Cálculo A - Prova 1

1. Determine $f(x)$ se $f(x^2+1) = x^4 + 5x^2 + 8$ 1.0

2. Dê o domínio de $f(x) = \sqrt{\frac{\sin x + \cos x}{\sin x - \cos x}}$. 1.5

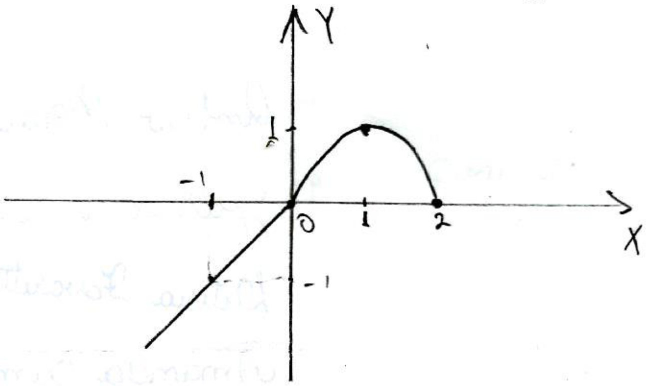
3. Dê a imagem de $f(x) = \log_2(4 - x^2)$. 1.0

4. Mostre que $\arctan x + \text{arccot} x = \frac{\pi}{2}$ 1.0

5. Sejam $a > 0, c > 0, b = \sqrt{ac}, a \neq 1, c \neq 1, c \neq 1$ e $N > 0$. Mostre que

$$\frac{\log_a N}{\log_c N} = \frac{\log_a N - \log_b N}{\log_b N - \log_c N}$$ 1.0

6. Seja f a função cujo gráfico é mostrado na figura 1.0



Construa o gráfico de $|f(2|x|+1)| + 1$.

$\operatorname{sen}^2 x + \cos^2 x = 1$

$1 + \operatorname{tg}^2 x = \sec^2 x$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\operatorname{sen}^2 x = \frac{1-\cos 2x}{2}$

$\operatorname{sen} 2x = 2 \operatorname{sen} x \cos x$

$\cos 2x = \cos^2 x - \operatorname{sen}^2 x$

$\operatorname{sen}(a \pm b) = \operatorname{sen} a \cos b \pm \operatorname{sen} b \cos a$

$\cos(a \pm b) = \cos a \cos b \mp \operatorname{sen} a \operatorname{sen} b$

$\operatorname{tg}(a+b) = \frac{\operatorname{tg} a + \operatorname{tg} b}{1 - \operatorname{tg} a \operatorname{tg} b}$

$\operatorname{cotg}(a+b) = \frac{\operatorname{cotg} a \operatorname{cotg} b - 1}{\operatorname{cotg} a + \operatorname{cotg} b}$

$\log_a pq = \log_a p + \log_a q$

$\log_a p/q = \log_a p - \log_a q$

$\log_a p^q = q \log_a p$

$\log_{a^p} q = \frac{1}{p} \log_a q$

$\frac{\log_a b}{\log_a c} = \log_c b$

$\log_a b = \frac{1}{\log_b a}$

Im arctg $= (-\frac{\pi}{2}, \frac{\pi}{2})$; Im arc cotg $= (0, \pi)$

1.

$$f(x^2+1) = x^4+5x^2+8$$

$$= (x^2+1)^2 + 3x^2 + 7$$

$$= (x^2+1)^2 + 3(x^2+1) + 4$$

$$\therefore \; f(x) = x^2 + 3x + 4$$

1.0

2.

$$f(x) = \sqrt{\frac{\sin x + \cos x}{\sin x - \cos x}}$$

Devemos ter

$$\left.\begin{array}{ll} \sin x - \cos x \neq 0 & (*) \\ \text{e} & \\ \dfrac{\sin x + \cos x}{\sin x - \cos x} \geq 0 & (**) \end{array}\right\} \underline{\underline{0.25}}$$

$\underline{\underline{(*)}}$:

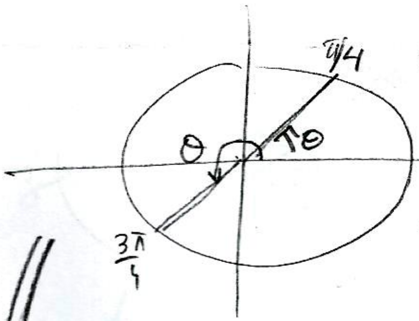
$$\sin x - \cos x \neq 0$$

$\therefore$ $\sin x \neq \cos x$

$\therefore$

$$\| x \neq \frac{\pi}{4} + n\pi \;;\; n \in \mathbb{Z} \| \quad (**A)$$

$\underline{\underline{0.5}}$



$\sin\theta = \cos\theta$

$\Leftrightarrow$

$$\left.\begin{array}{l} \theta = \dfrac{\pi}{4} + 2n\pi \\ \text{ou} \\ \theta = \dfrac{3\pi}{4} + 2n\pi \end{array}\right\}$$

$\therefore$ $\theta = \dfrac{\pi}{4} + n\pi$

$n \in \mathbb{Z}$

$(**)$

$$\frac{\sin x + \cos x}{\sin x - \cos x} \geq 0$$

$\therefore$

$$\frac{(\sin x + \cos x)}{(\sin x - \cos x)} \cdot \frac{(\sin x - \cos x)}{(\sin x - \cos x)} \geq 0$$

$\therefore$ $$\frac{\sin^2 x - \cos^2 x}{\underbrace{(\sin x - \cos x)^2}_{\text{sempre positivo}}} \geq 0$$

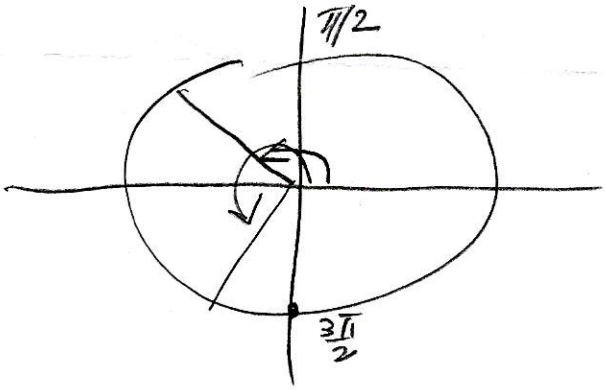
sempre positivo (já consideramos de (*) que $\sin x - \cos x \neq 0$)

$\therefore$ $\underbrace{\sin^2 x - \cos^2 x \geq 0}$

$-\cos 2x \geq 0$

$\cos 2x \leq 0 \implies \frac{\pi}{2} + 2n\pi \leq 2x \leq \frac{3\pi}{2} + 2n\pi$

$(n \in \mathbb{Z})$



$\therefore$ $\underbrace{\frac{\pi}{4} + n\pi \leq x \leq \frac{3\pi}{4} + n\pi}$

$\underline{\underline{0.5}}$ (****)

De (***) e (****) tem-se que

$$\frac{\pi}{4} + n\pi < x \leq \frac{3\pi}{4} + n\pi$$

$\therefore$ // $\mathrm{Dom}\, f = \left\{ x \in \mathbb{R} : \frac{\pi}{4} + n\pi < x \leq \frac{3\pi}{4} + n\pi ;\ n \in \mathbb{Z} \right\}$ //

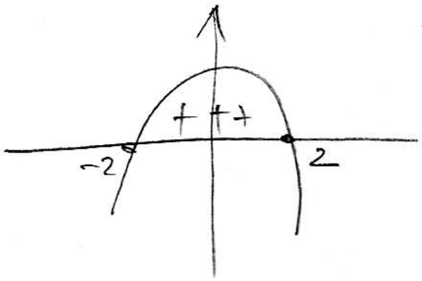
ou // $\mathrm{Dom}\, f = \bigcup_{n \in \mathbb{Z}} \left(\frac{\pi}{4} + n\pi , \frac{3\pi}{4} + n\pi \right]$ // 0.25

3. $f(x) = \log_2 (4 - x^2)$

Seja $z = 4 - x^2$.

Devemos ter $z > 0$

$\therefore\ 4 - x^2 > 0 \implies -2 < x < 2$ $\underline{0.25}$



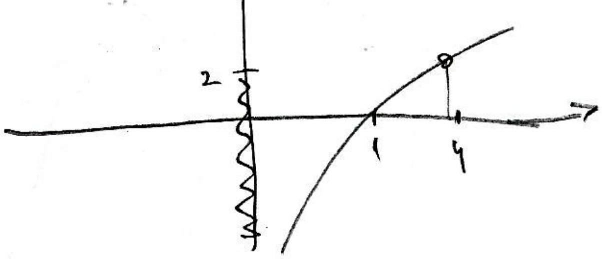
Daí, $\underline{0 < z \leq 4}$

Então $f(x) = \log_2 (4 - x^2)$

$\Leftrightarrow$

$f(z) = \log_2 z$ com $0 < z \leq 4$

↳ é função crescente



// $\operatorname{Im} f = (-\infty, 2]$ //

$\underline{0.75}$

4.

$$\operatorname{arc\,tg} x + \operatorname{arc\,cotg} x = \frac{\pi}{2}$$

Seja

$$\operatorname{cotg}\left(\operatorname{arc\,tg} x + \operatorname{arc\,cotg} x\right) =$$

$$= \frac{\operatorname{cotg} \operatorname{arc\,tg} x \cdot \overbrace{\operatorname{cotg}(\operatorname{arc\,cotg} x)}^{x} - 1}{\operatorname{cotg} \operatorname{arc\,tg} x + \underbrace{\operatorname{cotg} \operatorname{arc\,cotg} x}_{x}}$$

$$= \frac{(\operatorname{cotg} \operatorname{arc\,tg} x) \cdot x - 1}{\operatorname{cotg} \operatorname{arc\,tg} x + x} \quad (*)$$

0.5

Seja $y = \operatorname{arc\,tg} x$; $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\therefore \operatorname{tg} y = x$

$\therefore \operatorname{cotg} y = \frac{1}{x}$

Daí $\operatorname{cotg} \underbrace{\operatorname{arc\,tg} x} = \operatorname{cotg} \underbrace{y} = \frac{1}{x}$

$\therefore \operatorname{cotg} \operatorname{arc\,tg} x = \frac{1}{x}$ $(**)$.

Subst. $(**)$ em $(*)$ temos que:

$\cot g\left(\operatorname{arctg} x + \operatorname{arccotg} x\right) =$

$$= \frac{\frac{1}{x}\cdot x - 1}{\frac{1}{x} + x}$$

$$= \frac{0}{\frac{1}{x}+x}$$

$= 0$

$\therefore \cot g\left(\operatorname{arc\cdot tg} x + \operatorname{arc\,cotg} x\right) = 0$

$// \operatorname{arctg} x + \operatorname{arc\,cotg} x = \frac{\pi}{2} //$ $\underline{o.I}$

5.

$$\frac{\log_a N - \log_b N}{\log_b N - \log_c N} =$$

$$= \frac{\dfrac{1}{\log_N a} - \dfrac{1}{\log_N b}}{\dfrac{1}{\log_N b} - \dfrac{1}{\log_N c}} = \frac{\dfrac{\log_N b - \log_N a}{\log_N a \log_N b}}{\dfrac{\log_N c - \log_N b}{\log_N b \log_N c}}$$

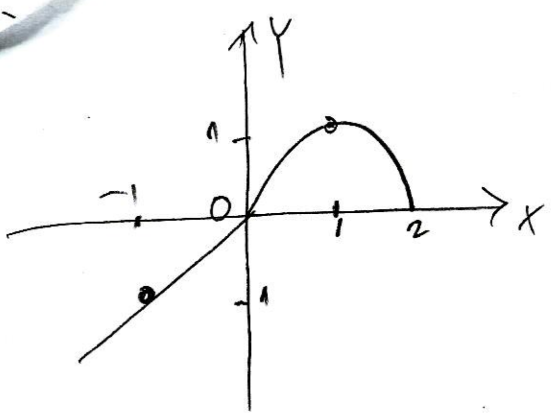
$$= \frac{\dfrac{\log_N b/a}{\log_N a \log_N b}}{\dfrac{\log_N c/b}{\log_N b \log_N c}} = \frac{\log_N b/a}{\log_N a \cancel{\log_N b}} \cdot \frac{\cancel{\log_N b} \log_N c}{\log_N c/b}$$

$$= \frac{\log_N \frac{\sqrt{ac}}{a} \; \log_N c}{\log_N a \; \log_N \frac{c}{\sqrt{ac}}} = \frac{\cancel{\log_N \sqrt{\frac{c}{a}}} \; \log_N c}{\log_N a \; \cancel{\log_N \sqrt{\frac{c}{a}}}}$$

$$= \frac{\log_N c}{\log_N a} = \frac{\dfrac{1}{\log_c N}}{\dfrac{1}{\log_a N}} = \frac{\log_a N}{\log_c N}$$

1.0

6.

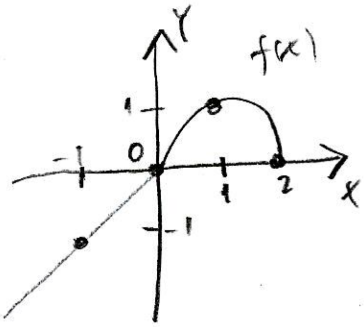


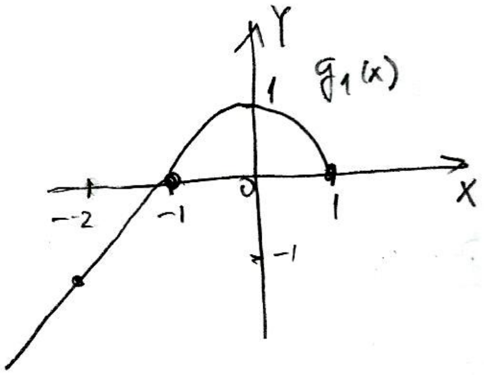
$$f(2|x|+1) = \begin{cases} f(2x+1) & \text{se } x \geq 0 \\ f(-2x+1) & \text{se } x < 0 \end{cases}$$

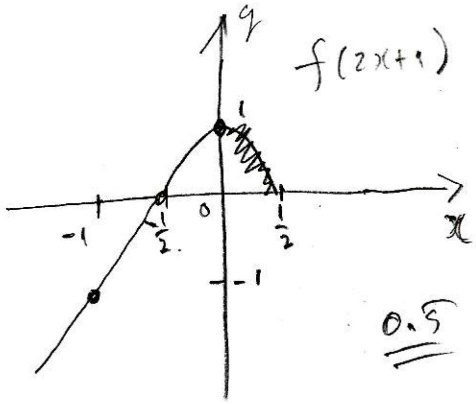
0.25

Construção do gráfico de $f(2x+1)$:

$f(x) \longrightarrow g_1(x) = f(x+1) \longrightarrow g_2(x) = g_1(2x) = f(2x+1)$







0.5

Construção do gráfico de $f(-2x+1)$

$f(x) \longrightarrow g_1(x) = f(-x) \longrightarrow g_2(x) = g_1(x-1) \longrightarrow g_3(x) = g_2(2x) = f(-2x+1)$

$\qquad = f(-x+1)$

ou $\searrow$ $g_1(x) = f(x+1) \longrightarrow g_2(x) = g_1(-x) \longrightarrow g_3(x) = g_2(2x) = f(-2x+1)$

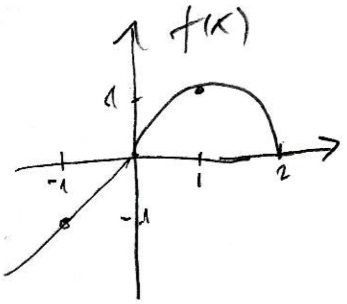
$\qquad = f(-x+1)$

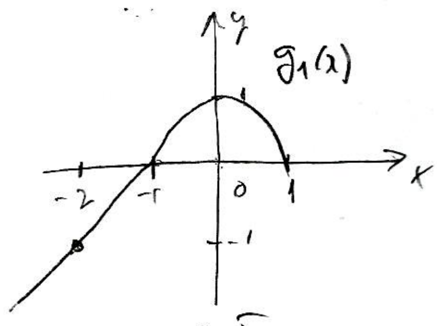
ou $\searrow$ $g_2(x) = g_1(2x) \longrightarrow g_3(x) = g_2(-x) = f(-2x+1)$

$\qquad = f(2x+1)$

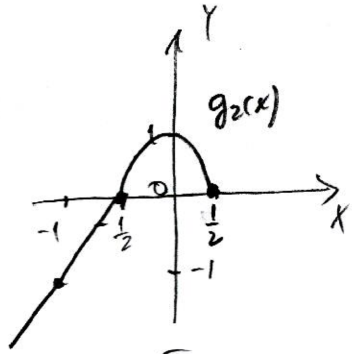
Usaremos a forma

$f(x) \longrightarrow g_1(x) = f(x+1) \longrightarrow g_2(x) = g_1(2x) \longrightarrow g_3(x) = g_2(-x)$

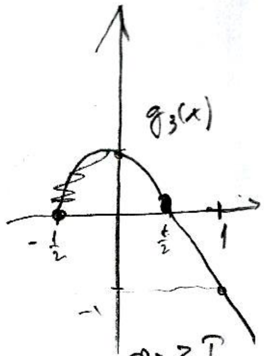




0.75



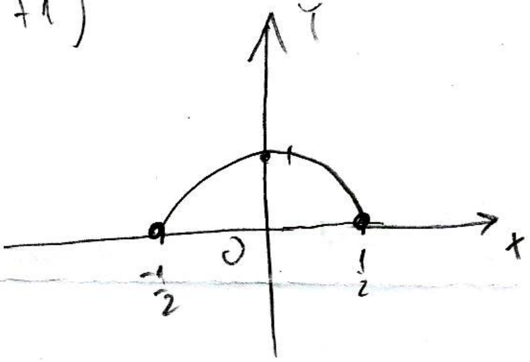
0.25



0.25

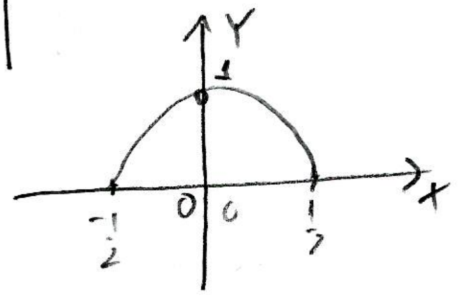
0.75

$f(2|x|+1)$

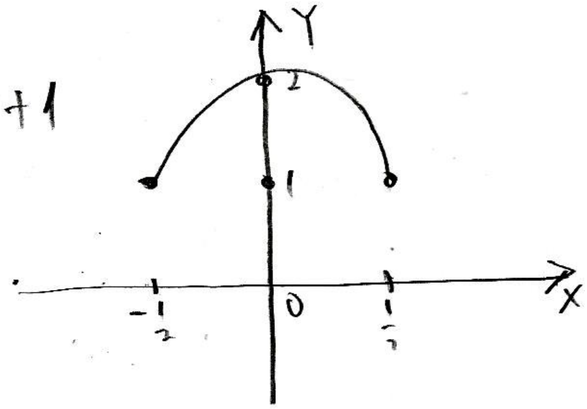


0.25

$|f(2|x|+1)|$



$|f(2|x|+1)|+1$



0.25