Cálculo A - Prova 2

(2.0) 1. Calcule

$$\lim_{x\to a^+}\frac{\sqrt{a-x}+\sqrt{a}-\sqrt{x}}{\sqrt{a^2-x^2}} \qquad (a>0)$$

(1.5) 2. Calcule

$$\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{1-\tan x}$$

3. Calcule as derivadas das funções

(1.5) a) $f(x)=x(\sin(\ln x)-\cos(\ln x))$

(1.5) b) $f(x)=\arctan\frac{x}{1+\sqrt{1-x^2}}$

(1.5) c) $f(x)=x^{\ln x}$

(2.0) 4. Uma pedra é lançada num lago produzindo uma onda circular que se propaga com velocidade de 25 cm/s. Encontre a razão com que a área do círculo varia após 4 s.

1. $$\lim_{x \to a^+} \frac{\sqrt{a-x} + \sqrt{a} - \sqrt{x}}{\sqrt{a^2 - x^2}}$$

fatora-se $\sqrt{a-x}$ no numerador e no denominador

$$= \lim_{x \to a^+} \frac{\sqrt{a-x}\left(1 + \frac{\sqrt{a} - \sqrt{x}}{\sqrt{a-x}}\right)}{\sqrt{a-x}\,\sqrt{a+x}}$$

$$= \lim_{x \to a^+} \left\{ \frac{1}{\sqrt{a+x}} + \frac{\sqrt{a} - \sqrt{x}}{\sqrt{a+x}\,\sqrt{a-x}} \right\}$$

$$= \lim_{x \to a^+} \frac{1}{\sqrt{a+x}} + \lim_{x \to a^+} \frac{\sqrt{a} - \sqrt{x}}{\sqrt{a+x}\,\sqrt{a-x}}$$

$$= \frac{1}{\sqrt{2a}} + \lim_{x \to a^+} \frac{1}{\sqrt{a+x}} \cdot \lim_{x \to a^+} \frac{\sqrt{a} - \sqrt{x}}{\sqrt{a-x}}$$

$$= \frac{1}{\sqrt{2a}} + \frac{1}{\sqrt{2a}} \cdot \lim_{x \to a^+} \frac{\sqrt{a} - \sqrt{x}}{\sqrt{a-x}}$$

$$= \frac{1}{\sqrt{2a}} + \frac{1}{\sqrt{2a}} \cdot \lim_{x \to a^+} \frac{(\sqrt{a} - \sqrt{x})}{\sqrt{a-x}} \frac{(\sqrt{a} + \sqrt{x})}{(\sqrt{a} + \sqrt{x})}$$

$$= \frac{1}{\sqrt{2a}} + \frac{1}{\sqrt{2a}} \cdot \lim_{x \to a^+} \frac{a - x}{\sqrt{a-x}} \frac{1}{\sqrt{a} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{2a}} + \frac{1}{\sqrt{2a}} \lim_{x \to a^+} \sqrt{a-x} \cdot \frac{1}{(\sqrt{a}+\sqrt{x})}$$

$$= \frac{1}{\sqrt{2a}} + 0$$

$$= \frac{1}{\sqrt{2a}}$$

2.

$$\lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{1 - \operatorname{tg} x}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{1 - \frac{\sin x}{\cos x}}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{\frac{\cos x - \sin x}{\cos x}}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{(\sin x - \cos x) \cdot \cos x}{\cos x - \sin x}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{-(\cos x - \sin x) \cos x}{(\cos x - \sin x)}$$

$$= \lim_{x \to \frac{\pi}{4}} -\cos x = -\frac{\sqrt{2}}{2}$$

3.

a) $f(x) = x(\sin \ln x - \cos \ln x)$

$$f'(x) = x'(\sin \ln x - \cos \ln x) + x(\sin \ln x - \cos \ln x)'$$

$$= 1.(\sin \ln x - \cos \ln x) + x\left((\sin \ln x)' - (\cos \ln x)'\right)$$

$$= \sin \ln x - \cos \ln x + x\left(\cos \ln x \, \frac{1}{x} - (-\sin \ln x)\frac{1}{x}\right)$$

$$= \sin \ln x - \cancel{\cos \ln x} + \cancel{\cos \ln x} + \sin \ln x$$

$$= 2 \sin \ln x$$

3b)

$$f(x) = \operatorname{arctg} \frac{x}{1+\sqrt{1-x^2}}$$

$$f'(x) = \frac{1}{1+\left(\frac{x}{1+\sqrt{1-x^2}}\right)^2} \cdot \left(\frac{x}{1+\sqrt{1-x^2}}\right)'$$

$$= \frac{1}{1+\frac{x^2}{1+2\sqrt{1-x^2}+1-x^2}} \cdot \left(\frac{1\cdot\left(1+\sqrt{1-x^2}\right) - x\left(\frac{1(-2x)}{2\sqrt{1-x^2}}\right)}{\left(1+\sqrt{1-x^2}\right)^2}\right)$$

$$= \frac{1}{\left(1+\frac{x^2}{2+2\sqrt{1-x^2}-x^2}\right)} \cdot \left(\frac{2\sqrt{1-x^2}+2(1-x^2)+2x^2}{\left(2\sqrt{1-x^2}\right)\left(1+\sqrt{1-x^2}\right)^2}\right)$$

$$= \frac{1}{\frac{2+2\sqrt{1-x^2}\cancel{-x^2}+\cancel{x^2}}{2+2\sqrt{1-x^2}-x^2}} \quad \frac{\left(2\sqrt{1-x^2}+2\right)}{2\sqrt{1-x^2}\left(1+\sqrt{1-x^2}\right)^2}$$

$$= \frac{\left(2+2\sqrt{1-x^2}-x^2\right)}{\cancel{\left(2+2\sqrt{1-x^2}\right)}} \quad \frac{\cancel{\left(2\sqrt{1-x^2}+2\right)}}{2\sqrt{1-x^2}\left(1+\sqrt{1-x^2}\right)^2}$$

$$= \frac{\cancel{\left(1+\sqrt{1-x^2}\right)^2}}{2\sqrt{1-x^2}\cancel{\left(1+\sqrt{1-x}\right)^2}} = \frac{1}{2\sqrt{1-x^2}}$$

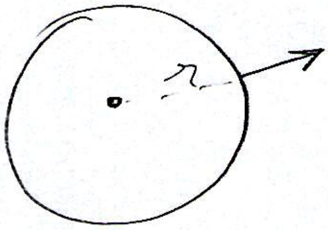
3c)

$$f(x) = x^{\ln x}$$
$$= e^{\ln x^{\ln x}}$$
$$= e^{\ln x \ln x}$$
$$= e^{(\ln x)^2}$$

$$f'(x) = \left(e^{(\ln x)^2}\right)'$$
$$= e^{(\ln x)^2}\left((\ln x)^2\right)'$$
$$= \underline{e^{(\ln x)^2}}\, 2 \ln x \frac{1}{x}$$
$$= x^{\ln x}\, 2 \ln x \frac{1}{x}$$

$$\|\ f'(x) = 2\frac{1}{x} \ln x\, x^{\ln x}\ \|$$

4.



$\frac{dr}{dt} = 25$ cm/s

$A(r) = \pi r^2$

$$\frac{dA}{dt} = \frac{d}{dt}\left(\pi r^2(t)\right)$$

$$= \pi \, 2r(t) \frac{dr}{dt} = 2\pi r(t) \, 25$$
$$= 50\pi \, r(t)$$

Mas sendo $\frac{dr}{dt} = 25$ cm/s temos que após 4 segundos tem-se $r(t) = 100$ cm.

Daí

$$\left.\frac{dA}{dt}\right|_{r=100} = \left. 50\pi \, r(t) \right|_{r=100}$$

$$= 5000\pi \text{ cm}^2/\text{s}.$$