Cálculo A - Prova 2

1. Calcule
$$\lim_{x\to 0^+} \frac{\arccos(1-x)}{\sqrt{x}}$$ 2.0

2. Calcule
$$\lim_{x\to 1} \frac{x^2-1}{\sqrt[5]{2x^5-1}-1}$$ 1.0

3. Mostre que $(\operatorname{arcsec} x)' = \frac{1}{x\sqrt{x^2-1}}$ 1.0

4. Um reservatório tem a forma de um cone com altura 6m, e o topo do cone tem raio 4m. Enche-se o reservatório com água a razão de 2m^3/segundo. Quão rápido aumenta o nível da água do reservatório quando a água atinge um nível de 4m? 1.5

5. Derive $y = \log_{f(x)} g(x)$ 1.5

Fórmulas:

- $(x^n)' = nx^{n-1}$, $n \in \mathbb{R}$
- $(e^x)' = e^x$ $\quad (a^x) = (\ln a)a^x$ $\quad (\ln x)' = \frac{1}{x}$ $\quad (\log_a x)' = \frac{1}{\ln a}\frac{1}{x}$
- $(\sin x)' = \cos x$ $\quad (\cos x)' = -\sin x$ $\quad (\tan x)' = \sec^2 x$

$(\cot x)' = -\csc^2 x$ $\quad (\sec x)' = (\tan x)\sec x$ $\quad (\csc x)' = -(\cot x)\csc x$

- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ $\quad (\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$ $\quad (\arctan x)' = \frac{1}{1+x^2}$

$(\text{arccot}x)' = \frac{-1}{1+x^2}$ $\quad (\text{arcsec}x)' = \frac{1}{x\sqrt{x^2-1}}$ $\quad (\text{arccosec}x)' = \frac{-1}{x\sqrt{x^2-1}}$

- $(f \pm g)' = f' \pm g'$ $\quad (fg)' = f'g + fg'$ $\quad \left(\frac{f(x)}{g(x)}\right) = \frac{f'g - fg'}{g^2}$
- $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1})$
- $\sin^2 x + \cos^2 x = 1$ $\quad 1 + \tan^2 x = \sec^2 x$ $\quad 1 + \cot^2 x = \csc^2 x$

$\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$ $\quad \cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$ $\quad \cos^2 x = \frac{1+\cos 2x}{2}$ $\quad \sin^2 x = \frac{1-\cos 2x}{2}$

$\sin 2x = 2\sin x \cos x$ $\quad \cos 2x = \cos^2 x - \sin^2 x$

1. $\lim_{x \to 0^+} \frac{\arccos(1-x)}{\sqrt{x}}$

Seja $y = \arccos(1-x)$

$\therefore$

$\cos y = 1-x$

$y \in [0, \pi]$

Quando $x \to 0^+$, $y \to \arccos 1 = 0^+$

0.5

Daí

$$\lim_{x \to 0^+} \frac{\arccos(1-x)}{\sqrt{x}} = \lim_{y \to 0^+} \frac{y}{\sqrt{1-\cos y}} =$$

$$= \lim_{y \to 0^+} \frac{y}{\sqrt{1-\cos y}} \frac{\sqrt{1+\cos y}}{\sqrt{1+\cos y}}$$

0.75

$$= \lim_{y \to 0^+} \frac{y\sqrt{1+\cos y}}{\sqrt{1-\cos^2 y}}$$

$$= \lim_{y \to 0^+} \frac{y\sqrt{1+\cos y}}{\sqrt{\sin^2 y}}$$

$$= \lim_{y \to 0^+} \frac{y\sqrt{1+\cos y}}{\sin y}$$

$$= \lim_{y \to 0^+} \frac{\sqrt{1+\cos y}}{\frac{\sin y}{y}} = \frac{\lim_{y \to 0^+} \sqrt{1+\cos y}}{\lim_{y \to 0^+} \frac{\sin y}{y}} = \frac{\sqrt{2}}{1}$$

$= \sqrt{2}$

2.0

2. $\lim_{x \to 1} \dfrac{x^2-1}{\sqrt[5]{2x^5-1}-1}$

$$z^5 - a^5 = (z-a)(z^4 + z^3 a + z^2 a^2 + z a^3 + a^4)$$

$\downarrow$

$$\underbrace{\left(\sqrt[5]{2x^5-1}\right)^5 - 1^5} = \left(\sqrt[5]{2x^5-1}-1\right)\left((2x^5-1)^{\frac{4}{5}} + (2x^5-1)^{\frac{3}{5}} + (2x^5-1)^{\frac{2}{5}} + (2x^5-1)^{\frac{1}{5}} + 1\right)$$

$\therefore$ $2x^5 - 1 - 1 = 2(x^5-1)$

$\therefore$ $$\sqrt[5]{2x^5-1} - 1 = \frac{2(x^5-1)}{\left\{(2x^5-1)^{\frac{4}{5}} + (2x^5-1)^{\frac{3}{5}} + (2x^5-1)^{\frac{2}{5}} + (2x^5-1)^{\frac{1}{5}} + 1\right\}}$$

$\therefore$

$$\lim_{x \to 1} \frac{x^2-1}{\sqrt[5]{2x^5-1}-1} =$$

$$= \lim_{x \to 1} \frac{(x^2-1)\left\{(2x^5-1)^{\frac{4}{5}} + (2x^5-1)^{\frac{3}{5}} + (2x^5-1)^{\frac{2}{5}} + (2x^5-1)^{\frac{1}{5}} + 1\right\}}{2(x^5-1)}$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)\left\{(2x^5-1)^{\frac{4}{5}} + (2x^5-1)^{\frac{3}{5}} + (2x^5-1)^{\frac{2}{5}} + (2x^5-1)^{\frac{1}{5}} + 1\right\}}{2\cancel{(x-1)}(x^4+x^3+x^2+x+1)}$$

$$= \lim_{x \to 1} \frac{(x+1)\left\{\overbrace{(2x^5-1)^{\frac{4}{5}}}^{1} + \overbrace{(2x^5-1)^{\frac{3}{5}}}^{1} + \overbrace{(2x^5-1)^{\frac{2}{5}}}^{1} + \overbrace{(2x^5-1)^{\frac{1}{5}}}^{1} + \overbrace{1}^{1}\right\}}{2(x^4+x^3+x^2+x+1)}$$

$$= \frac{\cancel{2}\cdot\cancel{(1+1+1+1+1)}}{\cancel{2}\,\cancel{(1+1+1+1+1)}} = 1 \;//$$

2.0

3. $$(\operatorname{arcsec} x)' = \lim_{z\to x} \frac{\operatorname{arcsec} z - \operatorname{arcsec} x}{z - x}$$

Seja

$$w = \operatorname{arcsec} z$$
$$\therefore$$
$$\sec w = z$$
$$w \in [0, \tfrac{\pi}{2}) \cup [\pi, \tfrac{3\pi}{2})$$

$$a = \operatorname{arcsec} x$$
$$\therefore$$
$$\sec a = x$$
$$a \in [0, \tfrac{\pi}{2}) \cup [\pi, \tfrac{3\pi}{2})$$

0.5

Então, quando $z \to x$ temos que $w \to a$ daí,

$$(\operatorname{arcsec} x)' = \lim_{w\to a} \frac{w - a}{\sec w - \sec a}$$

$$= \lim_{w\to a} \frac{1}{\frac{\sec w - \sec a}{w - a}}$$

$$= \frac{1}{\lim_{w\to a} \frac{\sec w - \sec a}{w - a}}$$

$$= \frac{1}{\sec'(a)}$$

1.0

$$= \frac{1}{\sec a \operatorname{tg} a}$$

$$= \frac{1}{\underbrace{\sec(\operatorname{arcsec} x)}_{x} \, tg(\operatorname{arcsec} x)}$$

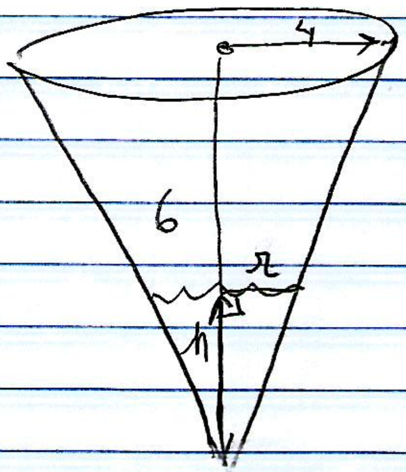
$$= \frac{1}{x\sqrt{1-\sec^2\underbrace{(\operatorname{arcsec} x)}_{x^2}}}$$

$$= \frac{1}{x\sqrt{1-x^2}}$$

$$\therefore \quad \boxed{(\operatorname{arcsec} x)' = \frac{1}{x\sqrt{1-x^2}}}$$

2.0

4.



Sejam

$h(t)$: nível de água do reservatório

$V = \frac{1}{3}\pi r^2 h$

Queremos obter $\frac{dh}{dt}$.

Temos que

$$\frac{6}{h} = \frac{4}{r}$$

0.5

$\therefore$

$$r = \frac{4h}{6} = \frac{2h}{3}$$

Daí $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \frac{4h^2}{9} h$

$$V = \frac{4\pi}{27} h^3$$

$\therefore$

$$\frac{dV}{dt} = \frac{4\pi}{27} 3h^2 \frac{dh}{dt}$$

$$\cancel{2} = \frac{\cancel{4}^{2}\pi}{\cancel{27}\,9} \cancel{3} h^2 \frac{dh}{dt}$$

$\therefore$

$$\frac{dh}{dt} = \frac{9}{2\pi}\frac{1}{h^2}$$

1.0

$$\left.\frac{dh}{dt}\right|_{h=4} = \frac{9}{2\pi}\frac{1}{16} = \frac{9}{32\pi} \text{ m/s}$$

5.

$$y = \log_{f(x)} g(x)$$

$$\Leftrightarrow$$

$$f(x)^{y} = g(x)$$ 0.5

$\therefore$

$$\left(f(x)^{y}\right)' = g'(x)$$

$$\left(e^{\ln f(x)^{y}}\right)' = g'(x)$$

$$\left(e^{y \ln f(x)}\right)' = g'(x)$$

$$\underbrace{e^{y \ln f(x)}} (y \ln f(x))' = g'(x)$$

$$f(x)^{y} \cdot \left(y' \ln f(x) + y \frac{1}{f(x)} f'(x)\right) = g'(x)$$

$$y' \ln f(x) + \frac{y}{f} f' = \frac{g'}{f^{y}}$$ 1.0

$\therefore$

$$y' \ln f = \frac{g'}{f^{y}} - \frac{y}{f} f'$$

$\therefore$

$$\| \; y' = \frac{1}{\ln f}\left(\frac{g'}{f^{y}} - \frac{y}{f} f'\right) \; \|$$

$$= \frac{1}{\ln f}\left(\frac{g'}{g} - \log_{f(x)} g(x) \frac{f'}{f}\right) \|$$

1.5

Outra Solução

$$\left(\log_{f(x)} g(x)\right)' = \left(\frac{\ln g(x)}{\ln f(x)}\right)'$$

$$= \frac{(\ln g(x))' \ln f(x) - \ln g(x) (\ln f(x))'}{(\ln f(x))^2}$$

$$= \frac{\frac{1}{g(x)} g'(x) \ln f(x) - \ln g(x) \frac{1}{f(x)} f'(x)}{(\ln f(x))^2}$$

$$= \frac{f g' \ln f - f' g \ln g}{f g (\ln f)^2} \quad //$$

1.5

Obs.: $$\frac{f g' \ln f - f' g \ln g}{f g (\ln f)^2} =$$

$$= \frac{g'}{g \ln f} - \frac{f' \ln g}{f (\ln f)^2}$$

$$= \frac{1}{\ln f} \frac{g'}{g} - \frac{f'}{f} \underbrace{\frac{\ln g}{\ln f}} \frac{1}{\ln f}$$

$$= \frac{1}{\ln f} \frac{g'}{g} - \frac{f'}{f} \ln_{f(x)} g(x) \frac{1}{\ln f}$$

$$= \frac{1}{\ln f} \left(\frac{g'}{g} - \log_f g \frac{f'}{f} \right) //$$