Cálculo 1 - Prova 1

1. Calcule os seguintes limites
   (a) $\lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$ **[1.5 ponto]**
   (b) $\lim_{x\to 1} \frac{x^{n+1}-(n+1)x+n}{(x-1)^2}$ **[1.5 ponto]**
   (c) $\lim_{x\to\frac{\pi}{3}} \frac{1-2\cos x}{\pi-3x}$ **[1.5 ponto]**

2. Usando a definição de derivada via limites calcule $\frac{d}{dx}\sin x$ **[1.5 ponto]**

3. Calcule as derivadas
   (a) $y = x^{3/2}e^{-2x} + \arctan\sqrt{1-x^2}$ **[1.0 ponto]**
   (b) $y = (\sin x)^x$ **[1.0 ponto]**

4. Assumindo que a relação

$$x^2 - xy^2 + y^3 = 13; \qquad (-1,2)$$

   define uma função implícita e diferenciável $y(x)$ determine a reta tangente ao gráfico de $y(x)$ no ponto $(-1,2)$.
   [1.0 ponto]

5. Um cilindro circular reto está sendo aquecido. Como consequência, seu raio aumenta a razão de 4 cm/s e sua altura aumenta a razão de 10 cm/s. Encontre a taxa de variação do volume do cilindro quando o raio é 50 cm e a altura é 30 cm. **[1.0 ponto]**

1a) $\lim_{x \to \infty} \frac{\sqrt{x}}{\sqrt{x + \sqrt{x + \sqrt{x}}}}$

$$= \lim_{x \to \infty} \frac{\sqrt{x}}{\sqrt{x + \sqrt{x\left(1 + \frac{\sqrt{x}}{x}\right)}}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{x}}{\sqrt{x + \sqrt{x\left(1 + \frac{1}{\sqrt{x}}\right)}}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{x}}{\sqrt{x + \sqrt{x}\sqrt{1 + \frac{1}{\sqrt{x}}}}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{x}}{\sqrt{x\left(1 + \frac{1}{\sqrt{x}}\sqrt{1 + \frac{1}{\sqrt{x}}}\right)}}$$

$$= \lim_{x \to \infty} \frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}\sqrt{1 + \frac{1}{\sqrt{x}}\sqrt{1 + \frac{1}{\sqrt{x}}}}}$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{1 + \frac{1}{\sqrt{x}} \to 0 \sqrt{1 + \frac{1}{\sqrt{x}} \to 0}}} = 1$$

1b)

$$\lim_{x\to 1} \frac{x^{n+1}-(n+1)x+n}{(x-1)^2}$$

$$\begin{array}{l|l}
\cancel{x^{n+1}}-(n+1)x+n & x-1 \\ \cline{2-2}
-\cancel{x^{n+1}}+x^{n} & x^{n}+x^{n-1}+x^{n-2}+\dots+ \\ \cline{1-1}
\cancel{x^{n}}-(n+1)x+n & +x^{2}+x-n \\
-\cancel{x^{n}}+x^{n-1} & \\ \cline{1-1}
x^{n-1}-(n+1)x+n & \\
\vdots & \\
\cancel{x^{2}}-(n+1)x+n & \\
-\cancel{x^{2}}+x & \\ \cline{1-1}
-\cancel{nx}+\cancel{n} & \\
+\cancel{nx}-\cancel{n} & \\ \cline{1-1}
0 &
\end{array}$$

$\therefore$

(*) $$\left\{ x^{n+1}-(n+1)x+n = (x-1)(x^{n}+x^{n-1}+\dots+x^{2}+x-n) \right.$$

$$
\begin{array}{l|l}
\cancel{x^n} + x^{n-1} + \cdots + x^2 + x - n & x-1 \\ \cline{2-2}
-\cancel{x^n} + x^{n-1} & x^{n-1} + 2x^{n-2} + \\ \cline{1-1}
2\cancel{x^{n-1}} + x^{n-2} + \cdots + x^2 + x - n & +3x^{n-3} + \cdots \\
-2\cancel{x^{n-1}} + 2x^{n-2} & +(n-2)x^2 + \\ \cline{1-1}
3\cancel{x^{n-2}} + x^{n-3} + \cdots + x^2 + x - n & +(n-1)x + n \\
-3\cancel{x^{n-2}} + 3x^{n-3} & \\ \cline{1-1}
4x^{n-3} + \cdots + x^2 + x - n & \\
\vdots & \\
(\cancel{n-1})x^2 + x - n & \\
-(\cancel{n-1})x^2 + (n-1)x & \\ \cline{1-1}
nx - n & \\
-nx + n & \\ \cline{1-1}
0 &
\end{array}
$$

$\therefore$

$$
(**) \left\{ \begin{array}{l} x^n + x^{n-1} + \cdots + x^2 + x - n = \\ = (x-1)\left(x^{n-1} + 2x^{n-2} + 3x^{n-3} + \cdots + (n-2)x^2 + (n-1)x + n\right) \end{array} \right.
$$

Substituindo $(**)$ em $(*)$ temos

$$
x^{n+1} - (n+1)x + n = (x-1)^2\left(x^{n-1} + 2x^{n-2} + 3x^{n-3} + \cdots + (n-2)x^2 + (n-1)x + n\right)
$$

$\therefore$

$$\lim_{x \to 1} \frac{x^{n+1} - (n+1)x + n}{(x-1)^2} =$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)^2}\left(x^{n-1} + 2x^{n-2} + 3x^{n-3} + \cdots + (n-1)x + n\right)}{\cancel{(x-1)^2}}$$

$$= \lim_{x \to 1} \left\{ x^{n-1} + 2x^{n-2} + 3x^{n-3} + \cdots + (n-2)x^2 + (n-1)x + n \right\}$$

$$= \underbrace{1 + 2 + 3 + \cdots + n-2 + n-1 + n}$$

$$= \frac{n(n+1)}{2}$$

1c) $\lim\limits_{x \to \frac{\pi}{3}} \dfrac{1 - 2\cos x}{\pi - 3x}$

$u = x - \frac{\pi}{3}$; $x \to \frac{\pi}{3}$, $u \to 0$

$\therefore$

$$\lim_{x \to \frac{\pi}{3}} \frac{1 - 2\cos x}{\pi - 3x} =$$

$$= \lim_{u \to 0} \frac{1 - 2\cos\left(u + \frac{\pi}{3}\right)}{\pi - 3\left(u + \frac{\pi}{3}\right)}$$

$$= \lim_{u \to 0} \frac{1 - 2\left[\cos u \cos\frac{\pi}{3} - \sin u \sin\frac{\pi}{3}\right]}{\cancel{\pi} - 3u - \cancel{\pi}}$$

$$= \lim_{u \to 0} \frac{1 - 2\left[\frac{1}{2}\cos u - \frac{\sqrt{3}}{2}\sin u\right]}{-3u}$$

$$= \lim_{u \to 0} \frac{1 - \cos u + \sqrt{3}\sin u}{-3u}$$

$$= -\frac{1}{3} \lim_{u \to 0} \underbrace{\left(\frac{1 - \cos u}{u}\right)}_{0} - \frac{\sqrt{3}}{3} \underbrace{\lim_{u \to 0} \frac{\sin u}{u}}_{1}$$

$$= -\frac{\sqrt{3}}{3}$$

2) $$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h+\sin h\cos x-\sin x}{h}$$

$$= \lim_{h\to 0}\left(\sin x\,\frac{(\cos h-1)}{h}+\cos x\,\frac{\sin h}{h}\right)$$

$$= \sin x\,\underbrace{\lim_{h\to 0}\frac{\cos h-1}{h}}_{0}+\cos x\,\underbrace{\lim_{h\to 0}\frac{\sin h}{h}}_{1}$$

$$= \cos x$$

$\therefore$ $$\frac{d}{dx}\sin x = \cos x$$

3a) $y = x^{3/2} e^{-2x} + \operatorname{arctg} \sqrt{1-x^2}$

$$y' = (x^{3/2})' e^{-2x} + x^{3/2} (e^{-2x})' + (\operatorname{arctg} \sqrt{1-x^2})'$$

$$= \frac{3}{2} x^{1/2} e^{-2x} - 2 e^{-2x} x^{3/2} +$$

$$+ \frac{1}{(1 + 1-x^2)} \cdot \frac{1}{2\sqrt{1-x^2}} (-2x)$$

$$\left\| y' = \frac{3}{2} e^{-2x} \sqrt{x} - 2 e^{-2x} x^{3/2} - \frac{x}{(2-x^2)\sqrt{1-x^2}} \right\|$$

3b) $y = (\sin x)^x = e^{\ln (\sin x)^x}$

$$= e^{x \ln \sin x}$$

$$y' = \left(e^{x \ln \sin x}\right)'$$

$$= \underline{e^{x \ln \sin x}} (x \ln \sin x)'$$

$$= (\sin x)^x \left(\ln \sin x + x \frac{1}{\sin x} \cos x \right)$$

$$\left\| y' = (\sin x)^x \left(\ln \sin x + x \operatorname{cotg} x \right) \right\|$$

4. $x^2 - xy^2 + y^3 = 13$; $(-1, 2)$

$$\frac{d}{dx}(x^2 - xy^2 + y^3) = 0$$

$$2x - y^2 - 2xyy' + 3y^2y' = 0$$

$$2x - y^2 + y'(-2xy + 3y^2) = 0$$

$$y'(-2xy + 3y^2) = y^2 - 2x$$

$$y' = \frac{y^2 - 2x}{3y^2 - 2xy}$$

<u>reta tangente em $(-1, 2)$</u>

$r: \quad y_r = mx + q$

$$m = \frac{4 - 2(-1)}{3.4 - 2(-1).2} = \frac{4+2}{12+4} = \frac{6}{16} = \frac{3}{8}$$

$$y_r = \frac{3}{8}x + q$$

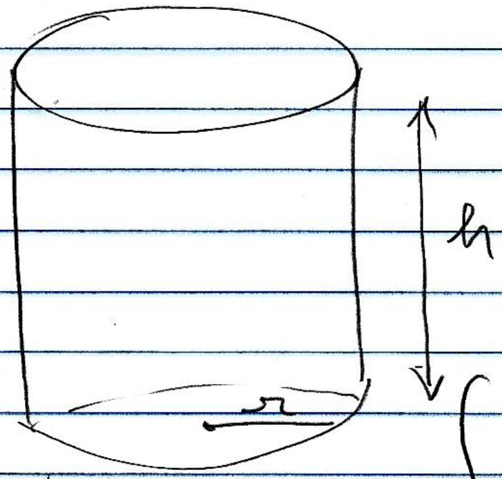
$(-1, 2) \in r$: $\quad 2 = \frac{3}{8}(-1) + q$

$$2 + \frac{3}{8} = q$$

$$q = \frac{19}{8}$$

$$// \; y_r = \frac{3}{8}x + \frac{19}{8} \; //$$

5.



$V = S_{base} \cdot h$

$V = \pi r^2 h$

$\frac{dr}{dt} = 4 \text{ cm/s}$

$\frac{dh}{dt} = 10 \text{ cm/s}$

$$\frac{dV}{dt} = \frac{d}{dt}(\pi r^2 h)$$

$$= \pi (r^2)' h + \pi r^2 h'$$

$$= 2\pi r r' h + \pi r^2 h'$$

$$= 2\pi r\, 4\, h + \pi r^2\, 10$$

$$= 8\pi r h + 10\pi r^2$$

$$\left.\frac{dV}{dt}\right|_{(r,h)=(50,30)} = 8\pi\, 50 \cdot 30 + 10\pi \cdot 2500$$

$$= \pi(12000 + 25000)$$

$$\boxed{\left.\frac{dV}{dt}\right|_{(r,h)=(50,30)} = 37000\,\pi \text{ cm}^3/\text{s}}$$