

Cálculo B - Lista 8

2. 8

Integrais impróprias II (Integrando é uma função ilimitada)3. -4

4. 4

Nos exercícios 1 a 15, determine se a integral imprópria é convergente ou divergente. Se for convergente, calcule-a.

5. $\frac{\pi}{3}$

6. diverge

7. ∞ (diverge)8. $\pi/2$ 9. ∞ (diverge)10. ∞ (diverge)11. ∞ (diverge)

12. 6

13. diverge

14. $\pi/4$ 15. ∞ (diverge)

1. $\int_0^1 \frac{dx}{\sqrt{1-x}}$

2. $\int_0^{16} \frac{dx}{x^{3/4}}$

3. $\int_{-5}^{-3} \frac{xdx}{\sqrt{x^2-9}}$

4. $\int_0^4 \frac{xdx}{\sqrt{16-x^2}}$

5. $\int_2^4 \frac{dt}{\sqrt{16-t^2}}$

6. $\int_{-4}^1 \frac{dz}{(z+3)^3}$

7. $\int_{\pi/4}^{\pi/2} \sec \theta d\theta$

8. $\int_{-2}^0 \frac{dx}{\sqrt{4-x^2}}$

9. $\int_0^\infty \frac{dx}{x^3}$

10. $\int_0^{\pi/2} \tan \theta d\theta$

11. $\int_0^{\pi/2} \frac{dy}{1-\sin y}$

12. $\int_0^2 \frac{dx}{(x-1)^{2/3}}$

13. $\int_0^4 \frac{dx}{x^2-2x-3}$

14. $\int_2^\infty \frac{dx}{x\sqrt{x^2-4}}$

15. $\int_0^\infty \ln x dx$

Resposta

1. 2

Lista 8 : Int. Improperos I

1 1

Exercício (Pg. 676)

$$1. \int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{t \rightarrow 1^-} \int_0^t \frac{dx}{\sqrt{1-x}}$$

$$= \lim_{t \rightarrow 1^-} -2\sqrt{1-x} \Big|_0^t$$

$$= \lim_{t \rightarrow 1^-} (-2\sqrt{1-t} + 2)$$

$$= 2 //$$

$$2. \int_0^{16} \frac{dx}{x^{3/4}} = \lim_{t \rightarrow 0^+} \int_t^{16} \frac{dx}{x^{3/4}}$$

$$= \lim_{t \rightarrow 0^+} 4x^{1/4} \Big|_t^{16}$$

$$\int \frac{dx}{x^{3/4}} = \frac{x^{-3/4+1}}{-3/4+1}$$

$$= \frac{x^{1/4}}{1/4}$$

$$= \lim_{t \rightarrow 0^+} (4 \cdot 16^{1/4} - 4t^{1/4})$$

$$= \lim_{t \rightarrow 0^+} (4 \cdot 2 - 4t^{1/4})$$

$$= 8 //$$

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$$3. \int_{-5}^{-3} \frac{x dx}{\sqrt{x^2-9}} = \text{ (} f(x) \text{ é descontinua em } x=-3 \text{)}$$

$$= \lim_{x \rightarrow -3^-} \int_{-5}^x \frac{x dx}{\sqrt{x^2-9}}$$

$$= \lim_{x \rightarrow -3^-} \left(\sqrt{x^2-9} \right)_{-5}^x$$

$$= \lim_{x \rightarrow -3^-} \left(\sqrt{x^2-9} - 4 \right)$$

$$= -4 //$$

$$4. \int_0^4 \frac{x dx}{\sqrt{6-x^2}} = \text{ (} f(x) \text{ é descontinua em } x=4 \text{)}$$

$$= \lim_{x \rightarrow 4^-} \int_0^x \frac{x dx}{\sqrt{6-x^2}}$$

$$= \lim_{x \rightarrow 4^-} \left(-\sqrt{6-x^2} \right)_{0}^x$$

$$= \lim_{x \rightarrow 4^-} \left(-\sqrt{6-x^2} + 4 \right)$$

$$= +4 //$$

$$5. \int_2^4 \frac{dt}{\sqrt{16-t^2}}$$

$$f(x) = \frac{1}{\sqrt{16-x^2}} \text{ descontinua em } x=4$$

$$\int_2^4 \frac{dt}{\sqrt{16-t^2}} = \lim_{x \rightarrow 4^-} \int_2^x \frac{dt}{\sqrt{16-t^2}}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} \quad \equiv \quad \lim_{x \rightarrow 4^-} \left[\sin^{-1} \frac{x}{4} \right]_2^x$$

$$= \lim_{x \rightarrow 4^-} \left(\sin^{-1} \frac{x}{4} - \sin^{-1} \frac{1}{2} \right)$$

$\sin^{-1} x$:

$$\text{Dom } \sin^{-1} = [-\pi/2, \pi/2]$$

$$\text{Im } \sin^{-1} = [-\pi/2, \pi/2]$$

$$= \lim_{x \rightarrow 4^-} \left(\sin^{-1} \frac{x}{4} - \frac{\pi}{6} \right)$$

$$= \left(\frac{\pi}{2} - \frac{\pi}{6} \right) = \frac{2\pi}{6} = \frac{\pi}{3}$$

$$6. \int_{-4}^1 \frac{dz}{(z+3)^3}$$

$$f(z) = \frac{1}{z+3} \text{ é descontinua em } z=-3.$$

$$= \lim_{t \rightarrow -3^-} \int_{-4}^t \frac{dz}{(z+3)^3} + \lim_{t \rightarrow -3^+} \int_t^1 \frac{dz}{(z+3)^3}$$

$$= \lim_{t \rightarrow -3^-} \left[\frac{-1}{2(z+3)^2} \right]_{-4}^t + \lim_{t \rightarrow -3^+} \left[\frac{-1/2}{(z+3)^2} \right]_t^1$$

$$= \lim_{t \rightarrow -3^-} \left(\frac{-1}{2(t+3)^2} + \frac{1}{2(-4+3)^2} \right) + \lim_{t \rightarrow -3^+} \left(\frac{-1}{2(4)^2} + \frac{1}{2(t+3)^2} \right)$$

$$= \lim_{x \rightarrow -3^-} \underbrace{\left(\frac{-1}{2(x+3)^2} + \frac{1}{2} \right)}_{-\infty} + \lim_{x \rightarrow -3^+} \left(\frac{-1}{32} + \frac{1}{2(x+3)^2} \right)$$

$$= -\infty + \infty$$

= Divergente

$$\therefore \int_{-4}^1 \frac{dx}{(x+3)^3} = \text{divergente}$$

$$7. \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sec \theta \, d\theta = \quad \sec \theta = \frac{1}{\cos \theta} \rightarrow \text{discontinuo} \\ \sin \theta = \frac{\pi}{2}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^-} \int_{\frac{\pi}{4}}^x \sec \theta \, d\theta \quad \sec \frac{\pi}{4} = \frac{1}{\cos \frac{\pi}{4}} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^-} \ln |\sec \theta + \tan \theta| \Big|_{\frac{\pi}{4}}^x$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^-} \left\{ \ln |\sec x + \tan x| - \ln |\sec \frac{\pi}{4} + \tan \frac{\pi}{4}| \right\}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^-} \left\{ \ln |\sec x + \tan x| - \ln \left| \frac{2}{\sqrt{2}} + 1 \right| \right\}$$

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$$= \ln |\sec x + \tan x| - \ln \left| \frac{1}{\cos x} + \frac{\sin x}{\cos x} \right| =$$

$$= \ln \left| \frac{1 + \sin x}{\cos x} \right|$$

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \ln \left| \frac{1 + \sin x}{\cos x} \right| = \ln \left| \frac{1+1}{0} \right| = +\infty$$

Uma vez que a limite anterior não existe temos que

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sec x dx = \text{diverge}$$

$$8. \int_{-2}^0 \frac{dx}{\sqrt{4-x^2}} = \text{fca} = \frac{1}{\sqrt{4-x^2}} \text{ diverge em } x = -2$$

$$= \lim_{x \rightarrow -2^+} \int_x^0 \frac{dx}{\sqrt{4-x^2}} \quad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a}$$

$$= \lim_{x \rightarrow -2^+} \left. \sin^{-1} \frac{x}{2} \right|_x^0$$

$$= \lim_{x \rightarrow -2^+} \left(\sin^{-1} \frac{0}{2} - \sin^{-1} \frac{x}{2} \right) \quad \sin^{-1} 0 = 0$$

$$= \lim_{x \rightarrow -2^+} \left(0 - \sin^{-1} \frac{x}{2} \right) \quad \sin^{-1} x \Rightarrow \text{Dom} = (-1, 1) \\ \text{Imag} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$= \lim_{x \rightarrow -2^+} -\sin^{-1} \frac{x}{2} = -\underbrace{\sin^{-1}(-1)}_{-\frac{\pi}{2}} = +\frac{\pi}{2} //$$

$$9. \int_0^{+\infty} \frac{dx}{x^3} =$$

$$= \lim_{t \rightarrow 0^+} \int_t^1 \frac{dx}{x^3} + \lim_{b \rightarrow +\infty} \int_1^b \frac{dx}{x^3}$$

$$= \lim_{t \rightarrow 0^+} \left. \frac{-1}{2x^2} \right|_t^1 + \lim_{b \rightarrow +\infty} \left. \frac{-1}{2x^2} \right|_1^b$$

$$= \lim_{t \rightarrow 0^+} \left(\frac{-1}{2} + \frac{1}{2t^2} \right) + \lim_{b \rightarrow +\infty} \left(\frac{-1}{2b^2} + \frac{1}{2} \right)$$

$$= +\infty + \frac{1}{2}$$

$$= +\infty$$

$$\int_0^{+\infty} \frac{dx}{x^3} = +\infty \quad \text{diverge}$$

$$10. \int_0^{\frac{\pi}{2}} \tan \theta \, d\theta$$

$\tan \theta$ diverge in $\theta = \frac{\pi}{2}$

$$= \lim_{x \rightarrow \frac{\pi}{2}^-} \int_0^x \tan \theta \, d\theta$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^-} \ln |\sec \theta| \Big|_0^x \quad \left. \begin{array}{l} \sec 0 = \frac{1}{\cos 0} = 1 \\ \sec \frac{\pi}{2} = \frac{1}{\cos \frac{\pi}{2}} = \frac{1}{0} = +\infty \end{array} \right\}$$

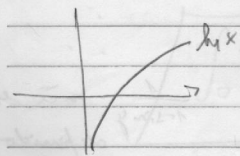
$$= \lim_{x \rightarrow \frac{\pi}{2}^-} (\ln |\sec x| - \ln \sec 0)$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^-} (\ln |\sec x| - \ln 1)$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^-} \ln |\sec x|$$

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \sec x = \frac{1}{\cos \frac{\pi}{2}} = \frac{1}{0} = +\infty$$

$= +\infty$



$$\therefore \int_0^{\frac{\pi}{2}} \tan \theta \, d\theta \quad \therefore \text{diverge}$$

11. $\int_0^{\frac{\pi}{2}} \frac{dy}{1-\sin y}$

$$\int \frac{dy}{1-\sin y} = \int \frac{(1+\sin y) dy}{(1-\sin y)(1+\sin y)}$$

$$= \int \frac{(1+\sin y) dy}{1-\sin^2 y}$$

$$= \int \frac{1+\sin y}{\cos^2 y} dy$$

$$= \int \frac{1}{\cos^2 y} dy + \int \sec y \tan y dy$$

$$= \int \sec^2 y dy + \int \sec y \tan y dy$$

$$= \tan y + \sec y$$

$$\int_0^{\frac{\pi}{2}} \frac{dy}{1-\sin y} = \lim_{t \rightarrow \frac{\pi}{2}^-} \int_0^t \frac{dy}{1-\sin y}$$

$\frac{1}{1-\sin y}$ não está definido em $y = \frac{\pi}{2}$

$$= \lim_{t \rightarrow \frac{\pi}{2}^-} [\tan y + \sec y]_0^t$$

$t_0 = 0$

$$u_0 = \frac{1}{1-\sin 0} = 1 = \lim_{t \rightarrow \frac{\pi}{2}^-} (\tan t + \sec t - \tan 0 - u_0)$$

$$t_1 = \frac{\pi}{2}^- \rightarrow \tan \rightarrow +\infty$$

$$u_1 = \frac{1}{1-\sin \frac{\pi}{2}^-} = +\infty$$

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$$\int_0^{\frac{\pi}{2}} \frac{dy}{1-\sin y} = \infty$$

diverge

12. $\int_0^2 \frac{dx}{(x-1)^{2/3}}$ $f(x) = \frac{1}{(x-1)^{2/3}}$ dehingga an x=1

$= \lim_{t \rightarrow 1^-} \int_0^t \frac{dx}{(x-1)^{2/3}} + \lim_{t \rightarrow 1^+} \int_t^2 \frac{dx}{(x-1)^{2/3}}$

$\frac{1}{3} + 1 = \frac{4}{3}$
 $= \lim_{t \rightarrow 1^-} \left[3(x-1)^{1/3} \right]_0^t + \lim_{t \rightarrow 1^+} \left[3(x-1)^{1/3} \right]_t^2$

$= \lim_{t \rightarrow 1^-} \left(3(t-1)^{1/3} - 3(-1)^{1/3} \right) + \lim_{t \rightarrow 1^+} \left(3 \cdot 1^{1/3} - 3(t-1)^{1/3} \right)$

$= \lim_{t \rightarrow 1^-} \left(3(t-1)^{1/3} + 3 \right) + \lim_{t \rightarrow 1^+} \left(3 - 3(t-1)^{1/3} \right)$

$= 3 + 3 = 6$

$\therefore \int_0^2 \frac{dx}{(x-1)^{2/3}} = 6$

13. $\int_0^4 \frac{dx}{x^2-2x-3}$ $\begin{cases} x^2-2x-3=0 \\ x = \frac{2 \pm \sqrt{4+12}}{2} \\ = \frac{2 \pm 4}{2} \rightarrow \textcircled{3} \rightarrow \text{mas} \\ \rightarrow -1 \text{ \small{inter}} \\ \text{definisi} \end{cases}$

$= \lim_{t \rightarrow 3^-} \int_0^t \frac{dx}{x^2-2x-3} + \lim_{t \rightarrow 3^+} \int_t^4 \frac{dx}{x^2-2x-3}$

May

$$\int \frac{dx}{x^2-2x-3} = \int \frac{dx}{(x-1)^2-4} \stackrel{y=x-1}{=} \int \frac{dy}{y^2-4}$$

$$\int \frac{du}{u^2-a^2} = \frac{1}{2a} \ln \left| \frac{u-a}{u+a} \right| \quad \left. \begin{array}{l} = \frac{1}{2 \cdot 2} \ln \left| \frac{y-2}{y+2} \right| \\ = \frac{1}{4} \ln \left| \frac{x-3}{x+1} \right| \end{array} \right\} y=x-1$$

$$\left. \begin{aligned} \lim_{x \rightarrow 5^-} \int_0^x \frac{dx}{x^2-2x-3} &= \lim_{x \rightarrow 3^-} \left[\frac{1}{4} \ln \left| \frac{x-3}{x+1} \right| \right]_0^x \\ &= \frac{1}{4} \lim_{x \rightarrow 3^-} \left(\ln \left| \frac{x-3}{x+1} \right| - \ln \left| \frac{-3}{1} \right| \right) \\ &= \frac{1}{4} \lim_{x \rightarrow 3^-} \left(\underbrace{\ln \left| \frac{x-3}{x+1} \right|}_{\rightarrow -\infty} - \ln 3 \right) \\ &= -\infty \end{aligned} \right\}$$

$$\left. \begin{aligned} \lim_{x \rightarrow 3^+} \int_x^4 \frac{dx}{x^2-2x-3} &= \lim_{x \rightarrow 3^+} \left[\frac{1}{4} \ln \left| \frac{x-3}{x+1} \right| \right]_x^4 \\ &= \frac{1}{4} \lim_{x \rightarrow 3^+} \left(\ln \frac{1}{5} - \underbrace{\ln \left| \frac{x-3}{x+1} \right|}_{\rightarrow -\infty} \right) \\ &= +\infty \end{aligned} \right\}$$

$$\int_0^4 \frac{dx}{x^2-2x-3} = \text{divergente}$$

14. $\int_2^{+\infty} \frac{dx}{x\sqrt{x^2-4}}$ $f(x)$ não está definida em $x=2$.

$= \lim_{t \rightarrow 2^+} \int_t^3 \frac{dx}{x\sqrt{x^2-4}} + \lim_{b \rightarrow +\infty} \int_3^b \frac{dx}{x\sqrt{x^2-4}}$

Mas

$\int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{2 \sec \theta \tan \theta d\theta}{2 \sec \theta \sqrt{4 \sec^2 \theta - 4}}$ $x = 2 \sec \theta$
 $\left. \begin{aligned} dx &= 2 \sec \theta \tan \theta d\theta \\ | \tan \theta &= \sqrt{\sec^2 \theta - 1} \end{aligned} \right\}$

$x = 2 \sec \theta$

$= \int \frac{\cancel{2} \sec \theta \tan \theta d\theta}{\sqrt{4} \cancel{2} \sec \theta}$

$\frac{x}{2} = \sec \theta$
 \Leftrightarrow

$= \int \frac{\cancel{2} \sec \theta \tan \theta d\theta}{2 \cancel{2} \sec \theta}$

$\frac{x}{2} = \sec \theta$

$= \frac{1}{2} \int \tan \theta d\theta = \frac{1}{2} \theta$

$\theta = \cos^{-1} \frac{2}{x}$

$= \frac{1}{2} \cos^{-1} \frac{2}{x} //$

D

$\lim_{x \rightarrow 2^+} \int_x^3 \frac{dx}{x\sqrt{x^2-4}} = \lim_{x \rightarrow 2^+} \left. \frac{1}{2} \cos^{-1} \frac{2}{x} \right|_x^3$

$\lim_{x \rightarrow 2^+} \cos^{-1} \frac{2}{x} = \cos^{-1} 1 = 0$

$= \lim_{x \rightarrow 2^+} \left(\frac{1}{2} \cos^{-1} \frac{2}{3} - \frac{1}{2} \cos^{-1} \frac{2}{x} \right)$

$\cos^{-1}: \text{Im} [0, \pi]$

$= \frac{1}{2} \cos^{-1} \frac{2}{3} //$

17

$\lim_{b \rightarrow +\infty} \int_3^b \frac{dx}{x\sqrt{x^2-4}} = \lim_{b \rightarrow +\infty} \left(\frac{1}{2} \cos^{-1} \frac{2}{x} \right) \Big|_3^b =$

$$\lim_{b \rightarrow \infty} \frac{1}{2} \cos^{-1} \frac{2}{x} \Big|_3^b =$$

$$= \lim_{b \rightarrow \infty} \left(\frac{1}{2} \cos^{-1} \frac{2}{b} - \frac{1}{2} \cos^{-1} \frac{2}{3} \right)$$

$$\lim_{b \rightarrow \infty} \cos^{-1} \frac{2}{b} = \cos^{-1} 0 = \frac{\pi}{2}$$

$$= \frac{1}{2} \frac{\pi}{2} - \frac{1}{2} \cos^{-1} \frac{2}{3}$$

$$= \frac{\pi}{4} - \frac{1}{2} \cos^{-1} \frac{2}{3} //$$

Das,

$$\int_2^{\infty} \frac{dx}{x\sqrt{x^2-4}} = \frac{1}{2} \cos^{-1} \frac{2}{3} + \frac{\pi}{4} - \frac{1}{2} \cos^{-1} \frac{2}{3}$$
$$= \frac{\pi}{4}$$

15. $\int_0^{+\infty} \ln x \, dx =$ $\ln x$ não está definida em $x=0$

$$= \lim_{t \rightarrow 0^+} \int_t^1 \ln x \, dx + \lim_{b \rightarrow +\infty} \int_1^b \ln x \, dx$$

Mas $\int \ln x \, dx = x \ln x - \int dx$

$$\left. \begin{array}{l} u = \ln x \rightarrow du = \frac{1}{x} dx \\ dv = dx \rightarrow v = x \end{array} \right\} \begin{array}{l} = x \ln x - x \\ = x(\ln x - 1) \end{array}$$

$$\therefore \lim_{t \rightarrow 0^+} \int_t^1 \ln x \, dx = \lim_{t \rightarrow 0^+} \left[x(\ln x - 1) \right]_t^1$$

$$= \lim_{t \rightarrow 0^+} \left[1 \cdot (\overset{0}{\ln 1} - 1) - t(\ln t - 1) \right]$$

$$\rightarrow = \lim_{t \rightarrow 0^+} (-1 - t \ln t + t)$$

Mas $\lim_{t \rightarrow 0^+} -t \ln t = - \lim_{t \rightarrow 0^+} \frac{\ln t}{\frac{1}{t}} \stackrel{\text{L'Hopital}}{=} \lim_{t \rightarrow 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}}$

$$= - \lim_{t \rightarrow 0^+} -t$$

$$= 0 //$$

$$\rightarrow = -1$$

$$// \lim_{t \rightarrow 0^+} \int_t^1 \ln x \, dx = -1 //$$

$$\lim_{b \rightarrow +\infty} \int_1^b \ln x \, dx =$$

$$= \lim_{b \rightarrow +\infty} x(\ln x - 1) \Big|_1^b$$

$$= \lim_{b \rightarrow +\infty} [b(\ln b - 1) - (\ln 1 - 1)]$$

$$= \lim_{b \rightarrow +\infty} [b(\ln b - 1) - (-1)]$$

Now,

$$\lim_{b \rightarrow +\infty} b(\ln b - 1) = \lim_{b \rightarrow +\infty} \infty \cdot \infty = \infty$$

$$\therefore \int_1^{\infty} \ln x \, dx = \infty$$

$$\int_0^{+\infty} \ln x \, dx = +\infty \text{ divergente}$$