Corrigir

Resposta 11 : $\frac{3\pi}{4}$

Cálculo C - Lista 8

Teorema de Stokes e Gauss

Use o teorema de Stokes para calcular $\int_\gamma \vec{F} \cdot d\vec{r}$ onde γ é a borda da superfície Ω. Assuma a orientação de γ como induzida pela orientação de Ω.

1. $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ e Ω é parte do parabolóide $z = 1 - x^2 - y^2$ no primeiro octante e com vetor normal com componente z negativa.

2. $\vec{F} = 2y\vec{i} + 3z\vec{j} - 2x\vec{k}$ e Ω é parte da esfera $x^2 + y^2 + z^2 = 1$ no primeiro octante e com vetor normal com componente z positiva.

3. $\vec{F} = y\vec{i} - x\vec{j} + z\vec{k}$ e Ω é composto de parte do cilindro $x^2 + y^2 = 1$ situado entre os planos $z = 0$ e $z = 1$ e da parte do plano $z = 1$ situado dentro do cilindro $x^2 + y^2 = 1$. O vetor normal é tal que na superfície do cilindro ele se afasta do eixo z, e no plano $z = 1$ ele aponta na direção positiva de z.

Use o teorema de Stokes para calcular $\int_\gamma \vec{F} \cdot d\vec{r}$ onde γ é orientada no sentido anti-horário.

4. $\vec{F} = xz\vec{i} + y^2\vec{j} + x^2\vec{k}$ onde γ é a interseção do plano $x + y + z = 5$ e do cilindro elíptico $x^2 + \frac{y^2}{4} = 1$.

5. $\vec{F} = y(x^2 + y^2)\vec{i} - x(x^2 + y^2)\vec{j}$ onde γ é o retângulo com vértices $(0,0,0), (1,0,0), (1,1,1), (0,1,1)$.

Use teorema de Stokes para calcular $\int_\Omega (\nabla \times \vec{F}) \cdot d\vec{S}$

6. $\vec{F} = x\vec{i} + (x^2 + y^2 + z^2)\vec{j} + z(y^4 - 1)\vec{k}$ e Ω é parte do cilindro $x^2 + z^2 = 1$ acima do plano xy e entre os planos $y = -1$ e $y = 1$ com vetor normal com componente z positiva.

7. $\vec{F} = xz^2\vec{i} + x^3\vec{j} + \cos xz\vec{k}$ onde Ω é parte do elipsóide $x^2 + y^2 + 3z^2 = 1$ abaixo do plano xy. Ω tem vetor normal apontando para fora.

8. Suponha que Ω é uma superfície orientada com borda γ. Seja $\vec{F}$ um campo vetorial constante definido em Ω. Mostre que $\int_\gamma \vec{F} \cdot d\vec{r} = 0$.

9. Seja Ω um elipsóide com vetor normal direcionado para fora. Seja $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ um campo vetorial cujas componentes F_1, F_2, F_3 tem derivadas parciais contínuas em Ω. Mostre que $\int_\Omega (\nabla \times \vec{F}) \cdot d\vec{S} = 0$.

Use o teorema de Gauss para calcular $\int_\Omega \vec{F} \cdot d\vec{S}$ onde Ω é orientada com vetor normal apontando para fora.

10. $\vec{F} = x^2\vec{i} + xy\vec{j} - 2xz\vec{k}$ e Ω é a superfície do tetraedro com vértices $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$.

11. $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ e Ω é a superfície que é borda da região sólida no primeiro octante, dentro do cilndro $x^2 + y^2 = 1$ e entre os planos $z = 0$ e $z = 1$.

12. $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ e Ω é composto do hemisfério $z = \sqrt{1 - x^2 - y^2}$ e o disco $x^2 + y^2 \leq 1$ no plano xy.

13. $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ e Ω é a borda da região sólida situada no interior do cilindro $x^2 + y^2 = 4$ e entre os planos $z = 0$ e $z = 2$.

14. $\vec{F} = y(x^2 + y^2)^{3/2}\vec{i} - x(x^2 + y^2)^{3/2}\vec{j} + (z + 1)\vec{k}$ e Ω é a borda da região sólida limitada acima pelo plano $z = 2x$ e abaixo pelo parabolóide $z = x^2 + y^2$.

15. Use o teorema de Gauss para calcular $\int_B \nabla \cdot F dV$ onde $\vec{F} = (x^2 + y^2 + z^2)(x\vec{i} + y\vec{j})$ e B é a região sólida definida por $x^2 + y^2 + z^2 \leq 1$

16. Seja $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ e seja B uma região sólida simples tendo por borda uma superfície Ω com vetor normal apontando para fora. Mostre que o volume V da região sólida é dado por

$$V = \frac{1}{3} \int_\Omega \vec{F} \cdot d\vec{S}$$

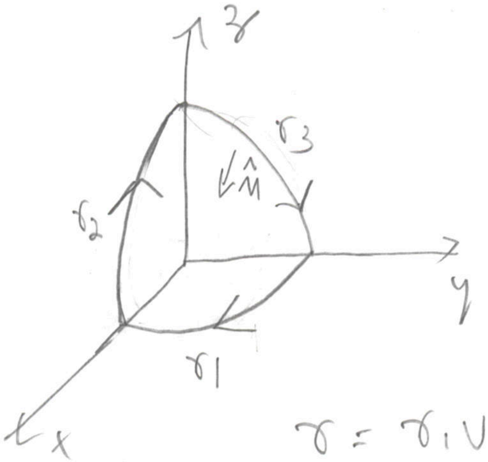
Respostas

1. $-4/3 - \pi/4$
2. $-3\pi/4$
3. -2π
4. 0
5. $-8/3$
6. 0
7. $-3\pi/4$

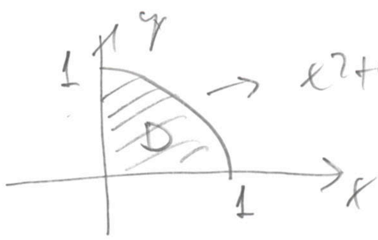
10. $1/24$
11. $3\pi/4$
12. 2π
13. 16π
14. $\pi/2$
15. $\frac{8\pi}{3}$

Cálculo C — Lista 8

1. $\oint_\gamma \vec{F}\cdot d\vec{r} = \int_\Omega (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$



$\Omega: \quad z = 1 - x^2 - y^2$



$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$

$$d\vec{S} = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right) dA$$

$$= (-2x, -2y, -1)\, dA$$

$$\vec{\nabla}\times\vec{F} = \epsilon_{ijk}\, \hat{e}_i\, \partial_j F_k$$

$$= \hat{e}_1(\partial_y F_3 - \partial_z F_2) + \hat{e}_2(\partial_z F_1 - \partial_x F_3)$$

$$+ \hat{e}_3(\partial_x F_2 - \partial_y F_x)$$

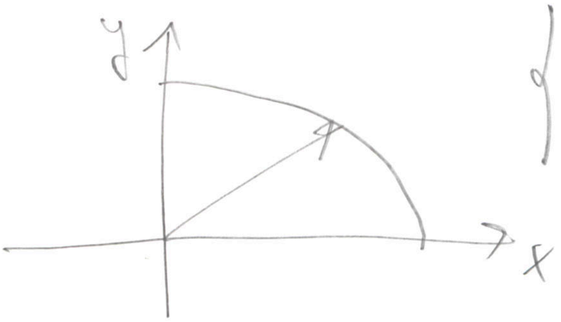
$$= \hat{\imath}(\partial_z y - \partial_z x) + \hat{\jmath}(\partial_z z - \partial_x y)$$

$$+ \hat{k}(\partial_x x - \partial_y z)$$

$$= \hat{\imath} + \hat{\jmath} + \hat{k}$$

$$\int_D (\hat{i}+\hat{j}+\hat{k}) \cdot (-2x\,\hat{i} - 2y\,\hat{j} - \hat{k})\, dA$$

$$= \int_D (-2x - 2y - 1)\, dx\, dy$$



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} ; \quad \begin{matrix} 0 \le r \le 1 \\ 0 \le \theta \le \frac{\pi}{2} \end{matrix}$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{\frac{\pi}{2}} (-2r\cos\theta - 2r\sin\theta - 1)\, r\,dr\,d\theta$$

$$= \int_{r=0}^{1} (-2r\sin\theta + 2r\cos\theta - \theta)\Big]_0^{\frac{\pi}{2}} r\,dr$$

$$= \int_{r=0}^{1} \left[-2r - \frac{\pi}{2} - (2r)\right] r\,dr$$

$$= \int_{r=0}^{1} \left(-4r - \frac{\pi}{2}\right) r\,dr$$

$$= \int_{r=0}^{1} -4r^2 - \frac{\pi}{2} r\, dr = -\frac{4r^3}{3}\Big]_0^1 - \frac{\pi}{2}\frac{r^2}{2}\Big]_0^1$$

$$= -\frac{4}{3} - \frac{\pi}{4}$$

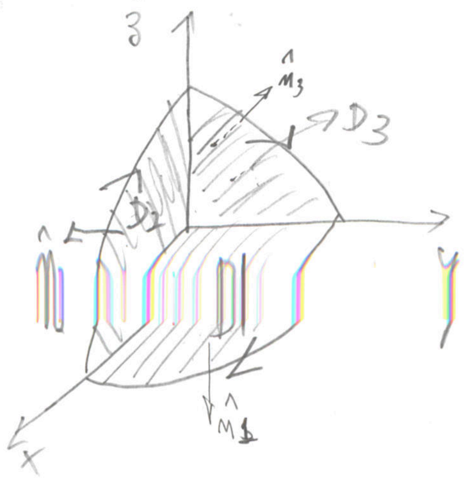
1. Cont.

Forma alternativa

Vimos $\gamma = \partial \Omega$ (parabolóide)

Podemos considerar γ a borda de

$$\gamma = D_1 \cup D_2 \cup D_3$$



$$\oint_\gamma \vec{F} \cdot d\vec{r} = \int_{D_1} \vec{\nabla} \times \vec{F} \cdot d\vec{S}_1 + \int_{D_2} \vec{\nabla} \times \vec{F} \cdot d\vec{S}_2 + \int_{D_3} \vec{\nabla} \times \vec{F} \cdot d\vec{S}_3$$

$$\int_{D_1} \vec{\nabla} \times \vec{F} \cdot d\vec{S}_1 = \int_{D_1} (\hat{i} + \hat{j} + \hat{k}) \cdot (0,0,-1)\, dS_1$$

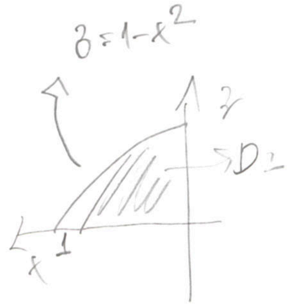
$$= \int_{D_1} - dS_1 = - \underbrace{\int_{D_1} dA_1}_{\frac{1}{4}\text{ Área do disco de raio } 1} = -\frac{1}{4}\pi \quad (*)$$

$D_1 = \{(x,y,z) \in \mathbb{R}^3 \mid z = 0;\ x \geq 0,\ y \geq 0 \text{ e } x^2 + y^2 \leq 1\}$

$$dS_1 = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy = dx\, dy$$

$$\int_{D_2} \vec{\nabla} \times \vec{F} \cdot d\vec{S_2} = \int_{D_2} \left(\hat{i} + \hat{j} + \hat{k}\right) \cdot (0,-1,0)\, dS_2$$

$$= -\int_{D_2} dS_2 = -\int_{D_2} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, dx\, dz$$



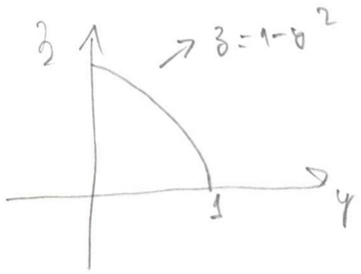
$$= -\int_{D_2} dx\, dz$$

$$= -\int_{x=0}^{1} dx \int_{z=0}^{1-x^2} dz$$

$\Omega_2 = \{ (x,y,z) \in \mathbb{R}^3 : y = 0 ;\ 0 \le x \le 1 ,\ 0 \le z \le 1 - x^2 \}$

$$= -\int_{x=0}^{1} dx\ z \Big]_0^{1-x^2}$$

$$= -\int_{x=0}^{1} dx\, (1 - x^2)$$



$$= -x + \frac{x^3}{3} \Big]_0^1$$

$$= -1 + \frac{1}{3} = -\frac{2}{3} \quad // \quad (**)$$

$$\int_{D_3} \vec{\nabla} \times \vec{F} \cdot d\vec{S_3} = \int_{D_3} \left(\hat{i} + \hat{j} + \hat{k}\right) \cdot (-1,0,0)\, dS_3$$

$$= -\int_{D_3} dS_3 = -\int_{D_3} \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\, dy\, dz$$

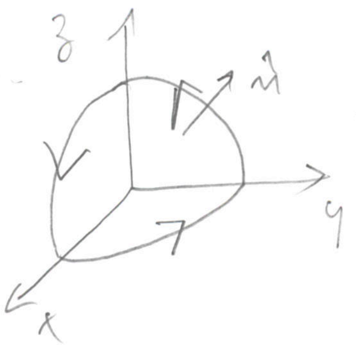
$\Omega_3 = \{ (x,y,z) \in \mathbb{R}^3 : x = 0 ,\ 0 \le y \le 1 ,\ 0 \le z \le 1 - y^2 \}$

cálculo análogo

$$= -\int_{D_3} dy\, dz = -\frac{2}{3} \quad // \quad (***)$$

$$\therefore \quad \oint_{\gamma} F \cdot d\vec{r} = * + ** + *** = -\frac{\pi}{4} - \frac{4}{3} \quad // \quad \text{ok!}$$

2. $\Omega : x^2+y^2+z^2=1 \Rightarrow z = \sqrt{1-x^2-y^2}$



$$d\vec{S} = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) dA$$

$$= \left(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1\right) dA$$

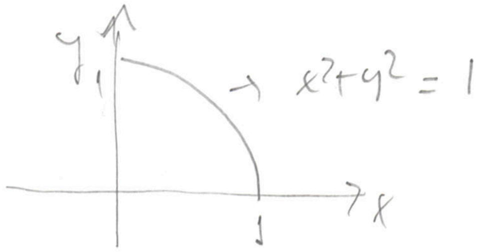
$$\vec{F} = (2y, 3z, -2x)$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2y & 3z & -2x \end{vmatrix} = -2\hat{k} - 3\hat{i} + 2\hat{j}$$

$$\oint_\gamma \vec{F}\cdot d\vec{r} = \int_\Omega (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$$

$$= \int_D (-3, 2, -2)\cdot\left(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1\right) dA$$

$$= \int_D \left(\frac{-3x}{\sqrt{1-x^2-y^2}} + \frac{2y}{\sqrt{1-x^2-y^2}} - 2\right) dx\,dy$$



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} ; \quad \begin{matrix} 0 \le r \le 1 \\ 0 \le \theta \le \frac{\pi}{2} \end{matrix}$$

$$= \int_{r=0}^{1} \int_{\theta=0}^{\frac{\pi}{2}} \left(\frac{-3r\cos\theta + 2r\sin\theta}{\sqrt{1-r^2}} - 2 \right) r\,dr\,d\theta$$

$$= \int_{r=0}^{1} \frac{-3r\sin\theta - 2r\cos\theta}{\sqrt{1-r^2}} - 2\theta \Bigg]_0^{\frac{\pi}{2}} r\,dr$$

$$= \int_{r=0}^{1} \left[\frac{-3r}{\sqrt{1-r^2}} - \pi - \left(\frac{-2r}{\sqrt{1-r^2}} \right) \right] r\,dr$$

$$= \int_0^1 \left(\frac{-r}{\sqrt{1-r^2}} - \pi \right) r\,dr$$

$$= \int_0^1 \left(\frac{-r^2\,dr}{\sqrt{1-r^2}} - \pi r \right) dr$$

(*) (**)

$$(*) = \int_0^1 -\frac{r^2}{\sqrt{1-r^2}}\,dr \quad ; \quad r = \sin\theta \, , \quad dr = \cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} -\frac{\sin^2\theta \cancel{\cos\theta}\,d\theta}{\cancel{\cos\theta}}$$

$$= \int_0^{\frac{\pi}{2}} -\frac{1-\cos 2\theta}{2}\,d\theta = -\frac{1}{2}\theta + \frac{\sin 2\theta}{4} \Bigg]_0^{\frac{\pi}{2}}$$

2- Cont.

$$(*) = -\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\Big]_0^{\pi/2}$$

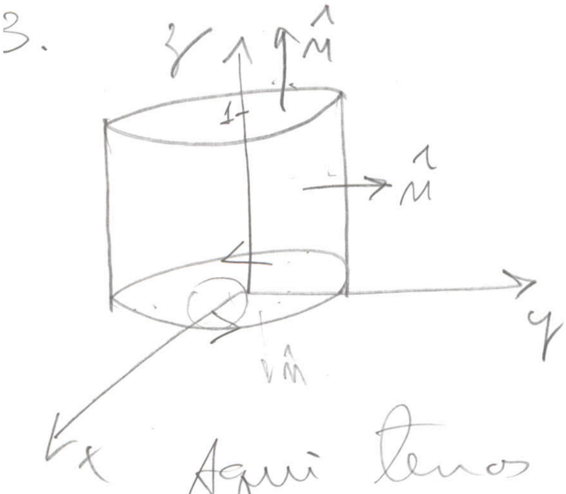
$$= -\frac{\pi}{4} - (0) = -\frac{\pi}{4}$$

$$(**) = -\int_0^1 \pi r \, dr$$

$$= -\frac{\pi r^2}{2}\Big]_0^1 = -\frac{\pi}{2}$$

$$\therefore \quad \boxed{\oint_\gamma F \cdot d\vec{r} = -\frac{\pi}{4} - \frac{\pi}{2} = -\frac{3\pi}{4}}$$

3.

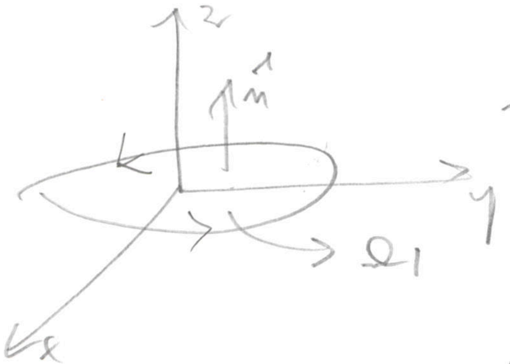


$$\oint_{\gamma} \vec{F}\cdot d\vec{r} = \int_{\Omega} (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$$

Aqui temos alternativamente que

$$\int_{\Omega} (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \int_{\Omega_1} (\vec{\nabla}\times\vec{F})\cdot d\vec{S}_1$$

onde Ω_1 é a superfície abaixo



$$\Omega_1 : \begin{cases} z = 0 \\ x^2+y^2 \leq 1 \end{cases}$$

$$d\vec{S}_1 = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) dA$$

Daí

$$= (0,0,1)\, dA$$

$$\vec{F} = (y, -x, z)$$

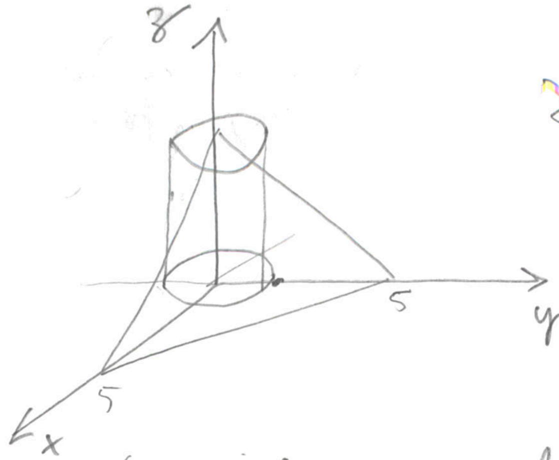
$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & -x & z \end{vmatrix} = -\hat{k} - \hat{k} = -2\hat{k}$$

$\therefore$

$$\int_{\Omega_1} (\vec{\nabla}\times\vec{F})\cdot d\vec{S}_1 = \int_{D_1} -2\hat{k}\cdot(0,0,1)\, dA = \int_{D_1} -2\, dA$$

$$= -2\int_{D_1} dA = \boxed{-2\pi}$$

4.

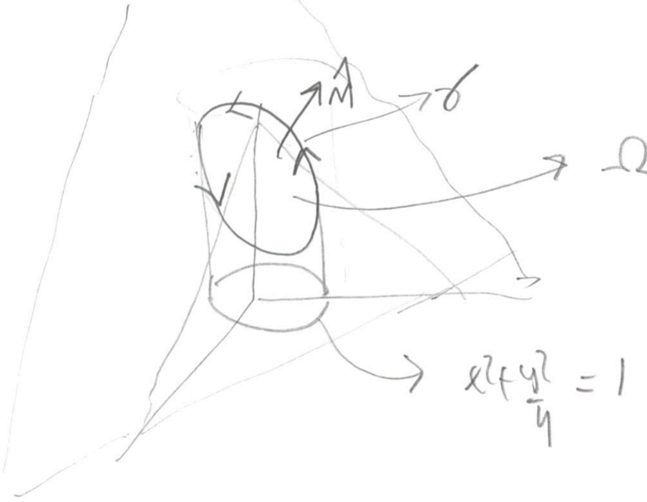


$$\gamma : \begin{cases} x+y+z=5 \\ x^2+\dfrac{y^2}{4}=1 \end{cases}$$

Usando o teorema de Stokes

$$\oint_\gamma \vec{F}\cdot d\vec{r} = \int_\Omega (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$$

onde Ω é a parte do plano



$$\Omega : z = 5-x-y$$

$$\hat{n} = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right)$$

$$d\vec{S} = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) dA$$

$$= (1, 1, 1)\, dx\,dy$$

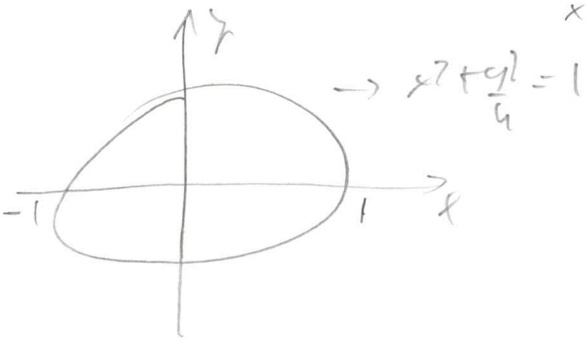
$$\vec{F} = (xz, y^2, x^2)$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xz & y^2 & x^2 \end{vmatrix} = \hat{j}x - 2x\hat{j} = -x\hat{j}$$

$$\int_{\Omega} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int_D (-x\,\hat{j}) \cdot (\hat{i} + \hat{j} + \hat{k})\, dx\,dy$$

$$= \int_D -x \, dx\,dy$$

$$= \int_{x=-1}^{1} \int_{y=-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} -x \, dx\,dy$$



$$= \int_{x=-1}^{1} -xy \Big]_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} dx$$

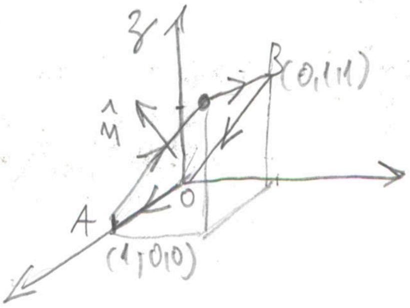
$$= \int_{x=-1}^{1} -4x\sqrt{1-x^2}\, dx$$

$$= -4 \frac{1}{\cancel{2}} \frac{\cancel{2}}{3} (1-x^2)^{3/2} \Big]_{-1}^{1}$$

$$= \frac{4}{3}(1-1)^{3/2} - \frac{4}{3}\left(1-(-1)^2\right)$$

$$= 0$$

5.



Ω : plano

$$\vec{u} = \vec{OA} = (1,0,0)$$

$$\vec{v} = \vec{OB} = (0,1,1)$$

$$\hat{n} \parallel \vec{u} \times \vec{v} = (1,0,0) \times (0,1,1)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= \hat{k} - \hat{j} = (0,-1,1)$$

$$\Omega : \quad 0x - y + z + k = 0$$

$(0,0,0) \in$ plano $\Longrightarrow$ $k = 0$

$$\therefore \quad \Omega : \quad -y + z = 0$$

$$\left\{ \begin{aligned} d\vec{S} &= \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) dA \\ &= (0, -1, 1)\, dA \end{aligned} \right.$$

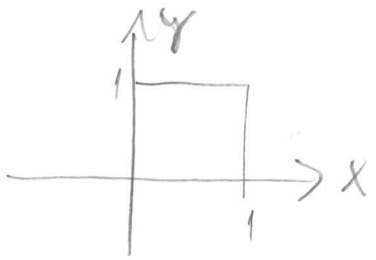
$$\vec{F} = y(x^2+y^2)\,\hat{i} - x(x^2+y^2)\,\hat{j} = (yx^2+y^3)\,\hat{i} - (x^3+xy^2)\,\hat{j}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ yx^2+y^3 & -x^3-xy^2 & 0 \end{vmatrix} = \hat{k}(-3x^2-y^2) - \hat{k}(x^2+3y^2)$$

$$= \hat{k}(-4x^2-4y^2)$$

$$\oint_\gamma \vec{F} \cdot d\vec{r} = \int_\Omega (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

$$= \int_D (-4x^2-4y^2)\,\hat{k} \cdot (-\hat{j} + \hat{k})\, dA$$



$$= \int_{x=0}^{1}\int_{y=0}^{1} (-4x^2-4y^2)\, dx\, dy$$

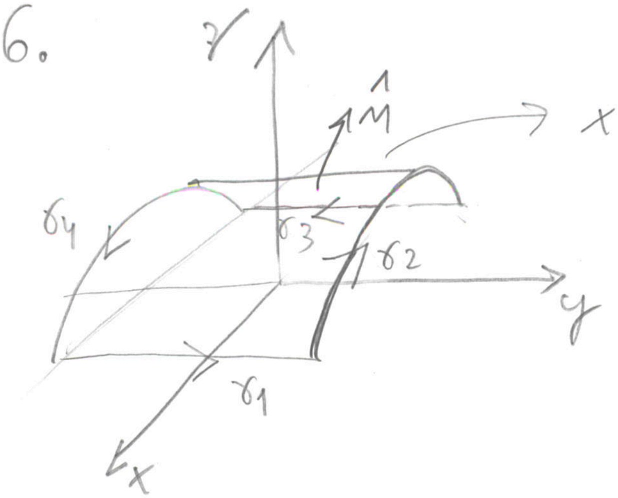
$$= \int_{x=0}^{1} -4x^2y - \frac{4y^3}{3}\Big]_0^1 \, dx$$

$$= \int_{x=0}^{1} \left[-4x^2 - \frac{4}{3}\right] dx$$

$$= -\frac{4x^3}{3} - \frac{4}{3}x\Big]_0^1$$

$$= -\frac{4}{3} - \frac{4}{3} = -\frac{8}{3}$$

6.



$$\int_{\Omega} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \oint_{\gamma} \vec{F} \cdot d\vec{r}$$

$\gamma_1 : \ \vec{r_1} = (1, y, 0) \quad ; \ -1 \leq y \leq 1$

$$\frac{d\vec{r_1}}{dy} = (0, 1, 0)$$

$$\therefore \int_{\gamma_1} \vec{F} \cdot d\vec{r_1} = \int_{-1}^{1} \left(x, \ x^2+y^2+z^2, \ z(y^4-1)\right) \cdot (0,1,0) \, dy$$

$$= \int_{-1}^{1} (x^2+y^2+z^2) \, dy$$

$$\downarrow 0$$

$$= \int_{-1}^{1} (1+y^2) \, dy$$

$$= y + \frac{y^3}{3} \Big]_{-1}^{1} = 1 + \frac{1}{3} - \left(-1 - \frac{1}{3}\right)$$

$$= 2 + \frac{2}{3}$$

γ_2 : $\vec{r}_2 = (x, 1, \sqrt{1-x^2})$ $\quad -1 \leq x \leq 1$

$\Leftrightarrow$ $\quad$ muda a orientação

$$\begin{cases} \vec{r}_2 \equiv (-x, 1, \sqrt{1-x^2}) \\ -1 \leq x \leq 1 \end{cases}$$

$$\frac{d\vec{r}_2}{dx} = \left(-1, 0, \frac{-x}{\sqrt{1-x^2}}\right)$$

$$\int_{\gamma_2} \vec{F} \cdot d\vec{r}_2 = \int_{-1}^{1} (x, x^2+y^2+z^2, z(y^4-1)) \cdot \left(-1, 0, \frac{-x}{\sqrt{1-x^2}}\right) dx$$

sobre γ_2 tem-se $y=1$, $z=\sqrt{1-x^2}$

$$= \int_{-1}^{1} -x - \frac{x z (y^4-1)}{\sqrt{1-x^2}} \, dx$$

$$= \int_{-1}^{1} -x - \cancelto{0}{\frac{x\sqrt{1-x^2}(1-1)}{\sqrt{1-x^2}}} \, dx$$

$$= \int_{-1}^{1} -x \, dx = -\frac{x^2}{2} \Big]_{-1}^{1} = -\frac{1}{2} - \left(-\frac{(-1)^2}{2}\right)$$

$$= -\frac{1}{2} + \frac{1}{2}$$

$$= 0 \;//$$

$\gamma_3 : \quad \vec{r_3} = (-1, y, 0) \quad ; \quad -1 \leq y \leq 1$

$\Big\downarrow$ invierte

$\vec{r_3} = (-1, -y, 0) \quad ; \quad -1 \leq y \leq 1$

$$\frac{d\vec{r_3}}{dy} = (0, -1, 0)$$

2° a)

$$\int_{\gamma_3} \vec{F} \cdot d\vec{r_3} = \int_{-1}^{1} (x, x^2+y^2+z^2, z(y^2-1)) \cdot (0,-1,0)\, dy$$

$$= \int_{-1}^{1} \left(-x^2-y^2-z^2\right) dy$$

en γ_3: $x=-1$, $z=0$

$$= \int_{-1}^{1} (-1-y^2)\, dy$$

$$= -y - \frac{y^3}{3} \Big|_{-1}^{1} = -1 - \frac{1}{3} - \left(-(-1) - \frac{(-1)^3}{3}\right)$$

$$= -1 - \frac{1}{3} - 1 - \frac{1}{3}$$

$$= -2 - \frac{2}{3}$$

$\underline{\gamma_4}$: $\vec{r}_4 = (x, -1, \sqrt{1-x^2})$; $-1 \le x \le 1$

$$\frac{d\vec{r}_4}{dx} = \left(+1, 0, \frac{-x}{\sqrt{1-x^2}}\right)$$

$$\int_{\gamma_4} \vec{F} \cdot d\vec{r}_4 = \int_{-1}^{1} \left(x, x^2+y^2+z^2, z(y^4-1)\right) \cdot \left(+1, 0, \frac{-x}{\sqrt{1-x^2}}\right) dx$$

$$= \int_{-1}^{1} \left[x - \frac{zx(y^4-1)}{\sqrt{1-x^2}}\right] dx$$

en γ_4
$y = -1$
$z = \sqrt{1-x^2}$

$$= \int_{-1}^{1} x \, dx$$

$$= \left.\frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$$

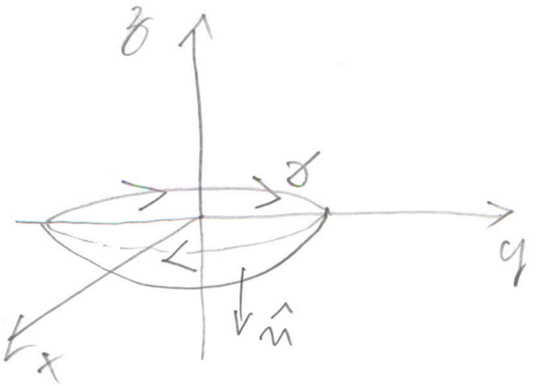
$\therefore$

$$\oint_{\Omega} (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} \vec{F} \cdot d\vec{r}$$

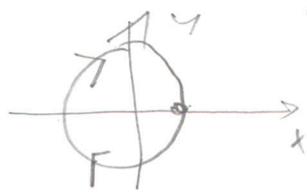
$$= 2 + \frac{2}{3} - 2 - \frac{2}{3}$$

$$= 0$$

7.



$$\gamma : \begin{cases} x^2+y^2=1 \\ x=\cos\theta \\ y=-\sin\theta \\ 0\le\theta\le 2\pi \end{cases}$$



$$\int_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \oint_\gamma \vec{F}\cdot d\vec{r}$$

$$\vec{r}(\theta)=(\cos\theta, -\sin\theta, 0)$$

$$\frac{d\vec{r}}{d\theta}=(-\sin\theta, -\cos\theta, 0)$$

$$= \int_0^{2\pi} (xz^2, x^3, \cos xz)\cdot(-\sin\theta, -\cos\theta, 0)\, d\theta$$

$$= \int_0^{2\pi} \left(-x\overset{0}{\cancel{z^2}}\sin\theta - x^3\cos\theta\right) d\theta$$

$$= \int_0^{2\pi} -\cos^3\theta\cos\theta\, d\theta$$

$$= \int_0^{2\pi} -\cos^4\theta\, d\theta$$

Mas

$$\int \cos^4\theta\, d\theta = \int \cos^3\theta\cos\theta\, d\theta$$

$$u=\cos^3\theta \rightarrow du=-3\cos^2\theta\sin\theta\, d\theta$$

$$dv=\cos\theta\, d\theta \rightarrow v=\sin\theta$$

$$\int \cos^4\theta \, d\theta = \cos^3\theta \sin\theta + 3\int \cos^2\theta \sin^2\theta \, d\theta$$

$$= \cos^3\theta \sin\theta + 3\int \cos^2\theta (1-\cos^2\theta) \, d\theta$$

$$\int \cos^4\theta \, d\theta = \cos^3\theta \sin\theta + 3\int \cos^2\theta \, d\theta - 3\int \cos^4\theta \, d\theta$$

$$\therefore 4\int \cos^4\theta \, d\theta = \cos^3\theta \sin\theta + 3\int \cos^2\theta \, d\theta$$

$$4\int \cos^4\theta \, d\theta = \cos^3\theta \sin\theta + 3\int \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= \cos^3\theta \sin\theta + \frac{3}{2}\theta + \frac{3}{4}\sin 2\theta$$

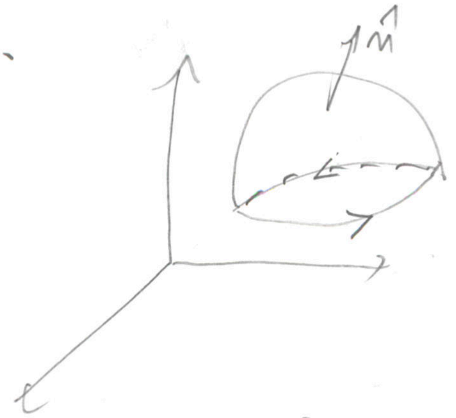
$$\therefore \int \cos^4\theta \, d\theta = \frac{1}{4}\cos^3\theta \sin\theta + \frac{3}{8}\theta + \frac{3}{16}\sin 2\theta$$

$$\therefore \int_0^{2\pi} -\cos^4\theta \, d\theta = -\frac{1}{4}\cos^3\theta \sin\theta - \frac{3}{8}\theta - \frac{3}{16}\sin 2\theta \Big]_0^{2\pi}$$

$$= -\frac{3}{\cancel{8}_4}\cancel{2}\pi$$

$$= -\frac{3\pi}{4}$$

8.



$\vec{F} = cte$

$$\oint_\gamma \vec{F} \cdot d\vec{r} = \int_\Omega (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

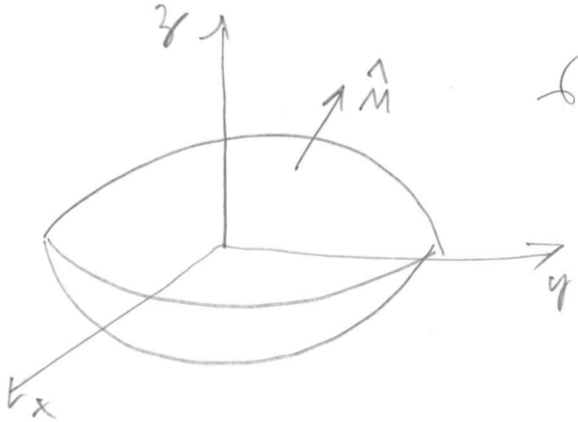
Sendo $\vec{F} = cte$ tem-se $\vec{\nabla} \times \vec{F} = 0$

logo

$$\int_\Omega (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = 0$$

$$\therefore \quad \boxed{\oint_\gamma \vec{F} \cdot d\vec{r} = 0}$$

9.



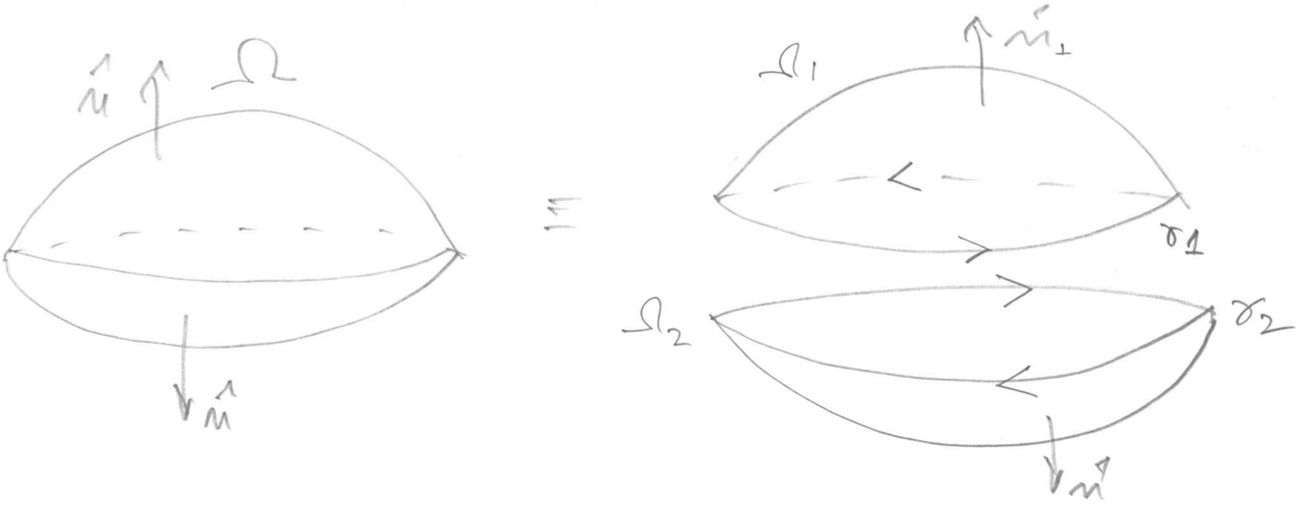
$$\Omega : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Divida a superficie em duas partes

$$\Omega = \Omega_1 \cup \Omega_2$$

$$\Omega_1 : \; z = +c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

$$\Omega_2 : \; z = -c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$



$$\oint_{\Omega} (\vec\nabla\times\vec F)\cdot d\vec S = \underbrace{\int_{\Omega_1}(\vec\nabla\times\vec F)\cdot d\vec{S_1}} + \underbrace{\int_{\Omega_2}(\vec\nabla\times\vec F)\cdot d\vec{S_2}}$$

$$= \oint_{\gamma_1}\vec F\cdot d\vec{r_1} + \oint_{\gamma_2}\vec F\cdot d\vec{r_2}$$

Mas $\gamma_1 : \vec r_1(t)$; $a < t < b$

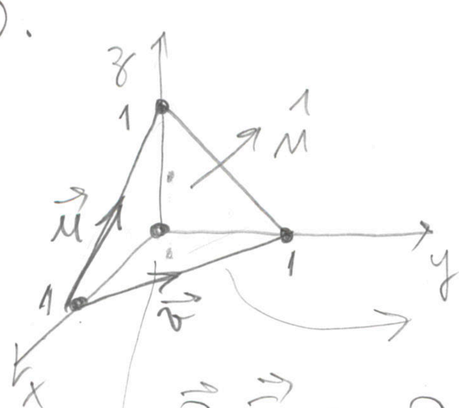
$\gamma_2 : \vec r_2(t) = \vec r_1(a+b-t)$

representa a mesma curva mas com orientações opostas.

Daí
$$\oint_{\gamma_1}\vec F\cdot d\vec{r_1} + \oint_{\gamma_2}\vec F\cdot d\vec{r_2} = 0$$

$$\therefore \boxed{\oint_{\Omega}(\vec\nabla\times\vec F)\cdot d\vec S = 0}$$

10.



$$\oint_{\Omega} \vec{F}\cdot d\vec{S} = \int_V \vec{\nabla}\cdot\vec{F}\, dV$$

$x+y=1$

$$\vec{\nabla}\cdot\vec{F} = \partial_x x^2 + \partial_y (xy) + \partial_z(-2xz)$$

$$= 2x + x - 2x$$

$$= \underline{x}$$

$\underline{\underline{V}}$: Equação do plano :

$\sigma : ax + by + zc + k = 0$

$\vec{u} = (-1, 0, 1)$

$\vec{v} = (-1, 1, 0)$

$$\Rightarrow \hat{n} = \vec{v} \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= \hat{i} + \hat{k} + \hat{j}$$

$\hat{n} = (1, 1, 1)$

$\sigma = x + y + z + k = 0$

$(1, 0, 0) \in \sigma \Rightarrow \underline{\underline{k = -1}}$

$\sigma : \underline{x + y + z - 1 = 0}$

Daí a região V se escreve como

$$V = \{ (x,y,z) \in \mathbb{R}^3 \mid 0 \leq z \leq 1-x-y, \; 0 \leq y \leq 1-x, \; 0 \leq x \leq 1 \}$$

$\therefore$

$$\oint_{\Omega} \vec{F} \cdot d\vec{S} = \int_V \vec{\nabla} \cdot \vec{F} \, dV = \int_V x \, dV$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} x \, dx\,dy\,dz$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} x\, z \Big]_0^{1-x-y} dy\,dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} x(1-x-y) \, dy\,dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} (x - x^2 - xy) \, dy\,dx$$

$$= \int_{x=0}^{1} xy - x^2 y - \frac{xy^2}{2} \Big]_0^{1-x} dx$$

$$= \int_{x=0}^{1} \left(x(1-x) - x^2(1-x) - \frac{x}{2}(1-x)^2 \right) dx$$

$$= \int_{x=0}^{1} \left[x - x^2 - x^2 + x^3 - \frac{x}{2}\left(1 - 2x + x^2\right) \right] dx$$

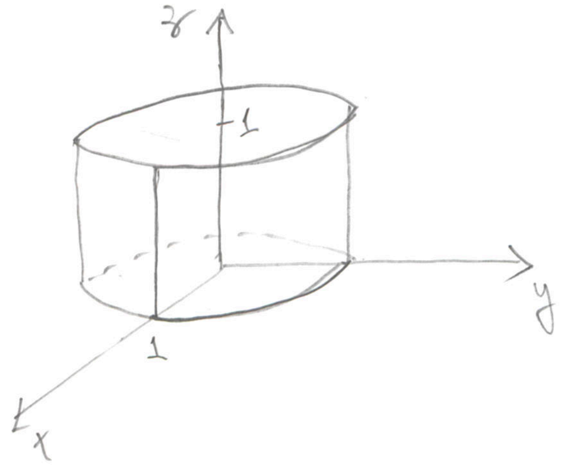
$$= \int_{x=0}^{1} \left[x - 2x^2 + x^3 - \frac{x}{2} + x^2 - \frac{x^3}{2} \right] dx$$

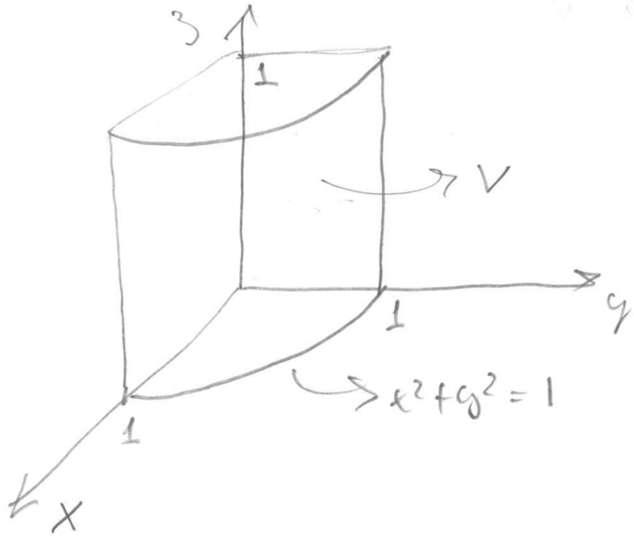
$$= \int_{x=0}^{1} \left[\frac{x}{2} - x^2 + \frac{x^3}{2} \right] dx$$

$$= \frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8} \Big]_0^1$$

$$= \frac{1}{4_{/6}} - \frac{1}{3_{/8}} + \frac{1}{8_{/3}} = \frac{6 - 8 + 3}{24} = \frac{1}{24}$$

11.





$$V = \{ (x,y,z) \mid 0 \le y \le \sqrt{1-x^2} , \quad 0 \le x \le 1 , \quad 0 \le z \le 1 \}$$

$$\oint_{\Omega} \vec{F} \cdot d\vec{S} = \int_V \vec{\nabla} \cdot \vec{F} \, dV \quad ; \quad \begin{cases} \vec{\nabla} \cdot \vec{F} = \partial_x x + \partial_y y + \partial_z z \\ \qquad = 3 \end{cases}$$

$$= \int_V 3 \, dV$$

$$= 3 \int_V dV = 3 \int_{x=0}^{1} \int_{z=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} dx \, dy \, dz$$

$$= 3 \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} 1 \, dx \, dy$$

$$= 3 \int_{x=0}^{1} y \Big|_0^{\sqrt{1-x^2}} dx$$

$$= 3 \int_{x=0}^{1} \sqrt{1-x^2} \, dx \qquad \longrightarrow$$

11. Cont.

$$\begin{cases} x = \sin\theta \longrightarrow dx = \cos\theta \, d\theta \\ x = 0 \rightarrow \theta = 0 \\ x = 1 \rightarrow \theta = \frac{\pi}{2} \end{cases}$$

$$\therefore \quad 3\int_{x=0}^{1} \sqrt{1-x^2}\, dx = 3\int_{\theta=0}^{\frac{\pi}{2}} \cos\theta \cos\theta \, d\theta$$

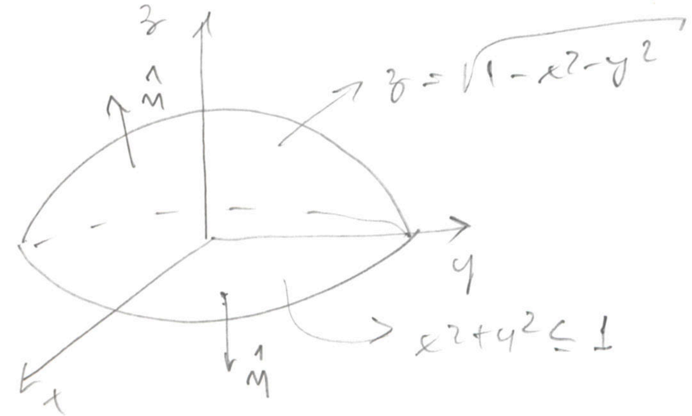
$$= 3\int_{\theta=0}^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

$$= 3\int_{\theta=0}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= \frac{3}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{3}{2}\left[\frac{\pi}{2}\right] = \frac{3\pi}{4}$$

12.



$$V \equiv \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; 0 \leq z \leq \sqrt{1-x^2-y^2} \quad (x,y) \in D : x^2+y^2 \leq 1 \right\}$$

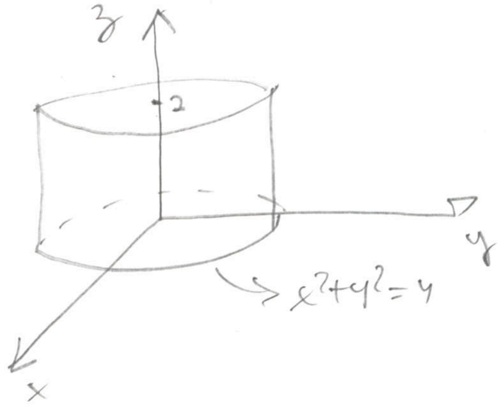
$$\oint_{\Omega} \vec{F} \cdot d\vec{S} = \int_V \left(\vec{\nabla} \cdot \vec{F}\right) dV$$

$$= 3 \int_V dV = 3(\text{volume do hemisfério})$$

$$= \cancel{3} \cdot \left(\frac{1}{2} \frac{4}{\cancel{3}} \pi 1^3 \right)$$

$$= 2\pi$$

13.



$$V = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; 0 \leq z \leq 2 \quad (x,y) \in D : x^2+y^2 \leq 4 \right\}$$

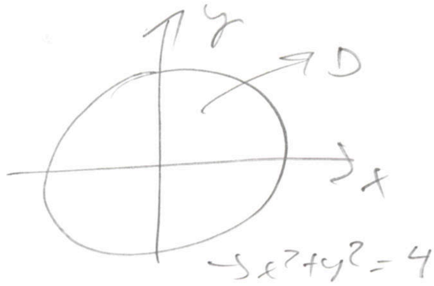
$$\oint_{\Omega} \vec{F} \cdot d\vec{S} = \int_V \vec{\nabla} \cdot \vec{F} \, dV = \int_V (2x+2y+2z) \, dV$$

13. Cont.

$$\int_{(x,y)\in D} dA \int_{z=0}^{2} (2x+2y+2z)\, dz =$$

$$= \int_{(x,y)\in D} dA \left[(2x+2y)z + z^2 \right]_{z=0}^{2}$$

$$= \int_{(x,y)\in D} dA \left[4x+4y+4 \right]$$



Seja $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $0 \le \theta \le 2\pi$
$0 \le r \le 2$

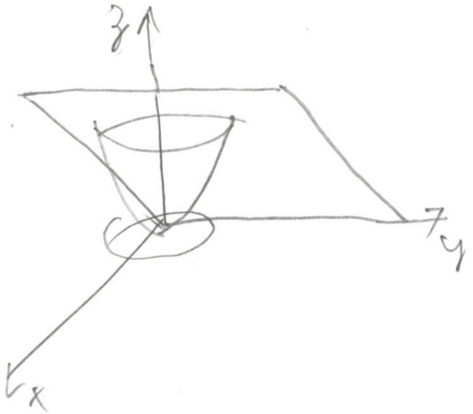
$$= \int_{r=0}^{2} \int_{\theta=0}^{2\pi} \left[4r\cos\theta + 4r\sin\theta + 4 \right] r\, dr\, d\theta$$

$$= \int_{r=0}^{2} 4r^2\sin\theta - 4r^2\cos\theta + 4r\theta \Big]_{\theta=0}^{2\pi} dr$$

$$= \int_{r=0}^{2} 4r \cdot 2\pi \, dr = \int_{r=0}^{2} 8\pi r\, dr$$

$$= 8\pi \frac{r^2}{2} \Big]_0^2 = 4\pi r^2 \Big]_0^2 = 16\pi$$

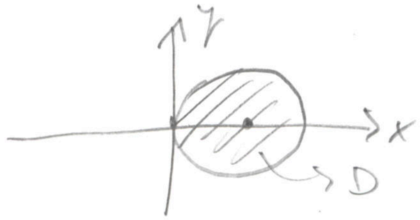
14.



$$z = x^2+y^2$$

$$V = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \le z \le 2x$$
$$(x,y) \in D :$$
$$(x-1)^2+y^2 \le 1 \}$$

$$\left.\begin{array}{l} z = x^2+y^2 \\ z = 2x \end{array}\right\} \Longrightarrow \quad 2x = x^2+y^2$$
$$\therefore \quad (x-1)^2+y^2 = 1$$



$$\vec{F} = y(x^2+y^2)^{3/2}\,\hat{i} - x(x^2+y^2)^{3/2}\,\hat{j} + (z+1)\,\hat{k}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}\left(y(x^2+y^2)^{3/2}\right) + \frac{\partial}{\partial y}\left(-x(x^2+y^2)^{3/2}\right) + \frac{\partial}{\partial z}(z+1)$$

$$= y\,\frac{3}{\cancel{2}}\sqrt{x^2+y^2}\;\cancel{2}x - x\,\frac{3}{\cancel{2}}\sqrt{x^2+y^2}\;\cancel{2}y + 1$$

$$= \cancel{3xy\sqrt{x^2+y^2}} - \cancel{3xy\sqrt{x^2+y^2}} + 1$$

$$= 1$$

$$\therefore \quad \oint_{\Omega} \vec{F}\cdot d\vec{S} = \int_V \vec{\nabla}\cdot\vec{F}\,dV = \int_V dV$$

$$= \int_{(x,y)\in D} dA \int_{x^2+y^2}^{2x} dz$$

$$\oint_{\Omega} \vec{F}\cdot d\vec{S} = \int_{(x,y)\in D} dA \quad z\Big]_{x^2+y^2}^{2x}$$

$$= \int_{(x,y)\in D} dA \left(2x - x^2 - y^2\right)$$

$$= \int_{(x,y)\in D} dA \left(-(x-1)^2 - y^2 + 1\right)$$

Seja

$$x-1 = r\cos\theta \qquad 0 \le r \le 1$$
$$y = r\sin\theta \quad ; \quad 0 \le \theta \le 2\pi$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} \left(-r^2\cos^2\theta - r^2\sin^2\theta + 1\right) r\,dr\,d\theta$$

$$= \int_{r=0}^{1}\underbrace{\int_{\theta=0}^{2\pi}}\left(-r^2+1\right) r\,dr\,\underbrace{d\theta}$$

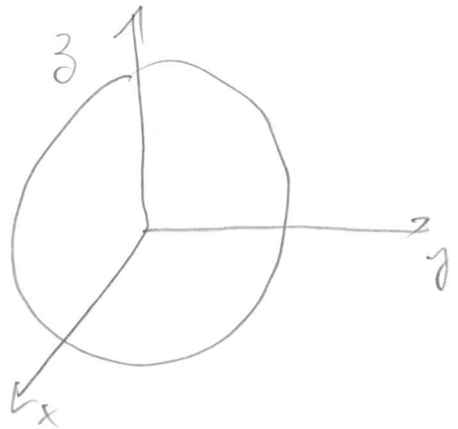
$$= \int_{r=0}^{1} \left(-r^2+1\right) r\,dr\;2\pi$$

$$= 2\pi \int_{r=0}^{1} \left(-r^3+r\right) dr$$

$$= 2\pi \left(-\frac{r^4}{4} + \frac{r^2}{2}\right)\Bigg|_0^1 = 2\pi\left(-\frac{1}{4}+\frac{1}{2}\right) =$$

$$= 2\pi\,\frac{1}{4} = \frac{\pi}{2}$$

15.



$$\vec{F} = (x^2+y^2+z^2)(x\vec{i} + y\vec{j})$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}\left((x^2+y^2+z^2)\,x\right) + \frac{\partial}{\partial y}\left((x^2+y^2+z^2)\,y\right)$$

$$= 3x^2+y^2+z^2 + x^2+z^2+3y^2$$

$$= 4x^2+4y^2+2z^2$$

$$\oint_{\Omega} \vec{F}\cdot d\vec{S} = \int_V \vec{\nabla}\cdot\vec{F}\, dV$$

$$= \int_V (4x^2+4y^2+2z^2)\, dV$$

Coordenadas esféricas

$$V = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} z = r\cos\theta \\ x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi\,; \\ \text{com } 0 \le r \le 1,\ 0 \le \theta \le \pi \\ 0 \le \varphi \le 2\pi \end{array} \right\}$$

15. Cont.

$$\int_V (4x^2+4y^2+2z^2)\, dV$$

$$= \int \left(4r^2\sin^2\theta\cos^2\varphi + 4r^2\sin^2\theta\sin^2\varphi + 2r^2\cos^2\theta\right).$$

$$r^2\sin\theta\, dr\, d\theta\, d\varphi$$

$$= \int \left(4r^2\sin^2\theta + 2r^2\cos^2\theta\right) r^2\sin\theta\, dr\, d\theta\, d\varphi$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \left(4r^4\sin^3\theta + 2r^4\cos^2\theta\sin\theta\right) dr\, d\theta\, d\varphi$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{\pi} \left(4r^4\sin^3\theta + 2r^4\cos^2\theta\sin\theta\right) 2\pi\, dr\, d\theta$$

$$= \underbrace{\int_{r=0}^{1}\int_{\theta=0}^{\pi} 8\pi r^4\sin^3\theta\, dr\, d\theta}_{(*)} + \underbrace{\int 4\pi r^4\cos^2\theta\sin\theta\, dr\, d\theta}_{(**)}$$

$$(*) = \int_{r=0}^{1}\int_{\theta=0}^{\pi} 8\pi r^4\sin^3\theta\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} 8\pi \frac{r^5}{5}\Big|_0^1 \sin^3\theta\, d\theta = \frac{8\pi}{5}\int_{\theta=0}^{\pi}\sin^3\theta\, d\theta =$$

$$= \frac{8\pi}{5}\int_0^{\pi} \sin^3\theta \, d\theta = \frac{8\pi}{5}\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi}$$

$$= \frac{8\pi}{5}\left[-(-1) + \frac{1}{3}(-1)^3 - \left(-1 + \frac{1}{3}\right)\right]$$

$$\int \sin^3\theta \, d\theta = \int \sin\theta \sin^2\theta \, d\theta$$

$$= \int \sin\theta\,(1-\cos^2\theta)\, d\theta$$

$$= \int \sin\theta \, d\theta - \int \cos^2\theta \sin\theta \, d\theta$$

$$= -\cos\theta + \frac{1}{3}\cos^3\theta$$

$$= \frac{8\pi}{5}\left[1 - \frac{1}{3} + 1 - \frac{1}{3}\right]$$

$$= \frac{8\pi}{5}\left[2 - \frac{2}{3}\right] = \frac{8\pi}{5}\,\frac{4}{3}$$

$$\therefore \; // \; (*) = \frac{32\pi}{15} \; //$$

$$(**) = \int_{r=0}^{1}\int_{\theta=0}^{\pi} 4\pi r^4 \cos^2\theta \sin\theta \, d\theta \, dr$$

$$= \int_{\theta=0}^{\pi} 4\pi \frac{r^5}{5}\Big]_0^1 \cos^2\theta \sin\theta \, d\theta$$

$$= \int_{\theta=0}^{\pi} 4\pi \frac{1}{5} \cos^2\theta \sin\theta \, d\theta$$

$$= \frac{4\pi}{5}\int_{\theta=0}^{\pi} \cos^2\theta \sin\theta \, d\theta$$

15. Cont.

$$(**) = \frac{4\pi}{5}\left(-\frac{1}{3}\cos^3\theta\right)\Big]_0^{\pi} = -\frac{4\pi}{15}\left[\cos^3\pi - \cos^3 0\right]$$

$$= -\frac{4\pi}{15}(-1-1)$$

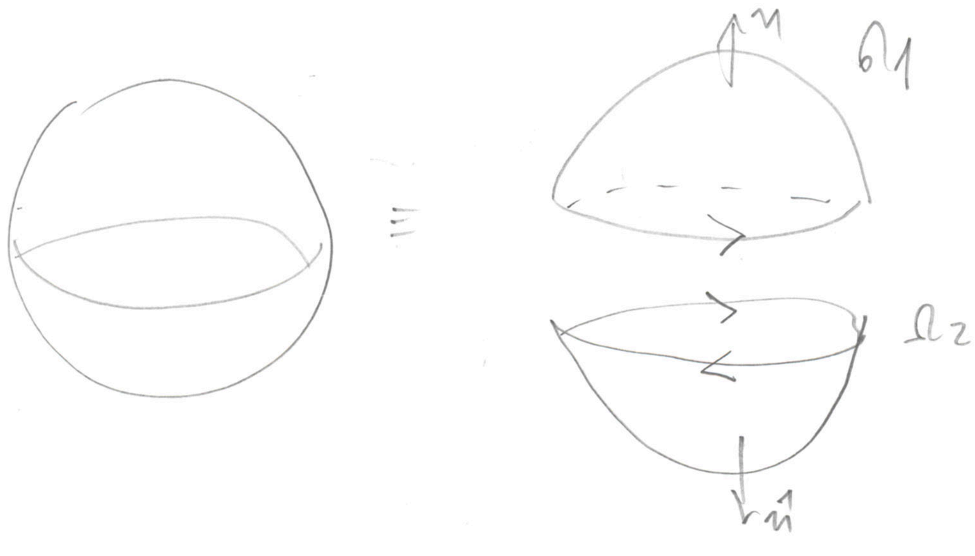
$$\therefore\ (**) = \frac{8\pi}{15}$$

$$\therefore\ \int \vec{\nabla}\cdot\vec{F}\, dV = \frac{32\pi}{15} + \frac{8\pi}{15} = \frac{40\pi}{15}$$

$$= \boxed{\frac{8\pi}{3}}$$

Usando o teorema de Gauss

$$\int_V \vec{\nabla}\cdot\vec{F}\, dV = \oint_{\Omega} \vec{F}\cdot d\vec{S} = \int_{\Omega_1} \vec{F}\cdot d\vec{S}_1 + \int_{\Omega_2} \vec{F}\cdot d\vec{S}_2$$



$$\int_{\Omega_1} \vec{F} \cdot d\vec{S_1}$$

$\Omega_1: \quad z = \sqrt{1-x^2-y^2}$

$$d\vec{S_1} = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) dA$$

$$= \left(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1\right) dA$$

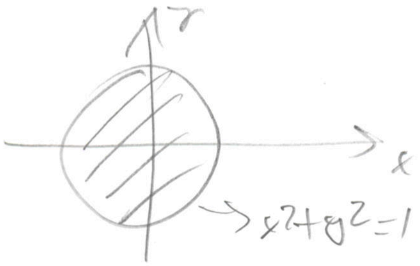
$$\therefore \int_{\Omega_1} \vec{F} \cdot d\vec{S_1} = \int_{D_1} \left((x^2+y^2+z^2)x, (x^2+y^2+z^2)y, 0\right) \cdot \left(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1\right) dx\,dy$$

$$= \int_{D_1} \left[\frac{(x^2+y^2+z^2)x^2}{\sqrt{1-x^2-y^2}} + \frac{(x^2+y^2+z^2)y^2}{\sqrt{1-x^2-y^2}}\right] dx\,dy$$

$$= \int_{D_1} \left[\frac{(\cancel{x^2}+\cancel{y^2}+1-\cancel{x^2}-\cancel{y^2})x^2}{\sqrt{1-x^2-y^2}} + \frac{(\cancel{x^2}+\cancel{y^2}+1-\cancel{x^2}-\cancel{y^2})y^2}{\sqrt{1-x^2-y^2}}\right] dx$$

$$= \int_{D_1} \left(\frac{x^2}{\sqrt{1-x^2-y^2}} + \frac{y^2}{1-x^2-y^2}\right) dx\,dy$$

$$= \int_{D_1} \frac{x^2+y^2}{\sqrt{1-x^2-y^2}} \, dx\, dy$$



$x = r\cos\theta$; $0 \le r \le 1$

$y = r\sin\theta$; $0 \le \theta \le 2\pi$

$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} \frac{r^2}{\sqrt{1-r^2}} \, r\, dr\, d\theta$$

$$= \int_{r=0}^{1} \frac{r^3}{\sqrt{1-r^2}} \, 2\pi \, dr$$

$$= 2\pi \int_{r=0}^{1} \frac{r^3}{\sqrt{1-r^2}} \, dr$$

$r = \sin u \quad \rightarrow \quad dr = \cos u \, du$

$$= 2\pi \int_{u=0}^{\frac{\pi}{2}} \frac{\sin^3 u}{\cancel{\cos u}} \, \cancel{\cos u} \, du$$

$$= 2\pi \int_{u=0}^{\frac{\pi}{2}} \sin^3 u \, du$$

$$= 2\pi \left| -\cos u + \frac{1}{3}\cos^3 u \right|^{\frac{\pi}{2}} = 2\pi\left(1 - \frac{1}{3}\right)$$

$= \frac{4\pi}{3}$ $\therefore$ $\boxed{\int_{\Omega_1} \vec{F} \cdot d\vec{S_1} = \frac{4\pi}{3}}$

Analogamente:

$$\int_{\Omega_2} \vec{F} \cdot d\vec{S_2}$$

$\Omega_2 : \quad z = -\sqrt{1-x^2-y^2}$

$$d\vec{S_2} = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1 \right) dA$$

$$= \left(\frac{+x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, -1 \right) dA$$

$$\therefore \int_{\Omega_2} \vec{F} \cdot d\vec{S_2} = \int_{D_2} \left((x^2+y^2+z^2)x, (x^2+y^2+z^2)y, 0 \right) \cdot \left(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, -1 \right) dx\,dy$$

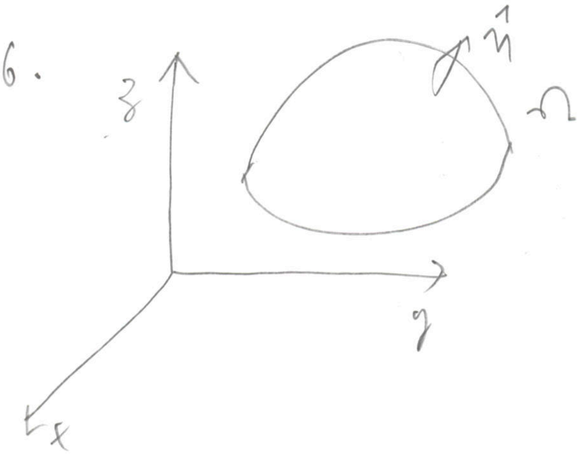
$$= \int_{D_2} \frac{(x^2+y^2+z^2)x^2 + (x^2+y^2+z^2)y^2}{\sqrt{1-x^2-y^2}} \, dx\,dy$$

tudo que é a mesma integral anterior, logo

$\therefore \displaystyle\int_{\Omega_2} \vec{F}\cdot d\vec{S}_2 = \frac{4\pi}{3}$

$$\therefore \boxed{\oint_{\Omega} \vec{F}\cdot d\vec{S} = \frac{4\pi}{3} + \frac{4\pi}{3} = \frac{8\pi}{3}}$$

16.



Temos

$$\int_{\Omega} \vec{F}\cdot d\vec{S} \overset{\text{Gauss}}{=} \int_{B} \underbrace{\vec{\nabla}\cdot\vec{F}}\, dV$$

$$= \underbrace{3}\int_{B} dV$$

$$= 3\,V \quad \text{(volume da região)}$$

$$\therefore \boxed{V = \frac{1}{3}\oint_{\Omega} \vec{F}\cdot d\vec{S}}$$