Cálculo A - Prova 2

1. Calcule
$$\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)$$ 2.0

2. Calcule
$$\lim_{x\to 0}\frac{\cos x-\cos 3x}{\sin x}$$ 2.0

3. Derive
$$y=\ln(\arccos\frac{1}{\sqrt{x}})$$ 2.0

4. Usando a definição de derivada como um limite calcule a derivada de $f(x)=\sec x$ 2.0

5. Uma pessoa de altura 2m está diante de um poste de luz cuja altura é 5m. Se a pessoa se afasta do poste caminhando com velocidade de 1.5m/s encontre a razão com que o comprimento de sua sombra varia quando ele está a 10m do poste. 2.0

Fórmulas:

- $(x^n)' = nx^{n-1},\ n\in\mathbb{R}$
- $(e^x)'=e^x \qquad (a^x)=(\ln a)a^x \qquad (\ln x)'=\frac{1}{x} \qquad (\log_a x)'=\frac{1}{\ln a}\frac{1}{x}$
- $(\sin x)'=\cos x \qquad (\cos x)'=-\sin x \qquad (\tan x)'=\sec^2 x$

$(\cot x)'=-\csc^2 x \qquad (\sec x)'=(\tan x)\sec x \qquad (\csc x)'=-(\cot x)\csc x$

- $(\arcsin x)'=\frac{1}{\sqrt{1-x^2}} \qquad (\arccos x)'=\frac{-1}{\sqrt{1-x^2}} \qquad (\arctan x)'=\frac{1}{1+x^2}$

$(\text{arccot}x)'=\frac{-1}{1+x^2} \qquad (\text{arcsec}x)'=\frac{1}{x\sqrt{x^2-1}} \qquad (\text{arccosec}x)'=\frac{-1}{x\sqrt{x^2-1}}$

- $(f\pm g)'=f'\pm g' \qquad (fg)'=f'g+fg' \qquad \left(\frac{f(x)}{g(x)}\right)=\frac{f'g-fg'}{g^2}$
- $x^n-a^n=(x-a)(x^{n-1}+x^{n-2}a+\ldots+xa^{n-2}+a^{n-1})$
- $\sin^2 x+\cos^2 x=1 \qquad 1+\tan^2 x=\sec^2 x \qquad 1+\cot^2 x=\csc^2 x$

$\sin(a\pm b)=\sin a\cos b\pm\sin b\cos a \qquad \cos(a\pm b)=\cos a\cos b\mp\sin a\sin b$

$\tan(a+b)=\frac{\tan a+\tan b}{1-\tan a\tan b} \qquad \cos^2 x=\frac{1+\cos 2x}{2} \qquad \sin^2 x=\frac{1-\cos 2x}{2}$

$\sin 2x=2\sin x\cos x \qquad \cos 2x=\cos^2 x-\sin^2 x$

1. $$\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)=$$

$$=\lim_{x\to 1}\left(\frac{1-x^3-3(1-x)}{(1-x)(1-x^3)}\right)$$

$$=\lim_{x\to 1}\left(\frac{1-x^3-3+3x}{(1-x)(1-x^3)}\right)$$

$$=\lim_{x\to 1}\left(\frac{-x^3+3x-2}{(1-x)(1-x^3)}\right)$$

$$\begin{array}{l|l} \cancel{x^3}-3x+2 & x-1 \\ \cancel{-x^3}+x^2 & x^2+x-2 \\ \hline \cancel{x^2}-3x+2 & \\ \cancel{-x^2}+x & \\ \hline -2x+2 & \\ +2x-2 & \\ \hline 0 & \end{array}$$

$$=\lim_{x\to 1}\frac{(-1)(x^3-3x+2)}{(1-x)(1-x^3)}$$

$$=\lim_{x\to 1}\frac{(-1)\cancel{(x-1)}(x^2+x-2)}{(-1)\cancel{(x-1)}(1-x^3)}$$

$$=\lim_{x\to 1}\frac{x^2+x-2}{(1-x^3)}$$

$$=\lim_{x\to 1}\frac{(x-1)(x+2)}{(-1)(x^3-1)}$$

$$=\lim_{x\to 1}\frac{\cancel{(x-1)}(x+2)}{(-1)\cancel{(x-1)}(x^2+x+1)}$$

$$=\lim_{x\to 1}\frac{x+2}{(-1)(x^2+x+1)}=\frac{\cancel{3}}{(-1)\cdot\cancel{(3)}}=-1\ //$$

2. $\lim\limits_{x \to 0} \dfrac{\cos x - \cos 3x}{\sin x} =$

$= \lim\limits_{x \to 0} \dfrac{\cos x - \cos(2x + x)}{\sin x}$

$= \lim\limits_{x \to 0} \dfrac{\cos x - [\cos 2x \cos x - \sin 2x \sin x]}{\sin x}$

$= \lim\limits_{x \to 0} \dfrac{\cos x - [(\cos^2 x - \sin^2 x)\cos x - 2\sin x \cos x \sin x]}{\sin x}$

$= \lim\limits_{x \to 0} \dfrac{\cos x - [\cos^3 x - \sin^2 x \cos x - 2\sin^2 x \cos x]}{\sin x}$

$= \lim\limits_{x \to 0} \dfrac{\cos x - \cos^3 x + 3\sin^2 x \cos x}{\sin x}$

$= \lim\limits_{x \to 0} \dfrac{\cos x(1 - \cos^2 x) + 3\sin^2 x \cos x}{\sin x}$

$= \lim\limits_{x \to 0} \dfrac{\cos x \sin^2 x + 3\sin^2 x \cos x}{\sin x}$

$= \lim\limits_{x \to 0} \cos x \sin x + 3\sin x \cos x$

$= \lim\limits_{x \to 0} 4\cos x \sin x$

$= 0.$

3. $y = \ln\left(\arccos \frac{1}{\sqrt{x}}\right)$

$$y' = \left[\ln\left(\arccos \frac{1}{\sqrt{x}}\right)\right]'$$

$$= \frac{1}{\arccos \frac{1}{\sqrt{x}}} \cdot \left(\arccos \frac{1}{\sqrt{x}}\right)'$$

$$= \frac{1}{\arccos \frac{1}{\sqrt{x}}} \cdot \frac{(-1)}{\sqrt{1-\left(\frac{1}{\sqrt{x}}\right)^2}} \cdot \left(\frac{1}{\sqrt{x}}\right)'$$

$$= \frac{1}{\arccos \frac{1}{\sqrt{x}}} \cdot \frac{-1}{\sqrt{1-\frac{1}{x}}} \cdot \left(\frac{-1}{2x^{3/2}}\right)$$

$$= \frac{1}{\arccos \frac{1}{\sqrt{x}}} \quad \frac{1}{2x^{3/2}\sqrt{1-\frac{1}{x}}}$$

$$= \frac{1}{\arccos \frac{1}{\sqrt{x}}} \cdot \frac{1}{2x^{3/2}\frac{\sqrt{x-1}}{\sqrt{x}}}$$

$$= \frac{1}{\arccos \frac{1}{\sqrt{x}}} \quad \frac{1}{2x\sqrt{x-1}}$$

$$= \frac{1}{2x\sqrt{x-1}} \quad \frac{1}{\arccos \frac{1}{\sqrt{x}}} \quad //$$

4. $f(x) = \sec x$

$$f'(x) = \lim_{z \to x} \frac{\sec z - \sec x}{z - x}$$

$$= \lim_{z \to x} \frac{\frac{1}{\cos z} - \frac{1}{\cos x}}{z - x}$$

$$= \lim_{z \to x} \left\{ \frac{\frac{\cos x - \cos z}{\cos z \cos x}}{z - x} \right\}$$

$$= \lim_{z \to x} \frac{\cos x - \cos z}{z - x} \frac{1}{\cos z \cos x}$$

$$= \lim_{z \to x} \frac{(-1)(\cos z - \cos x)}{(z - x)} \frac{1}{\cos z \cos x}$$

$$= -\lim_{z \to x} \frac{\cos z - \cos x}{z - x} \cdot \lim_{z \to x} \frac{1}{\cos z \cos x}$$

$$= -\cos' x \cdot \frac{1}{\cos^2 x}$$

$$= -(-)\sin x \cdot \frac{1}{\cos^2 x}$$

$$= \frac{\sin x}{\cos x} \frac{1}{\cos x}$$

$$= \operatorname{tg} x \sec x$$

Outra solução

$$(\sec x)' = \lim_{h\to 0} \frac{\sec(x+h) - \sec x}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1}{\cos(x+h)} - \dfrac{1}{\cos x}}{h}$$

$$= \lim_{h\to 0} \left\{ \frac{\dfrac{\cos x - \cos(x+h)}{\cos(x+h)\cos x}}{h} \right\}$$

$$= \lim_{h\to 0} \frac{\cos x - \cos(x+h)}{h\cos(x+h)\cos x}$$

$$= \lim_{h\to 0} \frac{\cos x - (\cos x\cos h - \sin x \sin h)}{h\cos(x+h)\cos x}$$

$$= \lim_{h\to 0} \frac{\cos x - \cos x\cos h + \sin x\sin h}{h\cos(x+h)\cos x}$$

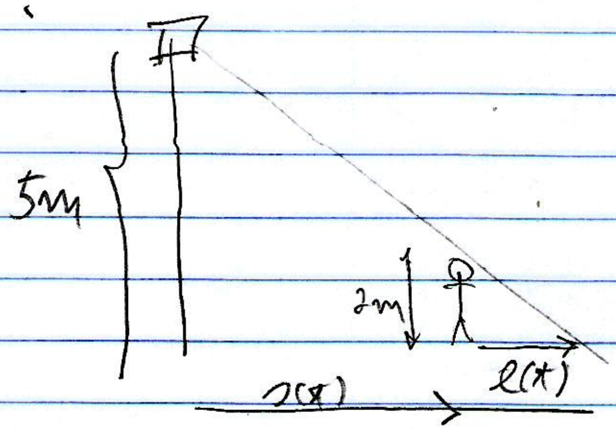
$$= \lim_{h\to 0} \frac{\cos x(1-\cos h) + \sin x\sin h}{h\cos(x+h)\cos x}$$

$$= \lim_{h\to 0} \frac{\cancel{\cos x}(1-\cos h)}{\cancel{\cos x}\, h\cos(x+h)} + \lim_{h\to 0} \underbrace{\frac{\sin x}{\cos x}}\, \frac{\sin h}{h}\, \frac{1}{\cos(x+h)}$$

$$= \underbrace{\lim_{h\to 0} \frac{(1-\cos h)}{h}}_{0} \cdot \underbrace{\lim_{h\to 0} \frac{1}{\cos(x+h)}}_{\frac{1}{\cos x}} + \underbrace{\operatorname{tg} x}\, \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{1}\, \underbrace{\lim_{h\to 0} \frac{1}{\cos(x+h)}}_{\frac{1}{\cos x}}$$

$$= \operatorname{tg} x \,\frac{1}{\cos x} = \operatorname{tg} x \cdot \sec x \quad //$$

5.



Seja

- $l(t)$: comprimento da sombra no instante t
- $s(t)$: posição da pessoa no instante t

Temos que

$$\frac{ds(t)}{dt} = 1{,}5 \text{ m/s}$$

$$\frac{5}{s(t)+l(t)} = \frac{2}{l(t)}$$

$$\therefore \quad 5l(t) = 2s(t) + 2l(t)$$

$$\therefore \quad 3l(t) = 2s(t)$$

$$l(t) = \frac{2}{3}s(t)$$

$$\therefore \quad \frac{dl(t)}{dt} = \frac{2}{3}\frac{ds}{dt} = \frac{2}{3}\,1{,}5$$

$$= 1 \text{ m/s} \;//$$