Cálculo A - Prova 1 [2014(1)]

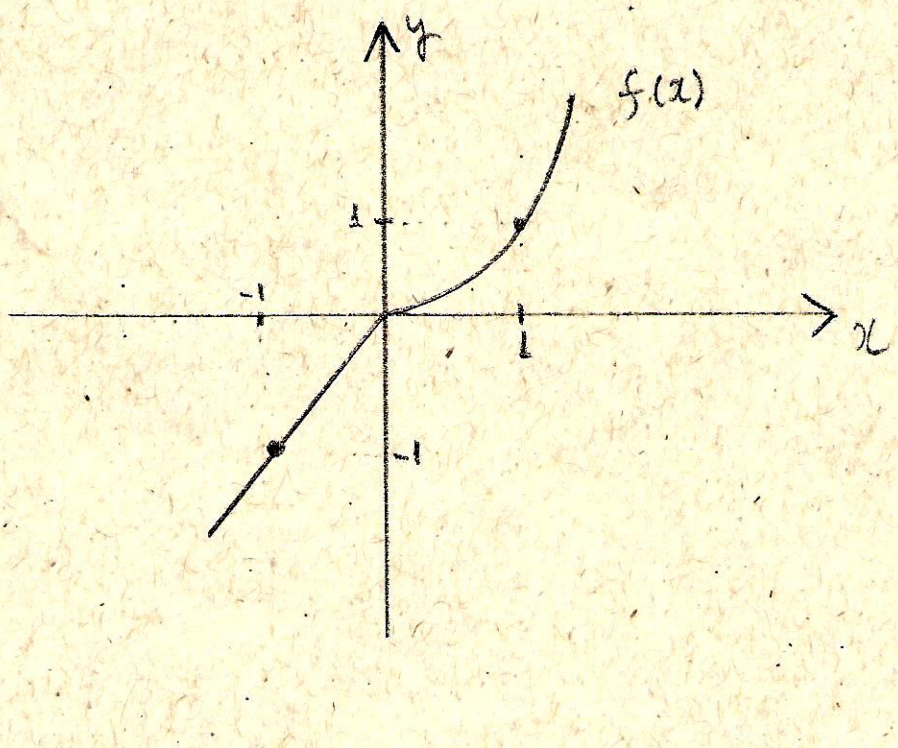
1. Dê o domínio de $f(x) = \sqrt{4 - \sqrt{1 + 9x^2}}$
2. Dê a imagem de $f(x) = \cos^2 x - \sin x$
3. Resolva

$$\log_x 49 = \log_{x^2} 7 + \log_{2x} 7$$

OU

$$\arccos x - \arcsin x = \arccos(\sqrt{3x})$$

4. Construa o gráfico de $f(2|x| + 1)$ a partir do gráfico da função $f(x)$ mostrado abaixo



22^2
$22 \cdot 22$
44
44
484

Prova 1 - Cálculo A

1. $f(x) = \sqrt{4 - \sqrt{1+9x^2}}$

$$\begin{cases} 4 - \sqrt{1+9x^2} \geq 0 & (*) \\ 1+9x^2 \geq 0 & (**) \end{cases}$$ 0.5

(**) é satisfeita para todo $x \in \mathbb{R}$

(*): $4 - \sqrt{1+9x^2} \geq 0$

$4 \geq \sqrt{1+9x^2} > 0$

$\therefore$ $16 \geq 1+9x^2$

$0 \geq 9x^2 - 15$ 0.5

$0 \geq x^2 - \frac{5}{3}$

$+ \quad 0 \quad - \quad 0 \quad +$

$-\sqrt{\frac{5}{3}} \qquad \sqrt{\frac{5}{3}}$

$\therefore$ $-\sqrt{\frac{5}{3}} \leq x \leq \sqrt{\frac{5}{3}}$ (1.0)

$\therefore$ $\text{Dom } f = \left[-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right]$

2. $f(x) = \cos^2 x - \sin x$

$y = \cos^2 x - \sin x$

$y = 1 - \sin^2 x - \sin x$

$y = -\sin^2 x - \sin x + 1$ (0.25)

Seja

$z = \sin x \quad ; \quad -1 \le z \le 1$

Então

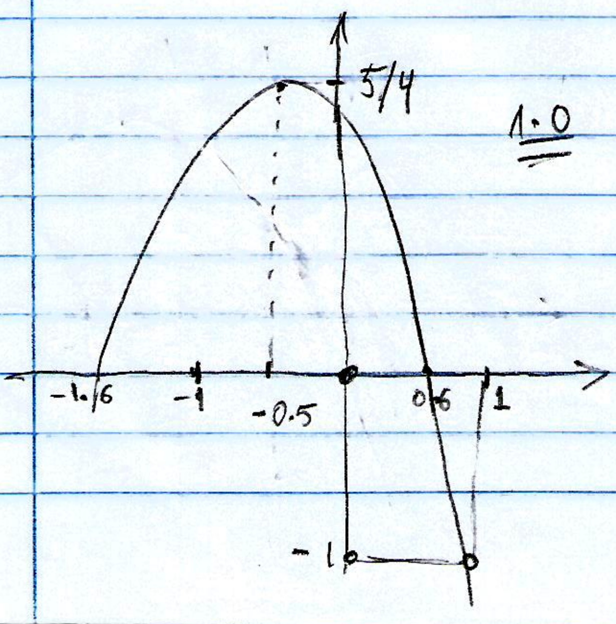
$$y = \underbrace{-z^2 - z + 1}_{\text{função do segundo grau em } z} \quad ; \quad -1 \le z \le 1 \qquad 0.25$$

$$-z^2 - z + 1 = 0 \quad \therefore \quad z = \frac{1 \pm \sqrt{1+4}}{-2}$$

$$= \frac{1 \pm \sqrt{5}}{-2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

$$= -0.5 \pm 1.1 \begin{cases} \nearrow 0.6 \\ \searrow -1.6 \end{cases}$$



1.0

$$\Delta = \left(-\frac{b}{2a}, -\frac{(b^2 - 4ac)}{4a}\right)$$

$$= \left(-\frac{1}{2}, +\frac{5}{4}\right)$$

Vemos do gráfico que os valores da imagem procurados estarão situados no intervalo

$$\left\|\ Im\, f = \left[-1\ ,\ \frac{5}{4}\right]\ \right\|$$

0,5

Outra Solução

$$y = -z^2 - z + 1$$

$\therefore$

$$z^2 + z + (y-1) = 0 \quad ; \quad -1 \leq z \leq 1$$

$$z = \frac{-1 \pm \sqrt{1-4(y-1)}}{2}$$

$$= \frac{-1 \pm \sqrt{1-4y+4}}{2}$$

$$= \frac{-1 \pm \sqrt{5-4y}}{2}$$

Assim, já vemos que para z estar definido é necessário que se tenha

$$5-4y \geq 0 \quad \therefore \quad \frac{5}{4} \geq y. \quad (*)$$

Mas $-1 \leq z \leq 1$

$\therefore$

$$-1 \leq \frac{-1 \pm \sqrt{5-4y}}{2} \leq 1$$

$\therefore$

$$-2 \leq -1 \pm \sqrt{5-4y} \leq 2$$

$$-1 \leq \pm\sqrt{5-4y} \leq 3$$

Podemos ter

$$(I)\begin{cases} y \leq 5/4 \\ \underline{e} \\ -1 \leq \sqrt{5-4y} \leq 3 \end{cases} \quad \text{ou} \quad (II)\begin{cases} y \leq 5/4 \\ \underline{e} \\ -1 \leq -\sqrt{5-4y} \leq 3 \end{cases}$$

Análise de (I)

$$\begin{cases} y \leq \frac{5}{4} \\ \underline{e} \\ \underbrace{-1 \leq \sqrt{5-4y}}_{\text{trivialmente verificada}} \leq 3 \end{cases} \quad \therefore \quad \begin{cases} y \leq 5/4 \quad (**) \\ \sqrt{5-4y} \leq 3 \end{cases}$$

$$\therefore \quad 5-4y \leq 9$$

$$-4 \leq 4y$$

$$-1 \leq y \quad (***)$$

De $(**)$ $\underline{e}$ $(***)$ temos

$$(I) \Longrightarrow \quad // -1 \leq y \leq \frac{5}{4} //$$

Análise de (II)

$$\begin{cases} y \leq \frac{5}{4} \\ \underline{e} \\ -1 \leq \underbrace{-\sqrt{5-4y} \leq 3}_{\text{Trivialmente verificado}} \end{cases}$$

$\therefore$

$$\begin{cases} y \le \frac{5}{4} \quad (4*) \\ \underline{\underline{e}} \\ -1 \le -\sqrt{5-4y} \end{cases}$$

$\therefore$

$$\begin{cases} y \le \frac{5}{4} \\ 1 \geq \sqrt{5-4y} \geq 0 \end{cases}$$

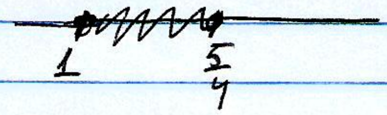
$\therefore$

$$1 \geq 5 - 4y$$

$$4y \geq 4$$

$$y \geq 1 \quad (5*)$$

De $(4*)$ e $(5*)$:



$\therefore$

$$(\mathrm{II}) \Longrightarrow \left\| \; 1 \le y \le \frac{5}{4} \; \right\|$$

Daí, (I) $\underline{\underline{\text{ou}}}$ (II)

$$-1 \le y \le \frac{5}{4} \quad \underline{\underline{\text{ou}}} \quad 1 \le y \le \frac{5}{4}$$

$\therefore$

$$-1 \le y \le \frac{5}{4} \qquad \therefore \left\| \; \mathrm{Im} f = \left[-1, \frac{5}{4}\right] \; \right\|$$

3. $\log_x 49 = \log_{x^2} 7 + \log_{2x} 7$ (*)

Temos

$$\log_x 49 = \log_x 7^2 = 2\log_x 7 \quad (2*)$$

1.0

$$\log_{x^2} 7 = \frac{1}{2}\log_x 7 \quad (3*)$$

$$\log_{2x} 7 = \frac{\log_x 7}{\log_x 2x} = \frac{\log_x 7}{\log_x 2 + \log_x x}$$

$$= \frac{\log_x 7}{\log_x 2 + 1} \quad (4*)$$

Daí, substituindo (4*), (3*), (2*) em (*):

$$2\log_x 7 = \frac{1}{2}\log_x 7 + \frac{\log_x 7}{\log_x 2 + 1}$$

$\therefore$

$$\frac{3}{2}\cancel{\log_x 7} = \frac{\cancel{\log_x 7}}{\log_x 2 + 1}$$

$\therefore$

$$\frac{3}{2} = \frac{1}{\log_x 2 + 1}$$

$$\frac{3}{2}\log_x 2 + \frac{3}{2} = 1$$

$$\frac{3}{2}\log_x 2 = -\frac{1}{2}$$

$$\log_x 2 = -\frac{1}{3}$$

$$\therefore \quad x^{-1/3} = 2$$

$$\frac{1}{x^{1/3}} = 2$$

$$x^{1/3} = \frac{1}{2}$$

$$x = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$$\therefore \quad x = \frac{1}{8}$$

1.0

$$\arccos x - \arcsin x = \arccos\sqrt{3x} \qquad (1)$$

Seja

$$z = \arccos x \quad \therefore \begin{cases} \cos z = x & (2) \\ z \in [0,\pi] & (3) \\ x \in [-1,1] & (4) \end{cases}$$

$$w = \arcsin x \quad \therefore \begin{cases} \sin w = x & (5) \\ w \in [-\frac{\pi}{2}, \frac{\pi}{2}] & (6) \\ x \in [-1,1] & (7) \end{cases}$$

$$y = \arccos\sqrt{3x} \quad \therefore \begin{cases} \cos y = \sqrt{3x} & (8) \\ y \in [0,\pi] & (9) \\ \sqrt{3x} \in [0,1] & \\ 0 \le \sqrt{3x} \le 1 & \\ 0 \le 3x \le 1 & \\ 0 \le x \le \frac{1}{3} & (10) \end{cases}$$

De (4), (7) e (10) segue-se que $0 \le x \le \frac{1}{3}$.

(1) : $z - w = y$

$\therefore \quad \cos(z-w) = \cos y$

$$\underline{\cos z}\cos w + \sin z \underline{\sin w} = \cos y$$

$$x \cos w + \sin z \ x = \sqrt{3x} \qquad (11)$$

Mas de (6):

$$w \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\therefore \quad \cos w = +\sqrt{1-\sin^2 w}$$

$$= \sqrt{1-x^2} \qquad (12)$$

De (3):

$$z \in [0, \pi]$$

$$\therefore$$

$$\sin z = +\sqrt{1-\cos^2 z}$$

$$= \sqrt{1-x^2} \qquad (13)$$

Daí, substituindo (12) e (13) em (11) temos:

$$x\sqrt{1-x^2} + \sqrt{1-x^2}\ x = \sqrt{3x}$$

$$2x\sqrt{1-x^2} = \sqrt{3x}$$

$$\Big\downarrow \quad 0 \leq x \leq \frac{1}{3} \text{ (só visto)}$$

$$4x^2(1-x^2) = 3x$$

$$4x^2(1-x^2) - 3x = 0$$

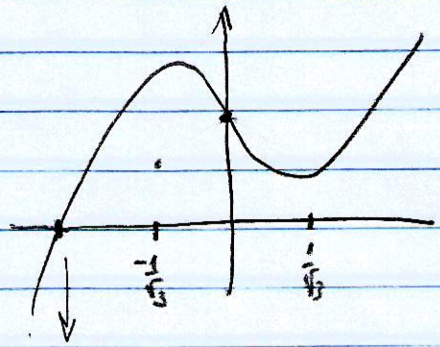
$$x\left[4x(1-x^2) - 3\right] = 0$$

$$\therefore$$

$x = 0$ ou $4x(1-x^2) = 3$

$-x^3 + x - \frac{3}{4} = 0$

$x^3 - x + 3/4 = 0$

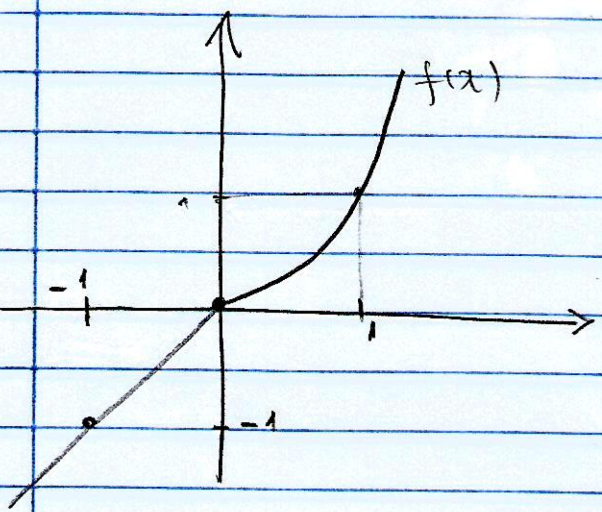


Há uma raiz negativa que não satisfaz a condição de estar no intervalo $[0, \frac{1}{3}]$

$\therefore$ $\| x = 0 \|$

4.

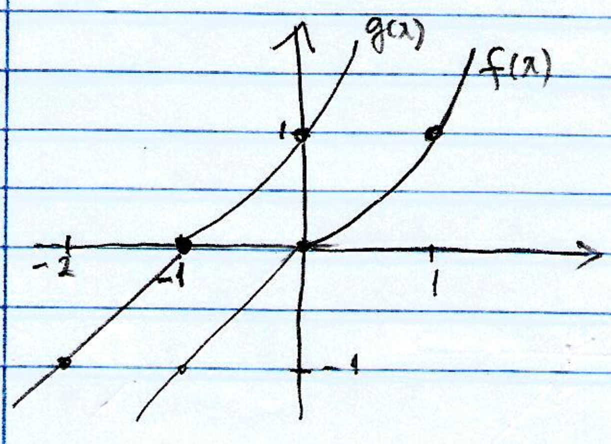
$f(2|x|+1)$

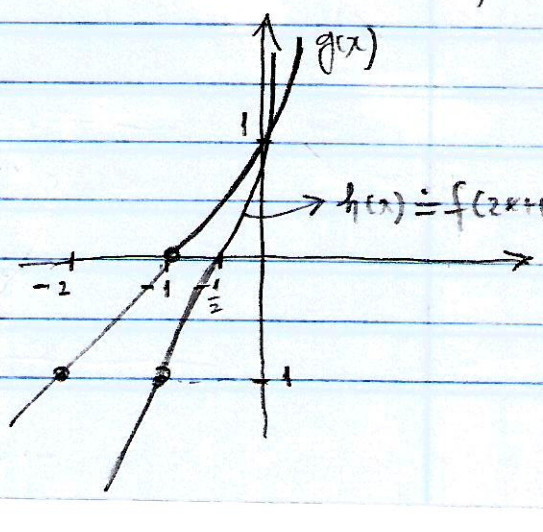


$$f(2|x|+1) = \begin{cases} f(2x+1) & , x \geq 0 \\ f(-2x+1) & , x < 0 \end{cases}$$

<u>Construção de $f(2x+1)$</u>

$f(x) \longrightarrow g(x) := f(x+1) \longrightarrow h(x) := g(2x) = f(2x+1)$





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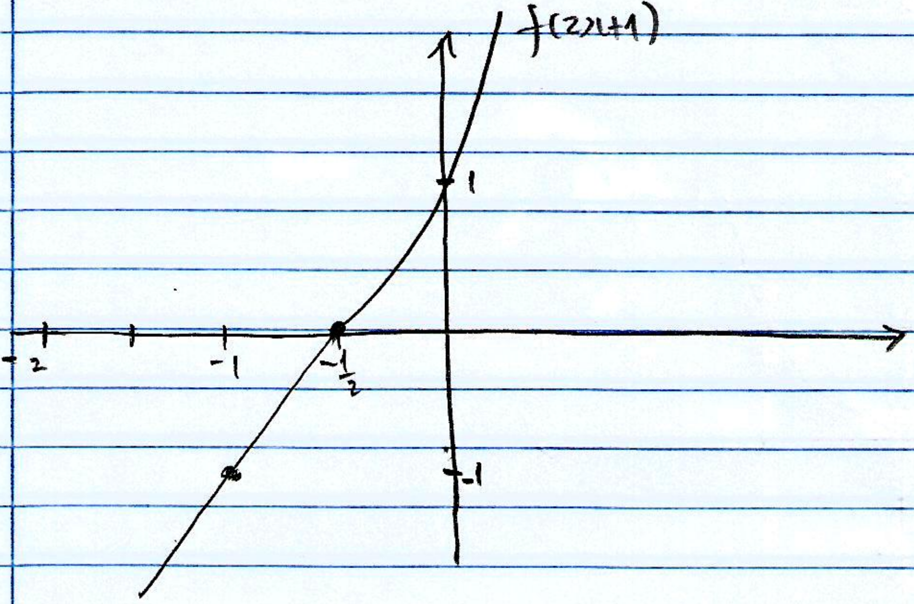
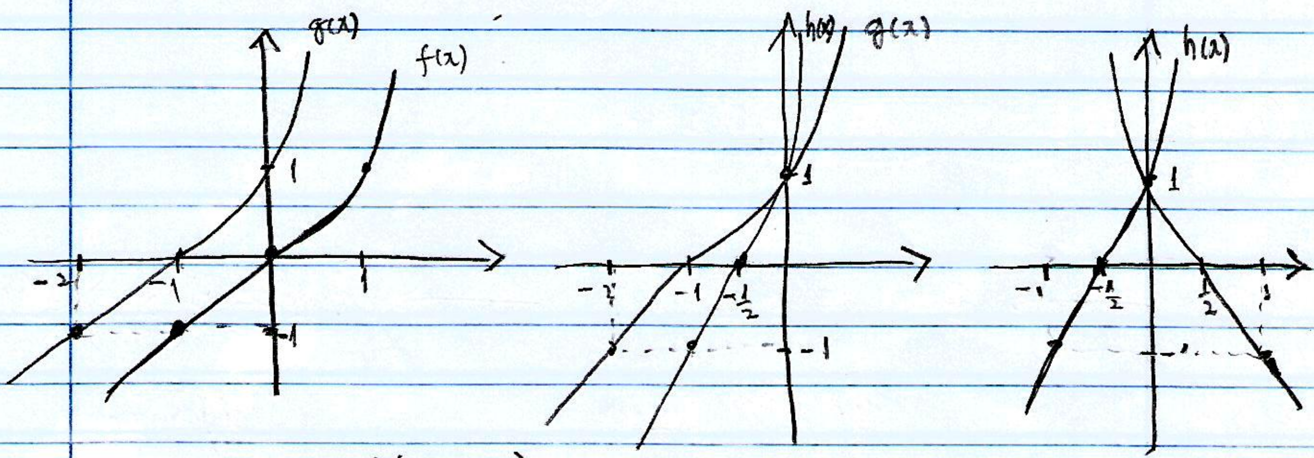


1.0

Construção de $f(-2x+1)$

$$f(x) \longrightarrow g(x) := f(x+1) \longrightarrow h(x) := g(2x) \longrightarrow \omega(x) := h(-x)$$
$$= f(2x+1) \qquad\qquad = f(-2x+1)$$



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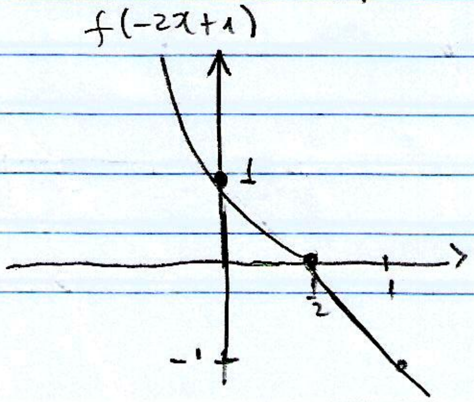
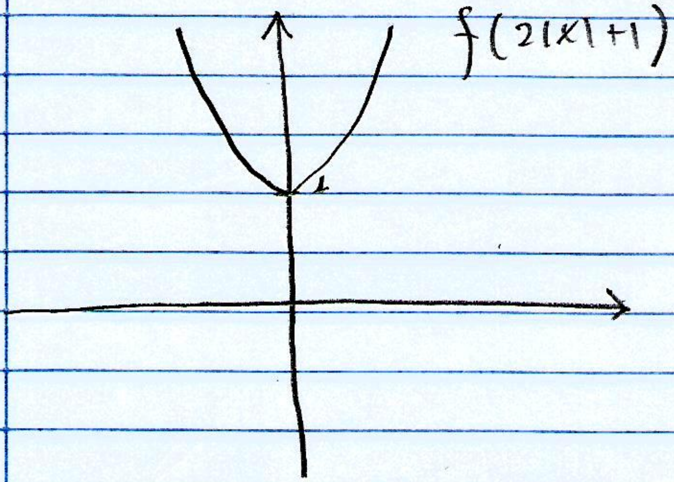


0.5

Dos gráficos de $f(2x+1)$ e $f(-2x+1)$ temos que o gráfico de $f(2|x|+1)$ é dado por



0.5