

Cálculo B - Lista 3

Integração por substituição trigonométrica

Resolva as integrais

$$1. \int \frac{dx}{x^2\sqrt{4-x^2}}$$

$$2. \int \frac{\sqrt{4-x^2}}{x^2} dx$$

$$3. \int \frac{dx}{x\sqrt{x^2+4}}$$

$$4. \int \frac{x^2 dx}{\sqrt{x^2+6}}$$

$$5. \int \frac{dx}{x\sqrt{25-x^2}}$$

$$6. \int \sqrt{1-u^2} du$$

$$7. \int \frac{dx}{\sqrt{x^2-a^2}}$$

$$8. \int \frac{dw}{w^2\sqrt{w^2-7}}$$

$$9. \int \frac{x^2 dx}{(x^2+4)^2}$$

$$10. \int \frac{dx}{(4+x^2)^{3/2}}$$

$$11. \int \frac{dx}{(4x^2-9)^{3/2}}$$

$$12. \int \frac{dx}{x^4\sqrt{16+x^2}}$$

$$13. \int \frac{2dt}{t\sqrt{t^4+25}}$$

$$14. \int \frac{x^3 dx}{(25-x^2)^2}$$

$$15. \int \frac{dx}{\sqrt{4x+x^2}}$$

$$16. \int \frac{dx}{\sqrt{4x-x^2}}$$

$$17. \int \frac{dx}{(5-4x-x^2)^{3/2}}$$

$$18. \int \frac{dx}{x\sqrt{x^4-4}}$$

$$19. \int \frac{\sec^2 x dx}{(4-\tan^2 x)^{3/2}}$$

$$20. \int \frac{e^{-x} dx}{(9e^{-2x}+1)^{3/2}}$$

21. $\int \frac{\ln^3 w \, dw}{w\sqrt{\ln^2 w - 4}}$
22. $\int \frac{dz}{(z^2 - 6z + 18)^{3/2}}$
23. $\int \frac{e^t dt}{(e^{2t} + 8e^t + 7)^{3/2}}$
24. $\int \frac{\sqrt{16 - e^{2x}}}{e^x} dx$

Respostas

1. $-\frac{\sqrt{4-x^2}}{4x} + C$
2. $-\frac{\sqrt{4-x^2}}{x} - \arcsin \frac{x}{2} + C$
3. $\frac{1}{2} \ln \left| \frac{\sqrt{x^2+4}-2}{x} \right| + C$
4. $\frac{x}{2} \sqrt{x^2+6} - 3 \ln |\sqrt{x^2+6} + x| + C$
5. $\frac{1}{5} \ln \left| \frac{5-\sqrt{25-x^2}}{x} \right| + C$
6. $\frac{1}{2} \arcsin u + \frac{1}{2} u \sqrt{1-u^2} + C$
7. $\ln \left| \frac{x+\sqrt{x^2-a^2}}{a} \right| + C$
8. $\frac{1}{7} \frac{\sqrt{w^2-7}}{w} + C$
9. $\frac{1}{4} \arctan \frac{x}{2} - \frac{x}{2(x^2+4)} + C$
10. $\frac{1}{4} \frac{x}{\sqrt{4+x^2}} + C$
11. $-\frac{1}{9} \frac{x}{\sqrt{4x^2-9}} + C$
12. $-\frac{1}{768} \frac{(16+x^2)^{3/2}}{x^3} + \frac{1}{256} \frac{\sqrt{16+x^2}}{x} + C$
13. $\frac{1}{5} \ln \left(\frac{\sqrt{t^4+25}-5}{t^2} \right) + C$
14. $\frac{x^2}{2(25-x^2)} + \ln \left(\frac{\sqrt{25-x^2}}{5} \right) + C$
15. $\ln |x + 2 + \sqrt{x^2 + 4x}| + C$
16. $\arcsin \frac{x-2}{2} + C$

$$17. \frac{x+2}{9\sqrt{5-4x-x^2}} + C$$

$$18. \frac{1}{4}\operatorname{arcsec}\frac{x^2}{2} + C$$

$$19. \frac{1}{4}\frac{\tan x}{\sqrt{4-\tan^2 x}} + C$$

$$20. -\frac{e^{-x}}{\sqrt{1+9e^{-2x}}} + C$$

$$21. \sqrt{\ln^2 w - 4} \left(\frac{8+\ln^2 w}{3} \right) + C$$

$$22. \frac{1}{9}\frac{z-3}{\sqrt{z^2-6z+18}} + C$$

$$23. -\frac{1}{9}\frac{e^t+4}{\sqrt{e^{2t}+8e^t+7}} + C$$

$$24. -\frac{\sqrt{16-e^{2x}}}{e^x} - \arcsin \frac{e^x}{4} + C$$

Mat. Trigonometria

Ex. (Pg. 550)

1. $\int \frac{dx}{x^2 \sqrt{4-x^2}}$; $\sqrt{a^2-x^2} \rightarrow x = a \sin \theta$

$$\begin{cases} x = 2 \sin \theta, & -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta = \sin^{-1} \frac{x}{2} \end{cases}$$

$$dx = 2 \cos \theta d\theta$$

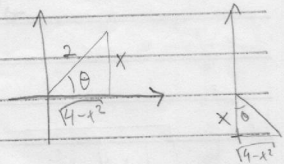
$$\int \frac{dx}{x^2 \sqrt{4-x^2}} = \int \frac{2 \cos \theta d\theta}{4 \sin^2 \theta \sqrt{4-4 \sin^2 \theta}} =$$

$$= \int \frac{2 \cos \theta d\theta}{4 \sin^2 \theta \cdot 2 \cos \theta}$$

$$= \frac{1}{4} \int \sec^2 \theta d\theta$$

$$= -\frac{1}{4} \cot \theta + C$$

$$= -\frac{1}{4} \frac{\sqrt{4-x^2}}{x} + C$$



$$\cot \theta = \frac{\sqrt{4-x^2}}{x}$$

$$\boxed{\int \frac{dx}{x^2 \sqrt{4-x^2}} = -\frac{\sqrt{4-x^2}}{4x} + C}$$

$$\frac{1}{x^2} \ln(\sqrt{x^2+25} - 5) + \text{constant}$$

2. $\int \frac{\sqrt{4-x^2}}{x^2} dx$ $\sqrt{a^2-x^2} \rightarrow x = a \sin \theta$

$2(25-x^2) = x$ $\left. \begin{aligned} x &= 2 \sin \theta & -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ \theta &= \sin^{-1} \frac{x}{2} \end{aligned} \right\}$

$dx = 2 \cos \theta d\theta$

$$\int \frac{\sqrt{4-x^2}}{x^2} dx = \int \frac{\sqrt{4(1-\sin^2 \theta)}}{4 \sin^2 \theta} 2 \cos \theta d\theta$$

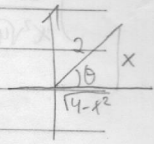
$$= \int \frac{2 \cos \theta \cdot 2 \cos \theta}{4 \sin^2 \theta} d\theta$$

$$= \int \frac{\cos^2 \theta}{\sin^2 \theta} d\theta$$

$$= \int \cot^2 \theta d\theta$$

$$= \int (\csc^2 \theta - 1) d\theta$$

$$= -\cot \theta - \theta + C$$



$$\int \frac{\sqrt{4-x^2}}{x^2} dx = -\frac{\sqrt{4-x^2}}{x} - \sin^{-1} \frac{x}{2} + C$$

22. $\frac{1}{9} \frac{0-3}{10^2-63+18} + C$

23. $-\frac{1}{9} \frac{e^2+4}{10^2+6e^2+9} + C$

24. $\frac{16-e^{2x}}{e^x} = 16e^{-x} - e^x + C$

tilibra

$$3. \int \frac{dx}{x\sqrt{x^2+4}} = \sqrt{x^2+a^2} \rightarrow x = a \operatorname{tg} \theta$$

$$dx = 2 \operatorname{tg} \theta \operatorname{ctg} \theta d\theta \quad \left\{ \begin{array}{l} x = 2 \operatorname{tg} \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta = \operatorname{tg}^{-1} \frac{x}{2} \end{array} \right.$$

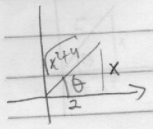
Daí,

$$\int \frac{dx}{x\sqrt{x^2+4}} = \int \frac{2 \operatorname{tg} \theta \operatorname{ctg} \theta d\theta}{2 \operatorname{tg} \theta \sqrt{4 \operatorname{tg}^2 \theta + 4}}$$

$$= \frac{1}{2} \int \frac{\operatorname{ctg} \theta d\theta}{\operatorname{tg} \theta}$$

$$= \frac{1}{2} \int \frac{1}{\operatorname{tg} \theta} \cdot \frac{1}{\operatorname{tg} \theta} d\theta$$

$$\operatorname{tg} \theta = \frac{x}{2}$$



$$\operatorname{cosec} \theta = \frac{1}{\operatorname{sen} \theta} = \frac{1}{\frac{x}{\sqrt{x^2+4}}} = \frac{\sqrt{x^2+4}}{x}$$

$$= \frac{1}{2} \int \operatorname{cosec} \theta d\theta$$

$$= \frac{1}{2} \ln | \operatorname{cosec} \theta - \operatorname{cotg} \theta | + C$$

$$\operatorname{cotg} \theta = \frac{2}{x}$$

$$= \frac{1}{2} \ln \left| \frac{\sqrt{x^2+4}}{x} - \frac{2}{x} \right| + C$$

$$= \frac{1}{2} \ln \left| \frac{\sqrt{x^2+4} - 2}{x} \right| + C$$

$$\boxed{\int \frac{dx}{x\sqrt{x^2+4}} = \frac{1}{2} \ln \left| \frac{\sqrt{x^2+4} - 2}{x} \right| + C}$$

$$4. \int \frac{x^2 dx}{\sqrt{x^2+6}} \quad \sqrt{x^2+a^2} \rightarrow x = a \tan \theta$$

$$dx = \sqrt{6} \sec^2 \theta d\theta \quad \left\{ \begin{array}{l} x = \sqrt{6} \tan \theta \\ \theta = \tan^{-1} \frac{x}{\sqrt{6}} \end{array} \right. \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\int \frac{x^2 dx}{\sqrt{x^2+6}} = \int \frac{6 \tan^2 \theta \sqrt{6} \sec^2 \theta d\theta}{\sqrt{6(1+\tan^2 \theta)}}$$

$$= \int \frac{6 \tan^2 \theta \sqrt{6} \sec^2 \theta d\theta}{\sqrt{6} \sec \theta}$$

$$= \int 6 \tan^2 \theta \sec \theta d\theta$$

$$= 6 \int (\sec^3 \theta - \sec \theta) d\theta$$

$$\begin{aligned} \sec \theta &= \frac{x}{\sqrt{6}} \\ \sec \theta &= \frac{1}{\cos \theta} = \frac{1}{\frac{\sqrt{6}}{\sqrt{6+x^2}}} \\ &= \frac{\sqrt{6+x^2}}{\sqrt{6}} \end{aligned} \quad \left| \begin{array}{l} = 6 \int \sec^3 \theta d\theta - 6 \int \sec \theta d\theta \\ = 6 \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] \\ - 6 \ln |\sec \theta + \tan \theta| + C \end{array} \right.$$

$$= 3 \sec \theta \tan \theta + 3 \ln |\sec \theta + \tan \theta| - 6 \ln |\sec \theta + \tan \theta| + C$$

$$= 3 \sec \theta \tan \theta - 3 \ln |\sec \theta + \tan \theta| + C$$

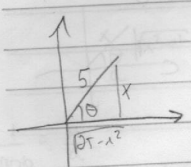
$$= 3 \frac{\sqrt{6+x^2}}{\sqrt{6}} \frac{x}{\sqrt{6}} - 3 \ln \left| \frac{\sqrt{6+x^2}}{\sqrt{6}} + \frac{x}{\sqrt{6}} \right| + C$$

$$\boxed{\int \frac{x^2 dx}{\sqrt{x^2+6}} = \frac{x}{2} \sqrt{6+x^2} - 3 \ln \left| \sqrt{6+x^2} + x \right| + C}$$

$$2. \int \frac{dx}{x\sqrt{25-x^2}} \quad \sqrt{a^2-x^2} \rightarrow x = a \sin \theta$$

$$dx = 5 \cos \theta d\theta \quad \left. \begin{array}{l} x = 5 \sin \theta \\ \theta = \sin^{-1} \frac{x}{5} \end{array} \right\} -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\int \frac{dx}{x\sqrt{25-x^2}} = \int \frac{5 \cos \theta d\theta}{5 \sin \theta \sqrt{25(1-\sin^2 \theta)}}$$



$$= \int \frac{dx d\theta}{\sin \theta 5 \cos \theta}$$

$$= \frac{1}{5} \int \sec \theta d\theta$$

$$\sec \theta = \frac{1}{\sin \theta} = \frac{5}{x}$$

$$\cot \theta = \frac{\sqrt{25-x^2}}{x}$$

$$= \frac{1}{5} \ln | \sec \theta - \cot \theta | + C$$

$$= \frac{1}{5} \ln \left| \frac{5}{x} - \frac{\sqrt{25-x^2}}{x} \right| + C$$

$$= \frac{1}{5} \ln \left| \frac{5 - \sqrt{25-x^2}}{x} \right| + C$$

$$\int \frac{dx}{x\sqrt{25-x^2}} = \frac{1}{5} \ln \left| \frac{5 - \sqrt{25-x^2}}{x} \right| + C$$

$$6. \int \sqrt{1-u^2} du \quad \left\{ \begin{array}{l} u = \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ \theta = \sin^{-1} u \end{array} \right.$$

$$du = \cos \theta d\theta$$

$$\therefore \int \sqrt{1-u^2} du = \int \sqrt{1-\sin^2 \theta} \cos \theta d\theta =$$

$$= \int \cos \theta \cos \theta d\theta$$

$$= \int \cos^2 \theta d\theta = \int \frac{1+\cos 2\theta}{2} d\theta$$

$$= \frac{1}{2} (\theta + \frac{\sin 2\theta}{2}) + C$$

$$\sin \theta = u$$

$$\cos \theta = \sqrt{1-u^2}$$

$$= \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta + C$$

$$= \frac{1}{2} \theta + \frac{1}{2} \sin \theta \cos \theta + C$$

$$\boxed{\int \sqrt{1-u^2} du = \frac{1}{2} \sin^{-1} u + \frac{1}{2} u \sqrt{1-u^2} + C}$$

$$7. \int \frac{dx}{\sqrt{x^2 - a^2}} \quad \left\{ \begin{array}{l} x = a \operatorname{arcc} \theta, \quad \theta \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi) \\ \theta = \operatorname{arcc} \frac{x}{a} \end{array} \right.$$

$$dx = a \operatorname{arcc} \theta \, d\theta$$

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 - a^2}} &= \int \frac{a \operatorname{arcc} \theta \, d\theta}{a \sqrt{a^2 \theta^2 - 1}} = \int \frac{\operatorname{arcc} \theta \, d\theta}{\sqrt{\theta^2 - 1}} \\ &= \ln | \operatorname{arcc} \theta + \sqrt{\theta^2 - 1} | + C \end{aligned}$$

$$\frac{x}{a} = \operatorname{arcc} \theta \quad \therefore \int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| \frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1} \right| + C$$

$$\left\{ \begin{array}{l} \operatorname{arcc} \theta = \frac{1}{a} = \frac{x}{a} \\ \theta = \sqrt{x^2 - a^2} / a \end{array} \right.$$

$$\boxed{\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| \frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1} \right| + C}$$

$$8. \int \frac{dw}{w^2 \sqrt{w^2 - 7}} \quad \left\{ \begin{array}{l} w = \sqrt{7} \operatorname{arcc} \theta \quad \theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi) \\ \operatorname{arcc} \theta = \frac{w}{\sqrt{7}} \end{array} \right.$$

$$dw = \sqrt{7} \operatorname{arcc} \theta \, d\theta$$

$$\int \frac{dw}{w^2 \sqrt{w^2 - 7}} = \int \frac{\sqrt{7} \operatorname{arcc} \theta \, d\theta}{7 \operatorname{arcc}^2 \theta \sqrt{7} \sqrt{\theta^2 - 1}} = \frac{1}{7} \int \frac{1}{\operatorname{arcc} \theta} \, d\theta$$

$$= \frac{1}{7} \int \cos \theta \, d\theta$$

$$= \frac{1}{7} \sin \theta + C$$

$$\left\{ \begin{array}{l} \operatorname{arcc} \theta = \frac{w}{\sqrt{7}} \\ \theta = \frac{\sqrt{w^2 - 7}}{\sqrt{7}} \end{array} \right.$$

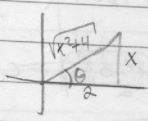
$$\boxed{\int \frac{dw}{w^2 \sqrt{w^2 - 7}} = \frac{1}{7} \frac{\sqrt{w^2 - 7}}{w} + C}$$

$$9. \int \frac{x^2 dx}{(x^2+4)^2}$$

$$\left. \begin{aligned} x &= 2 \tan \theta, & -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta &= \tan^{-1} \frac{x}{2} \end{aligned} \right\}$$

$$dx = 2 \sec^2 \theta d\theta$$

$$\int \frac{x^2 dx}{(x^2+4)^2} = \int \frac{4 \tan^2 \theta \cdot 2 \sec^2 \theta d\theta}{(4 \tan^2 \theta + 4)^2}$$



$$= \int \frac{8 \tan^2 \theta \sec^2 \theta d\theta}{16 (1 + \tan^2 \theta)^2}$$

$$= \frac{1}{2} \int \frac{\tan^2 \theta \sec^2 \theta d\theta}{(\sec^2 \theta)^2}$$

$$= \frac{1}{2} \int \frac{\tan^2 \theta \sec^2 \theta d\theta}{\sec^4 \theta}$$

$$= \frac{1}{2} \int \frac{\sin^2 \theta \cos^2 \theta d\theta}{\cos^4 \theta}$$

$$= \frac{1}{2} \int \sin^2 \theta d\theta$$

$$= \frac{1}{2} \int \frac{1 - \cos 2\theta}{2} d\theta$$

$$= \frac{1}{4} \left[\theta - \frac{1}{2} \sin 2\theta \right] + C$$

$$= \frac{1}{4} \left[\theta - \sin \theta \cos \theta \right] + C$$

$$= \frac{1}{4} \left[\tan^{-1} \frac{x}{2} - \frac{x}{2} \frac{2}{\sqrt{x^2+4} \sqrt{1+\frac{x^2}{4}}} \right] + C$$

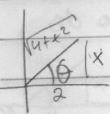
$$= \frac{1}{4} \tan^{-1} \frac{x}{2} - \frac{x}{2(x^2+4)} + C //$$

$$10. \int \frac{dx}{(4+x^2)^{3/2}} \quad \left. \begin{array}{l} x = 2 \tan \theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta = \tan^{-1} \frac{x}{2} \end{array} \right\}$$

$$dx = 2 \sec^2 \theta d\theta$$

$$\int \frac{dx}{(4+x^2)^{3/2}} = \int \frac{2 \sec^2 \theta d\theta}{(4+4 \tan^2 \theta)^{3/2}}$$

$$= \int \frac{2 \sec^2 \theta d\theta}{4^{3/2} (1+\tan^2 \theta)^{3/2}}$$



$$= \int \frac{2 \sec^2 \theta d\theta}{2^3 (\sec^2 \theta)^{3/2}}$$

$$= \frac{1}{4} \int \frac{\sec^2 \theta d\theta}{\sec^3 \theta}$$

$$= \frac{1}{4} \int \frac{1}{\sec \theta} d\theta$$

$$= \frac{1}{4} \int \cos \theta d\theta$$

$$= \frac{1}{4} \sin \theta + C$$

$$\boxed{\int \frac{dx}{(4+x^2)^{3/2}} = \frac{1}{4} \frac{x}{\sqrt{4+x^2}} + C}$$

$$11. \int \frac{dx}{(4x^2-9)^{3/2}} =$$

$$= \int \frac{dx}{[4(x^2-\frac{9}{4})]^{3/2}} = \int \frac{dx}{4^{3/2} (x^2-\frac{9}{4})^{3/2}}$$

$$= \frac{1}{8} \int \frac{dx}{(x^2-\frac{9}{4})^{3/2}} \quad \left. \begin{array}{l} x = \frac{3}{2} \sec \theta \\ \theta = \sec^{-1} \frac{2x}{3} \\ dx = \frac{3}{2} \sec \theta \tan \theta d\theta \end{array} \right\} (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi) \ni \theta$$

$$= \frac{1}{8} \int \frac{\frac{3}{2} \sec \theta \tan \theta d\theta}{(\frac{9}{4} \tan^2 \theta)^{3/2}} \quad dx = \frac{3}{2} \sec \theta \tan \theta d\theta$$

$$= \frac{1}{8} \int \frac{\frac{3}{2} \sec \theta \tan \theta d\theta}{(\frac{9}{4})^{3/2} \tan^3 \theta}$$

$$\sec \theta = \frac{2x}{3}$$

↓

$$\frac{\tan \theta}{3} \quad \cos \theta = \frac{3}{2x}$$

$$= \frac{1}{8} \times \frac{3}{2} \left(\frac{1}{\frac{9}{4}} \right)^{3/2} \int \frac{\sec \theta d\theta}{\tan^2 \theta}$$

$$\cos \theta = \frac{2x}{\sqrt{4x^2-9}}$$

$$= \frac{1}{8} \times \frac{3}{2} \left(\frac{2}{3} \right)^3 \int \frac{1}{\cancel{\cos}} \frac{1}{\cancel{\sin}^2} d\theta$$

$$= \frac{1}{18} \int \cos (\sin \theta)^{-2} d\theta$$

$$= \frac{1}{18} (-) \sin \theta^{-1} + C$$

$$= -\frac{1}{18} \cos \theta + C$$

$$= -\frac{1}{18} \frac{2x}{\sqrt{4x^2-9}} + C$$

$$\text{til ora } \int \frac{dx}{(4x^2-9)^{3/2}} = -\frac{x}{9\sqrt{4x^2-9}} + C$$

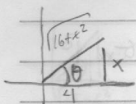
$$\int_a^b f(x) dx = F \Big|_a^b = F(b) - F(a)$$

$$12. \int \frac{dx}{x^4 \sqrt{16+x^2}} \quad \left. \begin{array}{l} x = 4 \tan \theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta = \tan^{-1} \frac{x}{4} \end{array} \right\}$$

$$dx = 4 \sec^2 \theta d\theta$$

$$\int \frac{dx}{x^4 \sqrt{16+x^2}} = \int \frac{4 \sec^2 \theta d\theta}{4^4 \tan^4 \theta \sqrt{16+16 \tan^2 \theta}}$$

$$= \frac{1}{4^3} \int \frac{\sec^2 \theta d\theta}{\tan^4 \theta \cdot 4 \sec \theta}$$



$$= \frac{1}{4^4} \int \frac{\sec \theta d\theta}{\tan^4 \theta}$$

$$\cos \theta = \frac{4}{\sqrt{16+x^2}} = \frac{4}{x}$$

$$= \frac{1}{4^4} \int \frac{1}{\cos \theta} \frac{1}{\sin^4 \theta} d\theta$$

$$= \frac{1}{4^4} \int \frac{\cos^3 \theta d\theta}{\sin^4 \theta} = \frac{1}{4^4} \int \cos \theta \frac{\cos^2 \theta}{\sin^4 \theta} d\theta$$

$$= \frac{1}{4^4} \int \cos \theta \frac{(1-\sin^2 \theta)}{\sin^4 \theta} d\theta$$

$$= \frac{1}{4^4} \int \cos \theta \sin^{-4} \theta d\theta = \frac{1}{4^4} \int \cos(\sin \theta)^{-2} d\theta$$

$$= \frac{1}{4^4} \frac{\sin^{-3} \theta}{-3} = \frac{1}{4^4} \frac{(\sin \theta)^{-1}}{-1} + C$$

$$= \frac{1}{768} \cos \theta + \frac{1}{256} \csc \theta + C$$

$$\int \frac{dx}{x^4 \sqrt{16+x^2}} = -\frac{1}{768} \frac{(16+x^2)^{3/2}}{x^3} + \frac{1}{256} \frac{\sqrt{16+x^2}}{x} + C$$

13.

$$\int \frac{2 dt}{x \sqrt{x^2+25}}$$

$$x^2 = u \rightarrow du = 2x dt$$

$$\Leftrightarrow$$

$$dt = \frac{1}{2x} du$$

$$= \frac{1}{2x} du$$

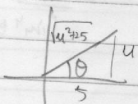
$$= \int \frac{2 \cdot \frac{1}{2x} du}{\sqrt{u} \sqrt{u^2+25}}$$

$$= \int \frac{du}{u \sqrt{u^2+25}}$$

$$\left. \begin{array}{l} u = 5 \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta = \tan^{-1} \frac{u}{5} \\ du = 5 \sec^2 \theta d\theta \end{array} \right\}$$

$$= \int \frac{5 \sec^2 \theta d\theta}{5 \tan \theta \sqrt{25 \tan^2 \theta + 25}}$$

$$= \int \frac{\sec \theta d\theta}{\tan \theta \cdot 5 \sec \theta} = \frac{1}{5} \int \frac{\sec \theta d\theta}{\tan \theta}$$



$$= \frac{1}{5} \int \frac{1}{\tan \theta} \frac{1}{\sec \theta} d\theta$$

$$= \frac{1}{5} \int \csc \theta d\theta$$

$$= \frac{1}{5} \ln |\csc \theta - \cot \theta| + C$$

$$= \frac{1}{5} \ln \left| \frac{\sqrt{u^2+25}}{u} - \frac{5}{u} \right| + C$$

$$(u = x^2)$$

$$= \frac{1}{5} \ln \left| \frac{\sqrt{x^2+25}}{x^2} - \frac{5}{x^2} \right| + C$$

$$\text{710 } \neq x$$

$$\int \frac{2 dt}{x \sqrt{x^2+25}} = \frac{1}{5} \ln \left(\frac{\sqrt{x^2+25} - 5}{x^2} \right) + C$$

1 1

$$4. \int \frac{x^3 dx}{(25-x^2)^2} \quad \left. \begin{array}{l} x = 5 \sin \theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta = \sin^{-1} \frac{x}{5} \\ dx = 5 \cos \theta d\theta \end{array} \right\}$$

$$\int \frac{x^3 dx}{(25-x^2)^2} = \int \frac{(5^3 \sin^3 \theta) (5 \cos \theta d\theta)}{(25(1-\sin^2 \theta))^2}$$

$$= \int \frac{5^4 \sin^3 \theta \cos \theta d\theta}{(25 \cos^2 \theta)^2}$$

$$\frac{5^4 x}{(25-x^2)^2}$$

$$= 5 \int \frac{\sin^3 \theta \cos \theta d\theta}{\cos^4 \theta}$$

$$\int \frac{x^3}{(25-x^2)^2} = \int \frac{\sin^3 \theta d\theta}{\cos^3 \theta} = \int (\tan \theta)^3 d\theta$$

$$= \int \tan \theta \tan^2 \theta d\theta$$

$$= \int \tan \theta (\sec^2 \theta - 1) d\theta$$

$$= \int \tan \theta \sec^2 \theta d\theta - \int \tan \theta d\theta$$

$$= \frac{\tan^2 \theta}{2} - \ln |\sec \theta| + C$$

$$= \frac{\tan^2 \theta}{2} + \ln \cos \theta + C$$

$$\int \frac{x^3 dx}{(25-x^2)^2} = \frac{x^2}{2(25-x^2)} + \ln \frac{\sqrt{25-x^2}}{5} + C$$

15.

$$\int \frac{dx}{\sqrt{4x+x^2}}$$

$$\begin{aligned} 4x+x^2 &= x^2+4x+4-4 \\ &= (x+2)^2-4 \end{aligned}$$

$$= \int \frac{dx}{\sqrt{(x+2)^2-4}}$$

$$u = x+2 \rightarrow du = dx$$

$$= \int \frac{du}{\sqrt{u^2-4}}$$

$$\left. \begin{aligned} u &= 2 \sec \theta \quad (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi) \\ \theta &= \sec^{-1} \frac{u}{2} \end{aligned} \right\}$$

$$= \int \frac{2 \sec \theta d\theta}{\sqrt{4(\sec^2 \theta - 1)}}$$

$$du = 2 \sec \theta \tan \theta d\theta$$

$$= \int \frac{\sec \theta d\theta}{\tan \theta}$$

$$\begin{aligned} \sec \theta &= \frac{u}{2} \\ \downarrow \\ \cos \theta &= \frac{2}{u} \end{aligned}$$

$$= \ln |\sec \theta + \tan \theta| + C$$

$$\tan \theta = \frac{\sqrt{u^2-4}}{2}$$

$$= \ln \left| \frac{u}{2} + \frac{\sqrt{u^2-4}}{2} \right| + C$$

$$= \ln \left| \frac{x+2}{2} + \frac{\sqrt{x^2+4x}}{2} \right| + C$$

$$\int \frac{dx}{\sqrt{4x+x^2}} = \ln \left| \frac{x+2 + \sqrt{x^2+4x}}{2} \right| + C$$

$$\boxed{\int \frac{dx}{\sqrt{4x+x^2}} = \ln |x+2 + \sqrt{x^2+4x}| + C'}$$

16.

$$\int \frac{dx}{\sqrt{4x-x^2}}$$

$$4x-x^2 = -(x^2-4x)$$

$$= -(x^2-4x+4) + 4$$

$$= -(x-2)^2 + 4$$

$$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}}$$

$$\text{Let } \int u = x-2 \\ du = dx$$

$$= \int \frac{du}{\sqrt{4-u^2}}$$

$$\int u = 2 \sin \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \int \frac{2 \cos \theta d\theta}{2\sqrt{1-\sin^2 \theta}}$$

$$\theta = \sin^{-1} \frac{u}{2} \\ du = 2 \cos \theta d\theta$$

$$= \int \frac{\cancel{2} \cos \theta d\theta}{\cancel{2} \sqrt{1-\sin^2 \theta}}$$

$$= \theta + C$$

$$= \sin^{-1} \frac{u}{2} + C$$

$$\boxed{\int \frac{dx}{\sqrt{4x-x^2}} = \sin^{-1} \left(\frac{x-2}{2} \right) + C}$$

$$17. \int \frac{dx}{(5-4x-x^2)^{3/2}}$$

$$5-4x-x^2 = 9-4-4x-x^2 \\ = 9-(x+2)^2$$

$$= \int \frac{dx}{(9-(x+2)^2)^{3/2}}$$

$$= \int \frac{du}{(9-u^2)^{3/2}}$$

$$u = x+2 \rightarrow du = dx$$

$$= \int \frac{3 \cos \theta d\theta}{(9(1-\sin^2 \theta))^{3/2}}$$

$$u = 3 \sin \theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$0 = \sin^{-1} \frac{u}{3}$$

$$du = 3 \cos \theta d\theta$$

$$= \int \frac{3 \cos \theta d\theta}{3^3 \cos^3 \theta}$$

$$= \frac{1}{9} \int \frac{1}{\cos^2 \theta} d\theta$$

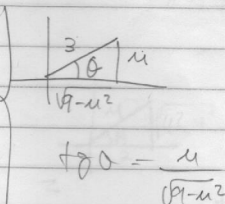
$$= \frac{1}{9} \int \sec^2 \theta d\theta$$

$$= \frac{1}{9} \tan \theta + C$$

$$= \frac{1}{9} \frac{u}{\sqrt{9-u^2}} + C$$

$$u = x+2$$

$$= \frac{1}{9} \frac{x+2}{\sqrt{9-(x+2)^2}} + C$$



$$\int \frac{dx}{(5-4x-x^2)^{3/2}} = \frac{x+2}{9\sqrt{5-4x-x^2}} + C$$

18.

$$\int \frac{dx}{x\sqrt{x^2-4}}$$

$$x^2 = t \rightarrow 2x dx = dt$$

$$= \int \frac{\frac{1}{2} \frac{dt}{t}}{\sqrt{t} \sqrt{t^2-4}}$$

$$dx = \frac{1}{2} \frac{dt}{t}$$

$$= \frac{1}{2} \int \frac{dt}{t\sqrt{t^2-4}}$$

$$x = 2 \sec \theta, [0, \pi) \cup (\pi, 3\pi/2]$$

$$\theta = \sec^{-1} \frac{t}{2}$$

$$= \frac{1}{2} \int \frac{2 \sec \theta \tan \theta d\theta}{2 \sec \theta \sqrt{4(\sec^2 \theta - 1)}}$$

$$dt = 2 \sec \theta \tan \theta d\theta$$

$$= \frac{1}{2} \int \frac{\cancel{2} \tan \theta d\theta}{2 \cancel{2} \tan \theta}$$

$$= \frac{1}{4} \int 1 d\theta$$

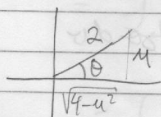
$$= \frac{1}{4} \sec^{-1} \frac{t}{2}$$

$$\int \frac{dx}{x\sqrt{x^2-4}} = \frac{1}{4} \sec^{-1} \frac{x^2}{2} + C$$

$$19. \int \frac{\sec^2 x \, dx}{(4 - \tan^2 x)^{3/2}}$$

Seja $u = \tan x \rightarrow du = \sec^2 x \, dx$

$$\int \frac{\sec^2 x \, dx}{(4 - \tan^2 x)^{3/2}} = \int \frac{du}{(4 - u^2)^{3/2}} \leftarrow$$



$$\tan \theta = \frac{u}{\sqrt{4-u^2}}$$

$$u = 2 \sin \theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\theta = \sin^{-1} \frac{u}{2}$$

$$du = 2 \cos \theta \, d\theta$$

$$= \int \frac{2 \cos \theta \, d\theta}{(4(1 - \sin^2 \theta))^{3/2}} \leftarrow$$

$$= \int \frac{2 \cos \theta \, d\theta}{2^3 \cos^3 \theta} = \frac{1}{4} \int \frac{1}{\cos^2 \theta} \, d\theta$$

$$= \frac{1}{4} \int \sec^2 \theta \, d\theta$$

$$= \frac{1}{4} \tan \theta + C$$

$$= \frac{1}{4} \frac{u}{\sqrt{4-u^2}} + C$$

$$\int \frac{\sec^2 x \, dx}{(4 - \tan^2 x)^{3/2}} = \frac{1}{4} \frac{\tan x}{\sqrt{4 - \tan^2 x}} + C \quad \left. \vphantom{\int} \right\} u = \tan x$$

20. $\int \frac{e^{-x} dx}{(9e^{-2x} + 1)^{3/2}} \left\{ \begin{array}{l} u = 3e^{-x} \\ du = -3e^{-x} dx \end{array} \right.$

$= \int \frac{-\frac{1}{3} du}{(u^2 + 1)^{3/2}}$

$= -\frac{1}{3} \int \frac{du}{(u^2 + 1)^{3/2}} \left\{ \begin{array}{l} u = \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta = \arctan u \\ du = \sec^2 \theta d\theta \end{array} \right.$

$= -\frac{1}{3} \int \frac{\sec^2 \theta d\theta}{(\tan^2 \theta + 1)^{3/2}}$



$= -\frac{1}{3} \int \frac{\sec^2 \theta d\theta}{\sec^3 \theta} = -\frac{1}{3} \int \cos \theta d\theta$

$\sin \theta = \frac{u}{\sqrt{1+u^2}}$

$= -\frac{1}{3} \sin \theta + C$

$= -\frac{1}{3} \frac{u}{\sqrt{1+u^2}} + C$

$= -\frac{1}{3} \frac{3e^{-x}}{\sqrt{1+9e^{-2x}}} + C$

$\int \frac{e^{-x} dx}{(9e^{-2x} + 1)^{3/2}} = -\frac{e^{-x}}{\sqrt{1+9e^{-2x}}} + C$

$$21. \int \frac{\ln^3 w \, dw}{w \sqrt{w^2 - 4}}$$

$$\left. \begin{aligned} u &= \ln w \\ du &= \frac{1}{w} dw \end{aligned} \right\}$$

$$= \int \frac{u^3 \, du}{\sqrt{u^2 - 4}}$$

$$\left. \begin{aligned} u &= 2 \sec \theta \quad \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right) \\ \theta &= \sec^{-1} \frac{u}{2} \\ du &= 2 \sec \theta \tan \theta \, d\theta \end{aligned} \right\}$$

$$= 8 \int \frac{u^3 \, d\theta}{\sqrt{4(\sec^2 \theta - 1)}} \quad \begin{matrix} \cancel{2 \sec \theta} \\ \cancel{2 \sec \theta} \end{matrix}$$

$$= 8 \int \sec^4 \theta \, d\theta$$

$$= 8 \int \sec^2 \theta \tan^2 \theta \, d\theta$$

$$= 8 \int \sec^2 \theta (1 + \tan^2 \theta) \, d\theta$$

$$= 8 \int \sec^2 \theta \, d\theta + 8 \int \sec^2 \theta \tan^2 \theta \, d\theta$$

$$= 8 \tan \theta + \frac{8}{3} \tan^3 \theta + C$$

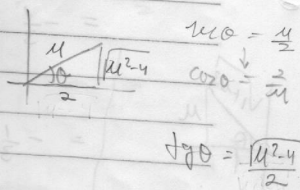
$$= 8 \frac{\sqrt{u^2 - 4}}{2} + \frac{8}{3} \left(\frac{\sqrt{u^2 - 4}}{2} \right)^3 + C$$

$$= 4 \sqrt{u^2 - 4} + \frac{1}{3} (u^2 - 4)^{3/2} + C$$

$$= 4 \sqrt{w^2 - 4} + \frac{1}{3} (w^2 - 4)^{3/2} + C$$

$$= \sqrt{w^2 - 4} \left[4 + \frac{w^2 - 4}{3} \right] + C$$

$$= \sqrt{w^2 - 4} \left(\frac{8 + w^2}{3} \right) + C //$$



22. $\int \frac{dz}{(z^2 - 6z + 18)^{3/2}}$

$$z^2 - 6z + 18 = (z-3)^2 + 9$$

$$= \int \frac{dz}{[(z-3)^2 + 9]^{3/2}}$$

$$\left. \begin{aligned} u &= z-3 \\ du &= dz \end{aligned} \right\}$$

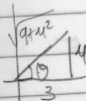
$$= \int \frac{du}{(u^2 + 9)^{3/2}}$$

$$\left. \begin{aligned} u &= 3 \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta &= \tan^{-1} \frac{u}{3} \end{aligned} \right\}$$

$$= \int \frac{3 \sec^2 \theta d\theta}{[9(\tan^2 \theta + 1)]^{3/2}} \quad du = 3 \sec^2 \theta d\theta$$

$$= \int \frac{3 \sec^2 \theta d\theta}{3^3 \sec^3 \theta} = \frac{1}{9} \int \cos \theta d\theta$$

$$= \frac{1}{9} \sin \theta + C$$



$$\tan \theta = \frac{u}{3}$$

$$\sin \theta = \frac{u}{\sqrt{u^2 + 9}}$$

$$= \frac{1}{9} \frac{u}{\sqrt{u^2 + 9}} + C$$

$$= \frac{1}{9} \frac{(z-3)}{\sqrt{9+(z-3)^2}} + C$$

$$u = z-3$$

$$\int \frac{dz}{(z^2 - 6z + 18)^{3/2}} = \frac{1}{9} \frac{z-3}{\sqrt{z^2 - 6z + 18}} + C$$

Thomas Clarkson

1785

$$23. \int \frac{e^t dt}{(e^{2t} + 8e^t + 7)^{3/2}} =$$

$$= \int \frac{e^t dt}{(e^{2t} + 8e^t + 16 - 9)^{3/2}}$$

$$= \int \frac{e^t dt}{((e^t + 4)^2 - 9)^{3/2}}$$

$$= \int \frac{du}{(u^2 - 9)^{3/2}} \quad \left. \begin{array}{l} u = e^t + 4 \\ du = e^t dt \end{array} \right\}$$

$$= \int \frac{3 \sec \theta \tan \theta d\theta}{(9(\sec^2 \theta - 1))^{3/2}} \quad \left. \begin{array}{l} u = 3 \sec \theta, \quad (0, \pi/2) \cup (\pi/2, \pi) \\ \theta = \sec^{-1} \frac{u}{3} \\ du = 3 \sec \theta \tan \theta d\theta \end{array} \right\}$$

$$= \int \frac{3 \sec \theta \tan \theta d\theta}{3^3 \tan^3 \theta}$$

$$= \frac{1}{9} \int \frac{\sec \theta d\theta}{\tan^2 \theta} = \frac{1}{9} \int \frac{1}{\cos \theta} \frac{\sin \theta d\theta}{\sin^2 \theta}$$

$$= \frac{1}{9} \int \sec \theta \csc^2 \theta d\theta$$

$$= \frac{1}{9} \frac{(\sec \theta)^{-1}}{-1} + C$$

$$= -\frac{1}{9} \operatorname{cosec} \theta + C$$

$$= -\frac{1}{9} \frac{u}{\sqrt{u^2 - 9}} + C$$

$$\int \frac{e^t dt}{(e^{2t} + 8e^t + 7)^{3/2}} = -\frac{1}{9} \frac{e^t + 4}{\sqrt{e^{2t} + 8e^t + 7}} + C$$

$$\sec \theta = \frac{u}{3}$$

$$\cos \theta = \frac{3}{u}$$

$$\frac{u}{3} \sqrt{u^2 - 9}$$

$$\sin \theta = \frac{\sqrt{u^2 - 9}}{u}$$

$$\operatorname{cosec} \theta = \frac{u}{\sqrt{u^2 - 9}}$$

$$24. \int \frac{\sqrt{16 - e^{2x}}}{e^x} dx ; \quad \begin{cases} u = e^x \\ du = e^x dx \end{cases}$$

$$= \int \frac{\sqrt{16 - u^2}}{u} \frac{du}{u}$$

$$= \int \frac{\sqrt{16 - u^2}}{u^2} du ; \quad \begin{cases} u = 4 \sin \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta = \sin^{-1} \frac{u}{4} \\ du = 4 \cos \theta d\theta \end{cases}$$

$$= \int \frac{\sqrt{16(1 - \sin^2 \theta)} \cdot 4 \cos \theta d\theta}{16 \sin^2 \theta}$$

$$= \int \frac{4 \cos \theta \cdot 4 \cos \theta d\theta}{16 \sin^2 \theta} \quad \begin{array}{|c|} \hline \frac{4}{16} \cdot \frac{4}{4} \\ \hline \frac{16}{16} \cdot u^2 \\ \hline \end{array}$$

$$= \int \cot^2 \theta d\theta$$

$$= \int (\csc^2 \theta - 1) d\theta$$

$$= \int \csc^2 \theta d\theta - \int d\theta$$

$$= -\cot \theta - \theta + C$$

$$= -\frac{\sqrt{16 - u^2}}{u} - \sin^{-1} \frac{u}{4} + C \quad | \quad u = e^x$$

$$\int \frac{\sqrt{16 - e^{2x}}}{e^x} dx = -\frac{\sqrt{16 - e^{2x}}}{e^x} - \sin^{-1} \frac{e^x}{4} + C$$