Cálculo A - Prova 2

1. Calcule
$$\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$$ 2.0

2. Seja $f(x) < g(x)$ para todo $x \neq a$ e assuma que $\lim_{x\to a} f(x) = L$ e $\lim_{x\to a} g(x) = M$. Mostre por meio de um exemplo que não se tem necessariamente $L < M$. 1.0

3. Derive $y = \arctan\frac{x}{1+\sqrt{1-x^2}}$
   Obs.: Deixe a expressão final na forma mais simplificada possível. 1.5

Fórmulas:

- $(x^n)' = nx^{n-1},\ n \in \mathbb{R}$
- $(e^x)' = e^x \qquad (a^x) = (\ln a)a^x \qquad (\ln x)' = \frac{1}{x} \qquad (\log_a x)' = \frac{1}{\ln a}\frac{1}{x}$
- $(\sin x)' = \cos x \qquad (\cos x)' = -\sin x \qquad (\tan x)' = \sec^2 x$
  $(\cot x)' = -\csc^2 x \qquad (\sec x)' = (\tan x)\sec x \qquad (\csc x)' = -(\cot x)\csc x$
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \qquad (\arccos x)' = \frac{-1}{\sqrt{1-x^2}} \qquad (\arctan x)' = \frac{1}{1+x^2}$
  $(\text{arccot} x)' = \frac{-1}{1+x^2} \qquad (\text{arcsec} x)' = \frac{1}{x\sqrt{x^2-1}} \qquad (\text{arccosec} x)' = \frac{-1}{x\sqrt{x^2-1}}$
- $(f \pm g)' = f' \pm g' \qquad (fg)' = f'g + fg' \qquad \left(\frac{f(x)}{g(x)}\right) = \frac{f'g-fg'}{g^2}$
- $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1})$
- $\sin^2 x + \cos^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$
  $\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a \qquad \cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$
  $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} \qquad \cos^2 x = \frac{1+\cos 2x}{2} \qquad \sin^2 x = \frac{1-\cos 2x}{2}$
  $\sin 2x = 2\sin x \cos x \qquad \cos 2x = \cos^2 x - \sin^2 x$

1. $\lim_{x \to +\infty} \left(\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x} \right)$

$= \lim_{x \to +\infty} \left(\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x} \right) \frac{\left(\sqrt{x + \sqrt{x + \sqrt{x}}} + \sqrt{x} \right)}{\left(\sqrt{x + \sqrt{x + \sqrt{x}}} + \sqrt{x} \right)}$

$= \lim_{x \to \infty} \frac{\cancel{x} + \sqrt{x + \sqrt{x}} - \cancel{x}}{\sqrt{x + \sqrt{x + \sqrt{x}}} + \sqrt{x}}$

$= \lim_{x \to \infty} \frac{\sqrt{x + \sqrt{x}}}{\sqrt{x + \sqrt{x + \sqrt{x}}} + \sqrt{x}}$ $\downarrow 0.25$

$= \lim_{x \to \infty} \frac{\sqrt{x + x^{1/2}}}{\sqrt{x + \sqrt{x + x^{1/2}}} + \sqrt{x}}$

$= \lim_{x \to \infty} \frac{\sqrt{x \left(1 + \frac{1}{x^{1/2}}\right)}}{\sqrt{x + \sqrt{x \left(1 + \frac{1}{x^{1/2}}\right)}} + \sqrt{x}}$

$= \lim_{x \to \infty} \frac{\sqrt{x} \sqrt{1 + \frac{1}{x^{1/2}}}}{\sqrt{x + \sqrt{x} \sqrt{1 + \frac{1}{x^{1/2}}}} + \sqrt{x}}$

$= \lim_{x \to \infty} \frac{\sqrt{x} \sqrt{1 + \frac{1}{x^{1/2}}}}{\sqrt{x + x^{1/2} \sqrt{1 + \frac{1}{x^{1/2}}}} + \sqrt{x}}$

$= \lim_{x \to \infty} \frac{\cancel{\sqrt{x}} \sqrt{1 + \frac{1}{x^{1/2}}}}{\cancel{\sqrt{x}} \sqrt{1 + \frac{1}{x^{1/2}} \sqrt{1 + \frac{1}{x^{1/2}}}} + \cancel{\sqrt{x}}}$

$$= \lim_{x \to \infty} \frac{\cancel{\sqrt{x}}\sqrt{1 + \frac{1}{x^{1/2}}}}{\cancel{\sqrt{x}}\sqrt{1 + \frac{1}{x^{1/2}}\sqrt{1 + \frac{1}{x^{1/2}}}} + \cancel{\sqrt{x}}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{1 + \cancel{\frac{1}{x^{1/2}}} \to 0}}{\sqrt{1 + \cancel{\frac{1}{x^{1/2}}}^{\,\to 0}\sqrt{1 + \cancel{\frac{1}{x^{1/2}}} \to 0}} + 1} = \frac{1}{2}$$

2.0

2. Seja

$$\begin{cases} f(x) = x^2 \\ g(x) = x^4 \end{cases}$$

Temos

$$\begin{cases} \lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 = 0 \\ \lim_{x \to 0} g(x) = \lim_{x \to 0} x^4 = 0 \end{cases}$$

e

$x^4 < x^2$ próximo de 0

1.0

3. $y = \operatorname{arctg} \dfrac{x}{1+\sqrt{1-x^2}}$

$$y' = \frac{1}{\left\{1+\left(\frac{x}{1+\sqrt{1-x^2}}\right)^2\right\}} \cdot \left(\frac{x}{1+\sqrt{1-x^2}}\right)'$$

$$= \frac{1}{\left(1+\frac{x^2}{\underbrace{1+2\sqrt{1-x^2}+(1-x^2)}_{2+2\sqrt{1-x^2}-x^2}}\right)} \cdot \left\{\frac{1+\sqrt{1-x^2} - x\frac{1}{2\sqrt{1-x^2}}(-2x)}{\left(1+\sqrt{1-x^2}\right)^2}\right\}$$

1.0

$$= \frac{1}{\left(\frac{2+2\sqrt{1-x^2}\cancel{-x^2}\cancel{+x^2}}{\left(1+\sqrt{1-x^2}\right)^2}\right)} \cdot \left\{\frac{\left(1+\sqrt{1-x^2}\right)2\sqrt{1-x^2}+2x^2}{2\sqrt{1-x^2}\left(1+\sqrt{1-x^2}\right)^2}\right\}$$

$$= \frac{\cancel{\left(1+\sqrt{1-x^2}\right)^2}}{2\left(1+\sqrt{1-x^2}\right)} \cdot \left\{\frac{2\sqrt{1-x^2}+2(1-\cancel{x^2})+\cancel{2x^2}}{2\sqrt{1-x^2}\cancel{\left(1+\sqrt{1-x^2}\right)^2}}\right\}$$

$$= \frac{\cancel{\left(2\sqrt{1-x^2}+2\right)}}{\cancel{2\left(1+\sqrt{1-x^2}\right)}2\sqrt{1-x^2}} = \frac{1}{2\sqrt{1-x^2}} \quad //$$

0.5