Cálculo B - Lista 7

Integrais impróprias I (intervalo de integração é ilimitado)

Nos exercícios 1 a 18, determine se a integral imprópria é convergente ou divergente. Se for convergente, calcule-a.

1. $\int_0^\infty e^{-x}dx$
2. $\int_{-\infty}^1 e^x dx$
3. $\int_{-\infty}^0 x5^{-x^2}dx$
4. $\int_1^\infty 2^{-x}dx$
5. $\int_0^\infty x2^{-x}dx$
6. $\int_5^\infty \frac{dx}{\sqrt{x-1}}$
7. $\int_{-\infty}^\infty x\cosh x dx$
8. $\int_{-\infty}^0 x^2e^x dx$
9. $\int_5^\infty \frac{xdx}{\sqrt[3]{9-x^2}}$
10. $\int_{-\infty}^\infty \frac{3xdx}{(3x^2+2)^3}$
11. $\int_{\sqrt{3}}^\infty \frac{3dx}{x^2+9}$
12. $\int_e^\infty \frac{dx}{x\ln x}$
13. $\int_{-\infty}^\infty e^{-|x|}dx$
14. $\int_{-\infty}^\infty xe^{-x^2}dx$
15. $\int_e^\infty \frac{dx}{x(\ln x)^2}$
16. $\int_{-\infty}^\infty \frac{dx}{16+x^2}$
17. $\int_1^\infty \ln x dx$
18. $\int_0^\infty e^{-x}\cos x dx$
19. Calcule se existir:
    (a) $\int_{-\infty}^\infty \sin x dx$
    (b) $\lim_{r\to\infty}\int_{-r}^r \sin x dx$
20. Prove que se $\int_{-\infty}^b f(x)dx$ for convergente, então $\int_{-b}^\infty f(-x)dx$ também será convergente e terá o mesmo valor.
21. Mostre que a integral imprópria
$$\int_{-\infty}^\infty x(1+x^2)^{-2}dx$$
é convergente e a integral imprópria
$$\int_{-\infty}^\infty x(1+x^2)^{-1}dx$$
é divergente.
22. Prove que a integral imprópria
$$\int_1^\infty \frac{dx}{x^n}$$
será convergente se e somente se $n > 1$.
23. (a) Suponha que f e g são contínuas em $[a,\infty)$. Mostre que se $\int_a^\infty f(x)dx$ e $\int_a^\infty g(x)dx$ convergem, então $\int_a^\infty (f(x)+g(x))dx$ converge.
    (b) Mostre que se $\int_a^\infty f(x)dx$ converge então $\int_a^\infty cf(x)dx$ também converge para todo c.
24. Sejam f e g contínuas em $[a,\infty)$ e assuma que $0 \le g(x) \le f(x)$ para $x \ge a$. Mostre que se $\int_a^\infty g(x)dx = \infty$, então $\int_a^\infty f(x)dx = \infty$ e consequentemente $\int_a^\infty f(x)dx$ diverge.
25. Usando o resultado (24) mostre que cada uma das integrais a seguir divergem.
    (a)
$$\int_1^\infty \frac{1}{1+x^{1/2}}\,dx$$
    (b)
$$\int_0^\infty \frac{1}{\sqrt{2+\sin x}}\,dx$$
    (c)
$$\int_2^\infty \frac{\ln x}{\sqrt{x^2-1}}\,dx$$
    (d)
$$\int_2^\infty \frac{1}{(1+x)\ln x}\,dx$$
26. Seja
$$I_n = \int_0^\infty x^n e^{-x}\,dx$$
$n \in N$. Usando integração por partes mostre que $I_n = nI_{n-1}$ para $n \ge 1$. Daí, mostre que $I_n = n(n-1)(n-2)\ldots 2.1$

Respostas

1. 1
2. e
3. $-\frac{1}{2\ln 5}$
4. $\frac{1}{2\ln 2}$
5. $\frac{1}{(\ln 2)^2}$
6. ∞ (divergente)
7. diverge
8. 2
9. $-\infty$ (diverge)
10. 0
11. $\frac{\pi}{3}$
12. ∞ (diverge)
13. 2
14. 0
15. 1
16. $\frac{\pi}{4}$
17. ∞ (diverge)
18. $\frac{1}{2}$
19. (a) $\nexists$

(b) 0

Lista 7 — Integrais Impróprias I

Leithold (pg. 672)

1. $\int_0^{+\infty} e^{-x}\,dx$

$= \lim_{b\to+\infty} \int_0^b e^{-x}\,dx$

$= \lim_{b\to+\infty} -e^{-x}\Big|_0^b$

$= \lim_{b\to+\infty} \left(-e^{-b} - (-e^0)\right)$

$= \lim_{b\to+\infty} \left(-e^{-b} + 1\right)$ $\quad \lim_{b\to+\infty} e^{-b} = 0$

$= 1$ //

2. $\int_{-\infty}^1 e^x\,dx =$

$= \lim_{a\to-\infty} \int_a^1 e^x\,dx$

$= \lim_{a\to-\infty} e^x\Big|_a^1$

$= \lim_{a\to-\infty} (e - e^a)$ $\quad \lim_{a\to-\infty} e^a = 0$

$= e$ //

/ /

3. $\int_{-\infty}^{0} x\, 5^{-x^2}\, dx =$

$$= \underbrace{\lim_{a \to -\infty} \int_a^0 x\, 5^{-x^2}\, dx}$$

$(e^x)' = e^x$

$(a^x)' = a^x \ln a$

$$= \lim_{a \to -\infty} -\frac{1}{2\ln 5} 5^{-x^2} \Big|_a^0$$

$$= \lim_{a \to -\infty} \left(-\frac{1}{2\ln 5} 5^{-0} + \frac{1}{2\ln 5} 5^{-a^2} \right)$$

$$= \lim_{a \to -\infty} \left(\frac{-1}{2\ln 5} + \frac{5^{-a^2}}{2\ln 5} \right)$$

Mas

$$\lim_{a \to -\infty} \frac{5^{-a^2}}{2\ln 5} = \text{"}\frac{5^{-\infty}}{2\ln 5}\text{"} = 0$$

$$= -\frac{1}{2\ln 5}$$

$\therefore$

$$\boxed{\int_{-\infty}^{0} x\, 5^{-x^2}\, dx = \frac{-1}{2\ln 5}}$$

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4. $\displaystyle\int_1^{+\infty} 2^{-x}\,dx$

$$= \lim_{b\to+\infty} \int_1^b 2^{-x}\,dx \qquad (2^{-x})' = -2^{-x}\ln 2$$

$$= \lim_{b\to+\infty} -\frac{2^{-x}}{\ln 2}\Big|_1^b$$

$$= \lim_{b\to+\infty}\left(-\frac{2^{-b}}{\ln 2} + \frac{1}{2\ln 2}\right)$$

$$= \cancel{\lim_{b\to+\infty} \frac{-2^{-b}}{\ln 2}}^{\,0} + \frac{1}{2\ln 2}$$

$$= \frac{1}{2\ln 2} \,//$$

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5. $\displaystyle\int_0^{+\infty} x\,2^{-x}\,dx =$

$$= \lim_{b\to+\infty} \int_0^b x\,2^{-x}\,dx$$

Mas

$$\int x\,2^{-x}\,dx = \frac{-x\,2^{-x}}{\ln 2} + \int \frac{2^{-x}}{\ln 2}\,dx$$

$u = x \to du = dx$

$dv = 2^{-x}dx \to v = \dfrac{2^{-x}}{-\ln 2}$

$$= \frac{-x\,2^{-x}}{\ln 2} + \frac{1}{\ln 2}\,\frac{2^{-x}}{(-\ln 2)}$$

$\Longrightarrow$

$$\int_0^b x\,2^{-x}\,dx = \left[-\frac{x2^{-x}}{\ln 2} - \frac{2^{-x}}{(\ln 2)^2}\right]_0^b$$

$$= -\frac{b2^{-b}}{\ln 2} - \frac{2^{-b}}{(\ln 2)^2} + \frac{1}{(\ln 2)^2}$$

$\therefore$

$$\lim_{b\to+\infty}\int_0^b x\,2^{-x}\,dx = \lim_{b\to+\infty}\left(-\frac{b2^{-b}}{\ln 2} - \frac{2^{-b}}{(\ln 2)^2} + \frac{1}{(\ln 2)^2}\right)$$

$$= \frac{1}{(\ln 2)^2}$$

$\therefore$

$$\boxed{\int_0^{+\infty} x\,2^{-x}\,dx = \frac{1}{(\ln 2)^2}}$$

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6. // $\displaystyle\int_5^{+\infty}\frac{dx}{\sqrt{x-1}} =$

$$= \lim_{b\to+\infty}\int_5^b \frac{dx}{\sqrt{x-1}}$$

$$= \lim_{b\to+\infty} 2\sqrt{x-1}\,\Big|_5^b$$

$$= \lim_{b\to+\infty}\left(2\sqrt{b-1} - 4\right)$$

$$= +\infty \quad //$$ (Divergente)

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7. $\int_{-\infty}^{+\infty} x \cosh x \, dx$

$$\int x \cosh x \, dx = \int x \frac{e^{x}+e^{-x}}{2} dx$$

$$= \frac{1}{2} \int (xe^{x} + xe^{-x}) dx$$

$$\int xe^{x} dx = xe^{x} - \int e^{x} dx$$

$u = x \to du = dx$

$dv = e^{x} dx \to v = e^{x}$

$$= xe^{x} - e^{x}$$

$$\int x e^{-x} dx = -xe^{-x} + \int e^{-x} dx$$

$u = x \to du = dx$

$dv = e^{-x} dx \to v = -e^{-x}$

$$= -xe^{-x} - e^{-x}$$

$\therefore$

$$= \frac{1}{2} (xe^{x} - e^{x} - xe^{-x} - e^{-x})$$

$\therefore$

$$\int x \cosh x \, dx = \frac{1}{2} e^{x}(x-1) - \frac{1}{2} e^{-x}(x+1)$$

$\Longrightarrow$

$$\int_{-\infty}^{+\infty} x\cosh x\,dx = \underbrace{\lim_{a\to-\infty}\int_a^0 x\cosh x\,dx}_{(*)} + \underbrace{\lim_{b\to+\infty}\int_0^b x\cosh x\,dx}_{(**)}$$

Mas

$$(*) = \lim_{a\to-\infty}\int_a^0 x\cosh x\,dx =$$

$$= \lim_{a\to-\infty}\left(\frac{1}{2}e^{x}(x-1) - \frac{1}{2}e^{-x}(x+1)\right)\Bigg|_a^0$$

$$= \lim_{a\to-\infty}\left(\underbrace{\frac{1}{2}(-1) - \frac{1}{2}(1)} - \frac{1}{2}e^{a}(a-1) + \frac{1}{2}e^{-a}(a+1)\right)$$

$$= \lim_{a\to-\infty}\left(-1 - \frac{1}{2}\cancel{e^{a}(a-1)}^{\,0} + \frac{1}{2}e^{-a}(a+1)\right)$$

$$= -\infty$$

$$\therefore \boxed{\int_{-\infty}^{+\infty} x\cosh x\,dx = \text{diverge}}$$

8. $\int_{-\infty}^{0} x^2 e^x \, dx =$

$$= \lim_{a \to -\infty} \int_a^0 x^2 e^x \, dx$$

Mas

$$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx \quad (*)$$

$u = x^2 \to du = 2x\,dx$

$dv = e^x dx \to v = e^x$

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x \quad (**)$$

$u = x \to du = dx$

$dv = e^x dx \to v = e^x$

Substituindo $(**) \to (*)$:

$$\int x^2 e^x \, dx = x^2 e^x - 2(x e^x - e^x)$$

$$= x^2 e^x - 2x e^x + 2e^x$$

$$= (x^2 - 2x + 2) e^x$$

Daí,

$$\int_a^0 x^2e^x\,dx = (x^2-2x+2)e^x \Big|_a^0$$

$$= 2-(a^2-2a+2)e^a$$

e

$$\lim_{a\to-\infty}\int_a^0 x^2e^x\,dx = \lim_{a\to-\infty}\left(2-(a^2-2a+2)\overset{0}{\cancel{e^a}}\right)$$

$$= 2$$

$\therefore$

$$\boxed{\int_{-\infty}^0 x^2e^x\,dx = 2}$$

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9. $$\int_5^{+\infty}\frac{x\,dx}{\sqrt[3]{9-x^2}}$$

$$= \lim_{b\to+\infty}\int_5^b \frac{x\,dx}{\sqrt[3]{9-x^2}}$$

$$= \lim_{b\to+\infty} \frac{-3}{4}(9-x^2)^{2/3}\Big]_5^b$$

$$= \frac{-3}{4}\lim_{b\to+\infty}\left((9-b^2)^{2/3}-4\sqrt[3]{2}\right)$$

$$= -\infty \quad \text{diverge} \;//$$

$$\left[-\frac{3}{4}(9-x^2)^{2/3}\right]' =$$

$$= -\frac{\cancel{3}}{4}\times\frac{2}{\cancel{3}}(9-x^2)^{\frac{2}{3}-1}(-2x)$$

$$= -\frac{1}{\cancel{2}}(-\cancel{2}x)(9-x^2)^{-1/3}$$

$$= \frac{x}{\sqrt[3]{9-x^2}}$$

$$(9-25)^{2/3} = -16^{2/3}$$

$$= \sqrt[3]{(-16)^2}$$

$$= \sqrt[3]{2^8} = 2^6 2$$

$$= 4\sqrt[3]{2}$$

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10. $$\int_{-\infty}^{+\infty} \frac{3x\,dx}{(3x^2+2)^3}$$

$$= \lim_{a\to-\infty} \int_a^0 \frac{3x}{(3x^2+2)^3}\,dx + \lim_{b\to+\infty} \int_0^b \frac{3x}{(3x^2+2)^3}\,dx$$

tomamos $c=0$

Mas

$$\int \frac{3x\,dx}{(3x^2+2)^3} = \frac{1}{-4\,(3x^2+2)^2}$$

$\therefore$

$$\lim_{a\to-\infty} \int_a^0 \frac{3x}{(3x^2+2)^3}\,dx = \lim_{a\to-\infty} -\frac{1}{4(3x^2+2)^2}\Bigg|_a^0$$

$$= \lim_{a\to-\infty} \left(-\frac{1}{4\cdot 4} + \frac{1}{4(3a^2+2)^2}\right)$$

$$= -\frac{1}{16}$$

$$\lim_{b\to+\infty} \int_0^b \frac{3x}{(3x^2+2)^3}\,dx = \lim_{b\to+\infty} -\frac{1}{4(3x^2+2)^2}\Bigg|_0^b$$

$$= \lim_{b\to+\infty} \left(-\frac{1}{4(3b^2+2)^2} + \frac{1}{4\cdot 4}\right)$$

$$= \frac{1}{16}$$

$\therefore$

$$\int_{-\infty}^{+\infty} \frac{3x}{(3x^2+2)^3}\,dx = -\frac{1}{16} + \frac{1}{16} = 0$$

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11. $\displaystyle\int_{\sqrt{3}}^{+\infty} \frac{3\,dx}{x^2+9} =$

$\displaystyle = \lim_{b\to+\infty} \int_{\sqrt{3}}^{b} \frac{3\,dx}{x^2+9}$

$\left\{ \displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\, tg^{-1}\frac{x}{a} \right.$

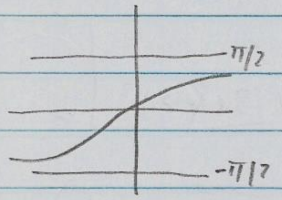
Mas,

$\displaystyle \int_{\sqrt{3}}^{b} \frac{3\,dx}{x^2+9} = 3\int_{\sqrt{3}}^{b} \frac{dx}{x^2+9}$

$\displaystyle = 3\,\frac{1}{3}\, tg^{-1}\frac{x}{3}\Big|_{\sqrt{3}}^{b}$

$\displaystyle = tg^{-1}\frac{b}{3} - \underbrace{tg^{-1}\frac{\sqrt{3}}{3}}$

$\displaystyle = tg^{-1}\frac{b}{3} - \frac{\pi}{3}$



Dom $tg^{-1} = (-\infty, +\infty)$

Im $tg^{-1} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$y = tg^{-1}x \Leftrightarrow x = tg\, y$

$y = tg^{-1}\frac{\sqrt{3}}{3} \Leftrightarrow \frac{\sqrt{3}}{3} = tg\, y$

$\Downarrow$

$y = \frac{\pi}{6}$

$\therefore$

$\displaystyle \lim_{b\to+\infty} \int_{\sqrt{3}}^{b} \frac{3\,dx}{x^2+9} =$

$\displaystyle = \lim_{b\to+\infty}\left(tg^{-1}\frac{b}{3} - \frac{\pi}{6}\right)$

$\displaystyle \lim_{b\to+\infty} tg^{-1}\frac{b}{3} = tg^{-1}\text{"}\infty\text{"} = \frac{\pi}{2}$

$\displaystyle = \frac{\pi}{2} - \frac{\pi}{6}$

$\displaystyle = \frac{3\pi - \pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$

$\therefore$

$$\boxed{\int_{\sqrt{3}}^{+\infty} \frac{3\,dx}{x^2+9} = \frac{\pi}{3}}$$

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12. $\int_e^{+\infty} \frac{dx}{x\ln x} =$

$$= \lim_{b\to+\infty} \int_e^b \frac{dx}{x\ln x}$$

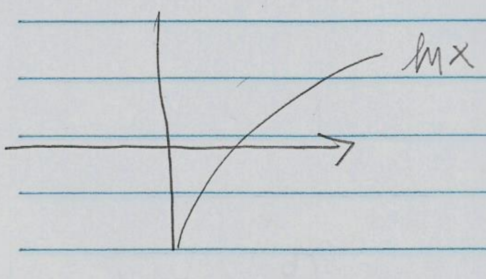
Mas

$$\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln|u| = \ln|\ln x|$$

$u = \ln x \to du = \frac{1}{x}dx$

$$\Rightarrow \lim_{b\to+\infty} \int_e^b \frac{dx}{x\ln x} = \lim_{b\to+\infty} \ln|\ln x| \Big|_e^b$$

$$= \lim_{b\to+\infty} \left(\ln|\ln b| - \ln|\ln e|\right)$$



$$= \lim_{b\to+\infty} \left(\ln|\ln b| - \ln 1\right)$$

$$= \lim_{b\to+\infty} \left(\ln|\ln b| - 0\right)$$

$$= \lim_{b\to+\infty} \ln|\ln b|$$

$$= +\infty$$

$$\Rightarrow \boxed{\int_e^{+\infty} \frac{dx}{x\ln x} = +\infty \quad \text{diverge}}$$

13. $\displaystyle\int_{-\infty}^{+\infty} e^{-|x|}\,dx =$

$$= \lim_{a\to-\infty} \int_a^0 \underbrace{e^{-|x|}}_{x<0} dx + \lim_{b\to+\infty} \int_0^b \underbrace{e^{-|x|}}_{x>0} dx$$

$$= \lim_{a\to-\infty} \int_a^0 e^{x}\,dx + \lim_{b\to+\infty} \int_0^b e^{-x}\,dx$$

$$= \lim_{a\to-\infty} e^{x}\Big|_a^0 + \lim_{b\to+\infty} -e^{-x}\Big|_0^b$$

$$= \lim_{a\to-\infty} (1-e^{a}) + \lim_{b\to+\infty} (-e^{-b}+1)$$

$$= 1 + 1$$

$$= 2$$

14. $\displaystyle\int_{-\infty}^{+\infty} x\,e^{-x^2}\,dx =$

$$= \lim_{a\to-\infty} \int_a^0 x\,e^{-x^2}dx + \lim_{b\to+\infty} \int_0^b x\,e^{-x^2}dx$$

$$= \lim_{a\to-\infty} -\frac{1}{2}e^{-x^2}\Big|_a^0 + \lim_{b\to+\infty} -\frac{1}{2}e^{-x^2}\Big|_0^b$$

$$= \lim_{a\to-\infty} \left(-\frac{1}{2} + \frac{1}{2}\cancelto{0}{e^{-a^2}}\right) + \lim_{b\to+\infty} \left(-\frac{1}{2}\cancelto{0}{e^{-b^2}} + \frac{1}{2}\right)$$

$= -\frac{1}{2} + \frac{1}{2} = 0$ → já era esperado pois $x\,e^{-x^2}$ é função ímpar.

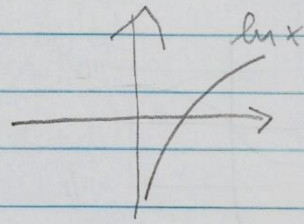
15. $\int_e^{+\infty} \frac{dx}{x(\ln x)^2} =$

$$= \lim_{b\to+\infty} \int_e^b \frac{dx}{x(\ln x)^2}$$

Mas:

$$\begin{cases} u = \ln x \rightarrow du = \frac{1}{x}dx \\ \therefore \int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{\ln x} \end{cases}$$

$$= \lim_{b\to+\infty} \left[-\frac{1}{\ln x} \right]_e^b$$



$$= \lim_{b\to+\infty} \left(-\cancel{\frac{1}{\ln b}}^{\,0} + 1 \right)$$

$\lim_{b\to+\infty} \frac{1}{\ln b} = \frac{1}{\infty} = 0$

$$= 1$$

$$\therefore \boxed{\int_e^{+\infty} \frac{dx}{x(\ln x)^2} = 1}$$

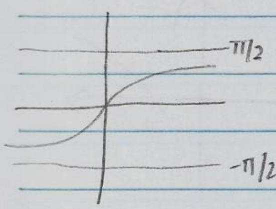
16. $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{16+x^2} =$

$$= \lim_{a\to-\infty} \int_a^0 \frac{dx}{16+x^2} + \lim_{b\to+\infty} \int_0^b \frac{dx}{16+x^2}$$

$$= \lim_{a\to-\infty} \frac{1}{4}\, tg^{-1}\frac{x}{4}\Big|_a^0 + \lim_{b\to+\infty} \frac{1}{4}\, tg^{-1}\frac{x}{4}\Big|_0^b$$

$\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\, tg^{-1}\frac{x}{a}$

$$= \lim_{a\to-\infty}\left(\cancel{\frac{1}{4}\, tg^{-1} 0}^{\,0} - \frac{1}{4}\, tg^{-1}\frac{a}{4}\right) + \lim_{b\to+\infty}\left(\frac{1}{4}\, tg^{-1}\frac{b}{4} - \cancel{\frac{1}{4}\, tg^{-1} 0}^{\,0}\right)$$



$$= \lim_{a\to-\infty} -\frac{1}{4}\, tg^{-1}\frac{a}{4} + \lim_{b\to+\infty} \frac{1}{4}\, tg^{-1}\frac{b}{4}$$

$$= -\frac{1}{4} \lim_{a\to-\infty} tg^{-1}\frac{a}{4} + \frac{1}{4} \lim_{b\to+\infty} tg^{-1}\frac{b}{4}$$

Dom $tg^{-1} = \mathbb{R}$

Im $tg^{-1} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$tg^{-1} x = y \Leftrightarrow x = tg\, y$

$$= -\frac{1}{4}\left(-\frac{\pi}{2}\right) + \frac{1}{4}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{8} + \frac{\pi}{8} = \frac{\pi}{4}$$

17. $\int_1^{+\infty} \ln x \, dx =$

$$= \lim_{b \to +\infty} \int_1^b \ln x \, dx$$

Mas

$$\int \ln x \, dx = x \ln x - \int dx$$

$u = \ln x \to du = \frac{1}{x} dx$ $\quad = x \ln x - x$

$dv = dx \to v = x$

$\therefore$

$$\lim_{b \to +\infty} \int_1^b \ln x \, dx = \lim_{b \to +\infty} (x \ln x - x)\Big|_1^b$$

$$= \lim_{b \to +\infty} (b \ln b - b - \cancel{x \ln 1}^{0} + 1)$$

$$= \lim_{b \to +\infty} (b \ln b - b + 1)$$

$$= \lim_{b \to +\infty} b \left(\ln b - 1 + \frac{1}{b}\right) = \infty \cdot \infty = \infty$$

$\therefore$

$$\boxed{\int_1^{+\infty} \ln x \, dx = +\infty \quad \text{diverge}}$$

18. $\displaystyle\int_0^{+\infty} e^{-x}\cos x\, dx$

$$= \lim_{b\to+\infty} \int_0^b e^{-x}\cos x\, dx$$

Mas,

$$\int e^{-x}\cos x\, dx = \sin x\, e^{-x} + \int \sin x\, e^{-x}\, dx \quad (*)$$

$u = e^{-x} \to du = -e^{-x}dx$

$dv = \cos x\, dx \to v = \sin x$

Mas

$$\int \sin x\, e^{-x}\, dx = -\cos x\, e^{-x} - \int \cos x\, e^{-x}\, dx \quad (**)$$

$u = e^{-x} \to du = -e^{-x}dx$

$dv = \sin x\, dx \to v = -\cos x$

Substituindo $(**) \to (*)$:

$$\int e^{-x}\cos x\, dx = \sin x\, e^{-x} - \cos x\, e^{-x} - \int \cos x\, e^{-x}\, dx$$

$$2\int e^{-x}\cos x\, dx = (\sin x - \cos x)\, e^{-x}$$

$$\therefore \quad // \int e^{-x}\cos x\, dx = \frac{1}{2}(\sin x - \cos x)\, e^{-x} //$$

$\Longrightarrow$

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$$\lim_{b \to +\infty} \int_0^b e^{-x} \cos x \, dx =$$

$$= \lim_{b \to +\infty} \frac{1}{2} (\sin x - \cos x) e^{-x} \Big|_0^b$$

$$= \lim_{b \to +\infty} \left(\frac{1}{2} (\sin b - \cos b) e^{-b} - \frac{1}{2} (\sin 0 - \cos 0) e^{-0} \right)$$

$$= \lim_{b \to +\infty} \left(\frac{1}{2} (\sin b - \cos b) e^{-b} - \frac{1}{2} (-1) \right)$$

$$= \lim_{b \to +\infty} \left(\frac{1}{2} (\sin b - \cos b) e^{-b} + \frac{1}{2} \right)$$

$b \to +\infty$: $\left| \frac{1}{2} (\sin b - \cos b) \right| < 2$

$$e^{-b} \to 0$$

$\therefore$

$$\lim_{b \to +\infty} \left(\frac{1}{2} (\sin b - \cos b) e^{-b} + \frac{1}{2} \right) = \frac{1}{2}$$

$\therefore$

$$\boxed{\int_0^{+\infty} e^{-x} \cos x \, dx = \frac{1}{2}}$$

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19) a) $\int_{-\infty}^{+\infty} \sin x \, dx =$

$$= \lim_{a \to -\infty} \int_a^0 \sin x \, dx + \lim_{b \to +\infty} \int_0^b \sin x \, dx$$

$$= \lim_{a \to -\infty} -\cos x \Big|_a^0 + \lim_{b \to +\infty} -\cos x \Big|_0^b$$

$$= \lim_{a \to -\infty} (-1 + \cos a) + \lim_{b \to +\infty} (-\cos b + 1)$$

uma vez que $\left\{ \begin{array}{l} \lim\limits_{a \to -\infty} \cos a \\ \lim\limits_{b \to \infty} \cos b \end{array} \right.$ não existem

temos que

$$\boxed{\int_{-\infty}^{+\infty} \sin x \, dx \quad \text{não existe}}$$

b) $\lim\limits_{r \to -\infty} \int_{-r}^{r} \sin x \, dx =$

$$= \lim_{r \to -\infty} \left[-\cos x \Big|_{-r}^{r} \right]$$

$$= \lim_{r \to -\infty} \left(-\cos r - (-\cos(-r)) \right)$$

$$= \lim_{r \to -\infty} \left(-\cos r + \cos r \right) = 0$$

20.

Seja $\int_{-\infty}^{b} f(x)\,dx$ convergente.

Temos de mostrar que

$$\int_{-b}^{+\infty} f(-x)\,dx = \int_{-\infty}^{b} f(x)\,dx .$$

De facto,

Por definição,

$$\int_{-b}^{+\infty} f(-x)\,dx = \lim_{a \to +\infty} \int_{-b}^{a} f(-x)\,dx \quad (*)$$

Façamos agora a mudança de variável $x \to y = -x$ na integral $\int_{-b}^{a} f(-x)\,dx$,

$$\int_{-b}^{a} f(-x)\,dx = \int_{b}^{-a} -f(y)\,dy$$

$x = -b \to y = b$

$x = a \to y = -a$

$$= -\int_{b}^{-a} f(y)\,dy$$

$$= \int_{-a}^{b} f(y)\,dy$$

$$= \int_{-a}^{b} f(x)\,dx$$

y é variável muda

$$\therefore \quad \int_{-b}^{a} f(-x)\,dx = \int_{-a}^{b} f(x)\,dx \quad (**)$$

Subst. (**) em (*) obtém-se,

$$\int_{-b}^{+\infty} f(-x)\,dx = \lim_{a\to+\infty} \int_{-b}^{a} f(-x)\,dx$$

$$= \underbrace{\lim_{a\to+\infty} \int_{-a}^{b} f(x)\,dx}$$

$$= \int_{-\infty}^{b} f(x)\,dx$$

$\therefore$ $\boxed{\int_{-b}^{+\infty} f(-x)\,dx = \int_{-\infty}^{b} f(x)\,dx}$

21.

a) $\int_{-\infty}^{+\infty} x(1+x^2)^{-2}\,dx$ é convergente

b) $\int_{-\infty}^{+\infty} x(1+x^2)^{-1}\,dx$ é divergente

De facto

a) $\int_{-\infty}^{+\infty} x(1+x^2)^{-2}\,dx =$

$$= \lim_{a\to-\infty} \int_{a}^{0} x(1+x^2)^{-2}\,dx + \lim_{b\to+\infty} \int_{0}^{b} x(1+x^2)^{-2}\,dx$$

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21. (continuação) $\int x(1+x^2)^{-2}\,dx = -\frac{1}{2}(1+x^2)^{-1}$

$$= \lim_{a\to-\infty} \left.\frac{-1}{2(1+x^2)}\right|_a^0 + \lim_{b\to+\infty} \left.\frac{-1}{2(1+x^2)}\right|_0^b$$

$$= \lim_{a\to-\infty}\left(-\frac{1}{2} + \cancel{\frac{1}{2(1+a^2)}}^{\,0}\right) + \lim_{b\to+\infty}\left(-\frac{1}{2} + \cancel{\frac{1}{2(1+b^2)}}^{\,0}\right)$$

$$= -\frac{1}{2} - \frac{1}{2} = -1$$

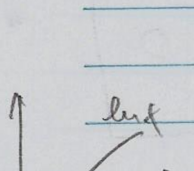
$\therefore$

$$\boxed{\int_{-\infty}^{+\infty} x(1+x^2)^{-2}\,dx = -1}$$

b) $\int_{-\infty}^{+\infty} x(1+x^2)^{-1}\,dx$

$$= \lim_{a\to-\infty}\int_a^0 x(1+x^2)^{-1}\,dx + \lim_{b\to+\infty}\int_0^b x(1+x^2)^{-1}\,dx$$

$$= \lim_{a\to-\infty} \left.\frac{1}{2}\ln|1+x^2|\right|_a^0 + \lim_{b\to+\infty} \left.\frac{1}{2}\ln|1+x^2|\right|_0^b$$



$$= \lim_{a\to-\infty}\left(\frac{1}{2}\cancel{\ln 1}^{\,0} - \frac{1}{2}\ln(1+a^2)\right) + \lim_{b\to+\infty}\left(\frac{1}{2}\ln 1 - \frac{1}{2}\ln(1+b^2)\right)$$

$$= \underbrace{\lim_{a\to-\infty}\left(-\frac{1}{2}\ln(1+a^2)\right)} + \underbrace{\lim_{b\to+\infty}\left(-\frac{1}{2}\ln(1+b^2)\right)}$$

$$= -\infty + -\infty = -\infty \Longrightarrow$$

$$\boxed{\int_{-\infty}^{+\infty} x(1+x^2)^{-1}\,dx = -\infty \quad \text{diverge}}$$

22. $\displaystyle\int_1^{+\infty} \frac{dx}{x^n}$

i) $\underline{\underline{n < 0}}$

$$\int_1^{+\infty} \frac{dx}{x^n} = \lim_{b \to +\infty} \int_1^b \frac{dx}{x^n}$$

$$= \lim_{b \to +\infty} \left.\frac{x^{-n+1}}{-n+1}\right|_1^b$$

$$= \lim_{b \to +\infty} \left(\frac{b^{-n+1}}{-n+1} - \frac{1}{-n+1} \right)$$

$n<0 \Rightarrow -n>0 \Rightarrow \lim_{b \to +\infty} b^{-n+1} = +\infty$

r.p. $\boxed{\displaystyle\int_1^{+\infty} \frac{dx}{x^n} = +\infty \text{ se } n<0}$

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ii) $\underline{n=0}$

$$\boxed{\int_1^{+\infty} \frac{dx}{x^0} = \int_1^{+\infty} dx = \infty}$$ diverge

iii) $n > 0$

$$\int_1^{+\infty} \frac{dx}{x^n} = \lim_{b \to +\infty} \left(\frac{b^{-n+1}}{-n+1} - \frac{1}{-n+1} \right)$$

Mas

$$\lim_{b \to +\infty} b^{-n+1} = 0 \qquad (n > 0)$$

logo

$$\int_1^{+\infty} \frac{dx}{x^n} = \lim_{b \to +\infty} \left(\frac{b^{-n+1}}{-n+1} - \frac{1}{-n+1} \right)$$

$$= -\frac{1}{-n+1}$$

$$\boxed{\int_1^{+\infty} \frac{dx}{x^n} = \frac{1}{n-1}} \qquad (n > 1)$$

converge