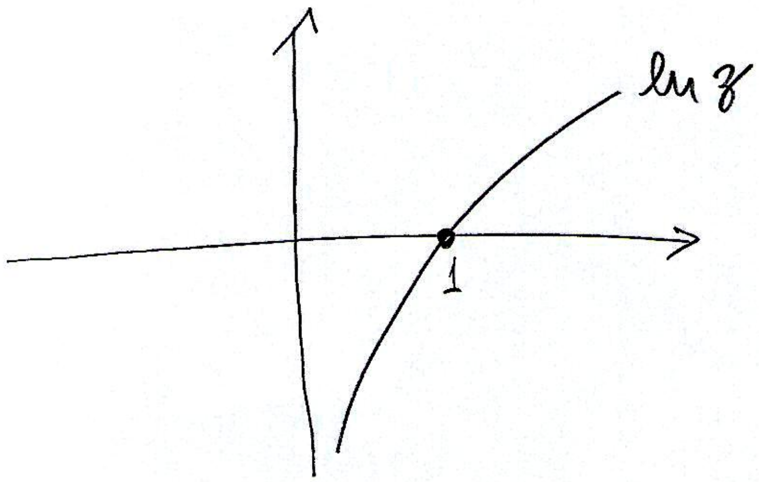
Daí $z(x)$ injetiva se $-\infty < x \leq \frac{1}{2}$ (*)

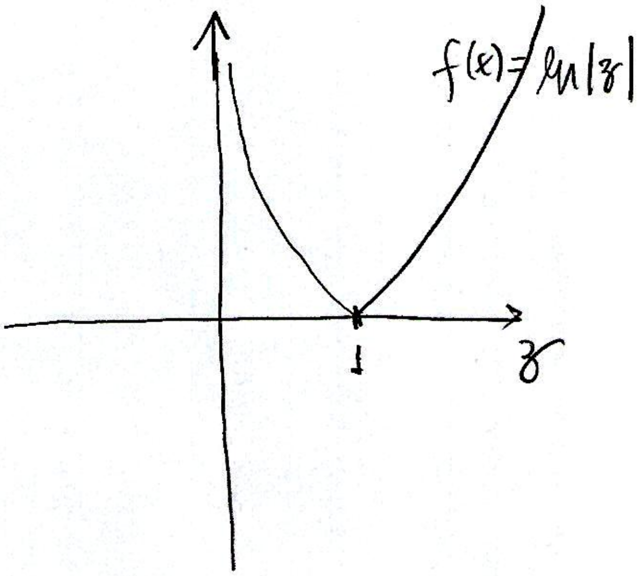
ou

$x \geq \frac{1}{2}$ (**)

→ Devemos agora analisar para cada intervalo anterior se $f(x) = |\ln z|$ é injetiva.

Seja o gráfico de $f(x) = | \ln z(x) |$





Vemos que f(x) será injetiva se

$0 < z \le 1$ (***)

ou

$z \geq 1$ (**)

→ Na faixa $-\infty < x \le \frac{1}{2}$ temos que

$\frac{3}{4} \le z$

e afim de termos verificado (***) e (**)

devemos fazer

$\frac{3}{4} \le z \le 1$ ou $z \geq 1$

Mas do gráfico de z(x) vemos que:

$\underline{0 \le x \le \frac{1}{2}} \Rightarrow \frac{3}{4} \le z \le 1$

$\underline{-\infty < X \le 0} \Rightarrow z \geq 1$

→ Na faixa $x \geq \frac{1}{2}$ temos que

$$\frac{3}{4} \leq z$$

e novamente devemos fazer

$$\frac{3}{4} \leq z \leq 1 \quad \text{ou} \quad z \geq 1$$

Mas, do gráfico $z(x)$ vemos que:

$$\underline{\frac{1}{2} \leq x \leq 1} \Longrightarrow \frac{3}{4} \leq z \leq 1$$

$$\underline{x \geq 1} \Longrightarrow z \geq 1 \, .$$

Temos então as possíveis escolhas para A que tornam $f(x) = |\ln(x^2 - x + 1)|$ injetiva:

$$\left\{ \begin{array}{l} A = [0, \frac{1}{2}] \text{ ou } A = (-\infty, 0] \text{ ou } A = [\frac{1}{2}, 1] \text{ ou} \\ A = [1, +\infty) \end{array} \right.$$

7. $f(x) = \ln(x^2+x+1)$, $x \in \mathbb{R}$

$\left\{ \begin{array}{l} h: \mathbb{R} \to \mathbb{R} \\ \text{par} \end{array} \right.$, $\begin{array}{l} g: \mathbb{R} \to \mathbb{R} \\ \text{ímpar} \end{array}$

$f(x) = g(x) + h(x)$, $x \in \mathbb{R}$

Seja

$$h(x) = \frac{f(x) + f(-x)}{2} \quad (h \text{ é par})$$

$$g(x) = \frac{f(x) - f(-x)}{2} \quad (g \text{ é ímpar})$$

Temos $h(x) + g(x) = f(x)$.

Daí:

$$h(x) = \frac{f(x) + f(-x)}{2} =$$

$$= \frac{1}{2}\{\ln(x^2+x+1) + \ln(x^2-x+1)\}$$

$$= \frac{1}{2}\ln(x^2+x+1)(x^2-x+1)$$

$$= \frac{1}{2}\ln(x^4 \cancel{-x^3} + \cancel{x^2} + \cancel{x^3} - \cancel{x^2} + \cancel{x} + x^2 - \cancel{x} + 1)$$

$$// h(x) = \frac{1}{2}\ln(x^4+x^2+1) //$$

$$g(x) = \frac{1}{2}\{\ln(x^2+x+1) - \ln(x^2-x+1)\}$$

$$/ g(x) = \frac{1}{2}\ln\left(\frac{x^2+x+1}{x^2-x+1}\right) //$$

8. $$\begin{cases} a^2+b^2 = 7ab \quad , \quad (ab > 0) \\ \log \dfrac{a+b}{3} = \dfrac{1}{2}(\lg a + \lg b) \end{cases}$$

Temos $a^2+b^2 = 7ab$

$$\therefore \quad a^2+b^2+2ab = 7ab+2ab$$

$$(a+b)^2 = 9ab$$

$$\therefore \quad \frac{(a+b)^2}{9} = ab > 0$$

$$\therefore \quad \ln\left[\frac{a+b}{3}\right]^2 = \ln ab$$

$$2\ln\frac{a+b}{3} = \ln a + \ln b$$

$$\Big/\!\!\Big/ \ln\frac{a+b}{3} = \frac{1}{2}(\ln a + \ln b) \Big/\!\!\Big/$$

a) $\log_a b \, \lg_b c = \log_a c$

Seja $x = \log_a b \rightarrow a^x = b$ $\textcircled{*}$

$y = \lg_b c \longrightarrow b^y = c$ $\textcircled{**}$

$\textcircled{*} \rightarrow \textcircled{**}$: $(a^x)^y = c$

$a^{xy} = c$

$\therefore \log_a c = \underset{\sim}{xy}$

$// \log_a c = \lg_a b \, \lg_b c //$

---

b) $\log_a b = \dfrac{1}{\lg_b a}$

Seja $x = \log_a b \therefore a^x = b$ $\textcircled{*}$

$y = \lg_b a \therefore b^y = a$ $\textcircled{**}$

$\textcircled{*} \rightarrow \textcircled{**}$: $(a^x)^y = a$

$a^{xy} = a^1$

$\therefore xy = 1 \therefore x = \dfrac{1}{y}$

$\therefore // \log_a b = \dfrac{1}{\log_b a} //$

10. $\log_2 5$ é irracional

Suponha que

$(*)\ \log_2 5 = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $q > 0$

Então

$$2^{\frac{p}{q}} = 5$$

$$\therefore \left(2^{\frac{p}{q}}\right)^q = 5^q$$

$$\therefore 2^p = 5^q \quad (**)$$

Mas 2 é par $\Longrightarrow$ 2^p é par

5 é ímpar $\Longrightarrow$ 5^q é ímpar

$\therefore$ $2^p =$ par $=$ ímpar $= 5^q$

o que é um absurdo. Logo, a suposição $(*)$ é falso, i.e.

$\log_2 5$ é irracional.

11. $\begin{cases} b, c, p, q > 0 \\ \frac{b}{c} = \frac{p}{q} \quad (*) \end{cases}$

$$\left(\ln b - \ln c = \ln p - \ln q \right)$$

$$// \ln b - \ln c = \ln \frac{b}{c} \overset{(*)}{=} \ln \frac{p}{q}$$
$$= \ln p - \ln q //$$

--- // --- // ---

12. $$f(x) = \frac{e^{x}}{e^{2x}+1}$$

$$f(-x) = \frac{e^{-x}}{e^{-2x}+1} = \frac{e^{-x}}{\frac{1}{e^{2x}}+1} =$$

$$= \frac{e^{-x}}{\frac{1+e^{2x}}{e^{2x}}} = \frac{e^{-x}\,e^{2x}}{1+e^{2x}}$$

$$= \frac{e^{x}}{e^{2x}+1}$$

$$= f(x)$$

$\therefore$ $\underline{f \text{ é par}}$.

13. $f(x) = \frac{1}{2}(a^x + a^{-x}) \quad (a > 0)$

$$\left(f(x+y) + f(x-y) = 2f(x)f(y) \right)$$

[illegible]

$$f(x+y) = \frac{1}{2}\left(a^{x+y} + a^{-(x+y)}\right)$$

$$f(x-y) = \frac{1}{2}\left(a^{x-y} + a^{-(x-y)}\right) = \frac{1}{2}\left(a^{x-y} + a^{y-x}\right)$$

[illegible]

$$/\!/\ f(x+y) + f(x-y) = \frac{1}{2}\left(a^{x+y} + a^{-(x+y)}\right) + \frac{1}{2}\left(a^{x-y} + a^{y-x}\right)$$

$$= \underline{\frac{1}{2}a^x a^y} + \underline{\frac{1}{2}a^{-x}a^{-y}} + \underline{\frac{1}{2}a^x a^{-y}} + \underline{\frac{1}{2}a^y a^{-x}}$$

$$= \underline{\frac{1}{2}a^x \overline{(a^y + a^{-y})}} + \underline{\frac{1}{2}a^{-x}\overline{(a^y + a^{-y})}}$$

$$= \frac{1}{2}(a^x + a^{-x})\ \overline{(a^y + a^{-y})}$$

$$= 2\ \underbrace{\frac{1}{2}(a^x + a^{-x})}\ \underbrace{\frac{1}{2}(a^y + a^{-y})}$$

$$= 2\ f(x)\ \ f(y) \ /\!/$$

14. $\dfrac{\log_a n}{\log_{am} n} = 1 + \log_a m$ $\quad (a > 0,\ m > 0,\ n > 0)$

Seja $x = \log_a n \quad \therefore \quad a^x = n \quad ⊛$

$y = \log_{am} n \quad \therefore \quad (am)^y = n$

$\therefore \quad a^y m^y = n \quad ⊛⊛$

$⊛ \rightarrow ⊛⊛ \quad : \quad a^y m^y = a^x$

$\therefore \quad m^y = a^{x-y}$

$m = a^{\frac{x-y}{y}}$

$\therefore \quad \log_a m = \dfrac{x-y}{y} = \dfrac{x}{y} - 1$

$\therefore \quad \dfrac{\textcircled{x}}{\boxed{y}} = 1 + \log_a m$

$\therefore \quad \left\| \; \dfrac{\textcircled{\log_a n}}{\boxed{\log_{am} n}} = 1 + \log_a m \; \right\|$

5. x, y, z :

$$\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\lg y} = \frac{z(x+y-z)}{\log z}$$

Mostrar que :

$$x^y y^x = z^y y^z = x^z z^x .$$

Seja

$$a = \frac{x(y+z-x)}{\lg x} \therefore \log x = \frac{x(y+z-x)}{a}$$

$$\therefore 10^{\frac{x(y+z-x)}{a}} = x \quad ①$$

$$b = \frac{y(z+x-y)}{\lg y} \therefore \log y = \frac{y(z+x-y)}{b}$$

$$\therefore 10^{\frac{y(z+x-y)}{b}} = y \quad ②$$

$$c = \frac{z(x+y-z)}{\log z} \therefore \log z = \frac{z(x+y-z)}{c}$$

$$\therefore 10^{\frac{z(x+y-z)}{c}} = z \quad ③$$

Então

$$x^y y^x = \left[10^{\frac{x(y+z-x)}{a}}\right]^y \left[10^{\frac{y(z+x-y)}{b}}\right]^x$$

$$= 10^{\frac{xy(y+z-x)}{a}} \, 10^{\frac{xy(z+x-y)}{b}} =$$

$$= 10^{\frac{xy(y+z-x)}{a}} \; 10^{\frac{xy(z+x-y)}{b}}$$

$$= 10^{\frac{xy}{a}(y+z-x)+\frac{xy}{b}(z+x-y)}$$

Mà $a=b=c$, do đó

$$// x^y y^x = 10^{\frac{xy}{a}(\cancel{y}+z-\cancel{x}+z+\cancel{x}-\cancel{y})}$$

$$= 10^{\frac{2xyz}{a}} \quad // \quad (*)$$

Tương tự:

$$// z^y y^z = \left(10^{\frac{z(x+y-z)}{c}}\right)^y \left(10^{\frac{y(x+z-y)}{b}}\right)^z$$

$$= 10^{\frac{zy(x+y-z)}{c}} \; 10^{\frac{yz(x+z-y)}{b}} \qquad (c=b)$$

$$= 10^{\frac{yz}{c}[x+\cancel{y}-\cancel{z}+x+\cancel{z}-\cancel{y}]}$$

$$= 10^{\frac{2xyz}{c}} = 10^{\frac{2xyz}{a}} \quad // \qquad (a=b=c) \quad (*1)$$

e

$$// x^z z^x = \left(10^{\frac{x(y+z-x)}{a}}\right)^z \left(10^{\frac{z(x+y-z)}{c}}\right)^x$$

$$= 10^{\frac{xz(y+z-x)}{a}} \; 10^{\frac{xz(x+y-z)}{c}}$$

$$= 10^{\frac{xz}{a}(y+\cancel{z}-\cancel{x}+\cancel{x}+y-\cancel{z})}$$

$$= 10^{\frac{2xyz}{a}} \quad // \quad (**1)$$

Do (*), (*1) e (**1): $\boxed{x^y y^x = z^y y^z = x^z z^x}$

16.

$$a^{\frac{\log(\log a)}{\log a}} = z \qquad \textcircled{*}$$

Mas

$$\frac{\log(\log a)}{\log a} = \log_a \log a \qquad \textcircled{**}$$

$\textcircled{**} \to \textcircled{*}$:

$$a^{\log_a \log a} = z$$

$$\therefore \quad \log_{\underbrace{a}} z = \log_{\underbrace{a}} (\log a)$$

$$\therefore \quad z = \log a$$

$$\therefore \quad \left\| \, a^{\frac{\log(\log a)}{\log a}} = \log a \, \right\|$$

17.

$$\begin{cases} y = 10^{\frac{1}{1-\log_{10} x}} \\ z = 10^{\frac{1}{1-\log_{10} y}} \end{cases}$$

$$y = 10^{\frac{1}{1-\log_{10} x}} \quad \therefore \quad \log_{10} y = \frac{1}{1-\log_{10} x} \qquad \textcircled{*}$$

$$z = 10^{\frac{1}{1-\log_{10} y}} \quad \therefore \quad \log_{10} z = \frac{1}{1-\log_{10} y} \qquad \textcircled{**}$$

$\textcircled{*} \rightarrow \textcircled{**}$:

$$\log_{10} z = \frac{1}{1 - \frac{1}{1-\log_{10} x}} = \frac{1}{\frac{1-\log_{10} x - 1}{1-\log_{10} x}}$$

$$\log_{10} z = \frac{1-\log_{10} x}{-\log_{10} x}$$

$$= -\frac{1}{\log_{10} x} + 1$$

$$\therefore \quad \frac{1}{\log_{10} x} = 1 - \log_{10} z$$

$$\log_{10} x = \frac{1}{1-\log_{10} z}$$

$$\therefore \left\| \, x = 10^{\frac{1}{1-\log_{10} z}} \, \right\|$$

18. $\begin{cases} a, b, c > 0 \\ a^2 + b^2 = c^2 \end{cases}$

$$\left(\log_{b+c} a + \log_{c-b} a = 2 \log_{b+c} a \, \log_{c-b} a \right)$$

Seja $x = \log_{b+c} a$ $\therefore$ $(b+c)^x = a$

$$(b+c) = a^{\frac{1}{x}} \quad (*)$$

$y = \log_{e-b} a$ $\therefore$ $(e-b)^y = a$

$$(c-b) = a^{\frac{1}{y}} \quad (**)$$

Do (*) e (**):

$$(b+c)(c-b) = a^{\frac{1}{x}} a^{\frac{1}{y}}$$

$$\underbrace{c^2 - b^2} = a^{\frac{1}{x}+\frac{1}{y}}$$

$$a^2 = a^{\frac{y+x}{xy}}$$

$$\therefore \quad 2 = \frac{x+y}{xy}$$

$$\therefore \quad \underbrace{x+y} = 2xy$$

$$\left\| \log_{b+c} a + \log_{c-b} a = 2 \log_{b+c} a \, \log_{c-b} a \right\|$$

19. $\begin{cases} a>0,\ e>0 \\ b=\sqrt{ac},\quad a\neq 1,\ c\neq 1,\ ac\neq 1,\ N>0 \end{cases}$

Seja

$$\left.\begin{aligned} x &= \log_a N \rightarrow a^x = N \\ y &= \log_b N \rightarrow b^y = N \\ z &= \log_c N \rightarrow e^z = N \end{aligned}\right\} a^x = b^y = c^z$$

$$\therefore \begin{cases} a = b^{\frac{y}{x}} \\ e = b^{\frac{y}{z}} \end{cases}$$

Mas $b = \sqrt{ac} \Rightarrow b^2 = ac = b^{\frac{y}{x}}\, b^{\frac{y}{z}}$

$$b^2 = b^{\frac{y}{x}+\frac{y}{z}}$$

$$b^2 = b^{\frac{yz+xy}{xz}}$$

$$\therefore \quad 2 = \frac{yz+xy}{xz}$$

$$2xz = y(z+x)$$

$$\therefore \quad y = \frac{2xz}{x+z} \quad \circledast$$

Daí :

19. cont.

$$\frac{\log_a N - \log_b N}{\log_b N - \log_c N} = \frac{x - y}{y - z}$$

$$\overset{(*)}{=} \frac{x - \frac{2xz}{x+z}}{\frac{2xz}{x+z} - z}$$

$$= \frac{\frac{x^2 + xz - 2xz}{\cancel{x+z}}}{\frac{2xz - zx - z^2}{\cancel{x+z}}}$$

$$= \frac{x^2 - xz}{-z^2 + xz}$$

$$= \frac{x(\cancel{x-z})}{z(\cancel{x-z})} = \frac{x}{z} = \frac{\log_a N}{\log_b N}$$

$$\therefore \left\| \frac{\log_a N - \log_b N}{\log_b N - \log_c N} = \frac{\log_a N}{\log_c N} \right\|$$

20.

$$\log_{a_1 \cdots a_m} x = \frac{1}{\frac{1}{\log_{a_1} x} + \cdots + \frac{1}{\log_{a_m} x}}$$

Temos que :

$$\frac{1}{\log_a x} = \log_x a$$

Então :

$$// \frac{1}{\frac{1}{\log_{a_1} x} + \cdots + \frac{1}{\log_{a_m} x}} = \frac{1}{\log_x a_1 + \cdots + \log_x a_m}$$

$$= \frac{1}{\log_x a_1 \cdots a_m}$$

$$= \log_{a_1 \cdots a_m} x \; //$$

21.

→ $a, a_1 \ldots a_n \ldots$; p.g. razão $q > 0$

$\therefore \quad a_n = aq^n$

→ $b, b_1, \ldots b_n \ldots$; p.a. diferença $r > 0$

$\therefore \quad b_n = b + nr$

Daí

$$\log_\beta a_n - b_n = \log_\beta a - b$$

$$\log_\beta (aq^n) - b - nr = \log_\beta a - b$$

$$\cancel{\log_\beta a} + \overbrace{n \log_\beta q} - \cancel{b} - \overbrace{nr} = \cancel{\log_\beta a} - \cancel{b}$$

$$n\left(\log_\beta q - r\right) = 0$$

$$n \neq 0 \Rightarrow \log_\beta q = r$$

$$\therefore \quad \beta^r = q$$

$$\therefore \quad \boxed{\beta = q^{1/r}}$$