Correção 3c)
3i)

15. Omitir

Cálculo A - Lista 3

Campos escalares e vetoriais

Gradiente, divergência e rotacional

1. Calcule a divergência e rotacional do campo vetorial abaixo

   (a) $\vec{F}(x,y,z) = x\vec{i} + y\vec{j}$

   (b) $\vec{F}(x,y,z) = y\vec{i} + z\vec{j} + x\vec{k}$

   (c) $\vec{F}(x,y,z) = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$

   (d) $\vec{F}(x,y,z) = -\frac{x}{z}\vec{i} - \frac{y}{z}\vec{j} + \frac{1}{z}\vec{k}$

   (e) $\vec{F}(x,y,z) = e^x \cos y\vec{i} + e^x \sin y\vec{j} + z\vec{k}$

2. Determine se $\vec{F}$ é o gradiente de uma certa função φ. Se for, determine φ.

   (a) $\vec{F}(x,y) = e^y\vec{i} + (xe^y + y)\vec{j}$

   (b) $\vec{F}(x,y) = (\sin xy)\vec{i} + (\cos xy)\vec{j}$

   (c) $\vec{F}(x,y,z) = 2xyz\vec{i} + x^2z\vec{j} + (x^2y+1)\vec{k}$

   (d) $\vec{F}(x,y,z) = xz\vec{i} + yz\vec{j} + xz\vec{k}$

   (e) $\vec{F}(x,y,z) = (y^2+x^2)\vec{i} + (z^2+y^2)\vec{j} + (x^2+z^2)\vec{k}$

3. Sejam f e g campos escalares e $\vec{F}$ e $\vec{G}$ campos vetoriais. Determine quais das expressões a seguir representam campos vetoriais, quais representam campos escalares e quais não tem sentido.

   (a) $\nabla(fg)$

   (b) $\nabla\vec{F}$

   (c) $\nabla \times (\nabla f)$

   (d) $\nabla(\nabla \cdot \vec{F})$

   (e) $\nabla \times (\nabla \times \vec{F})$

   (f) $\nabla \cdot (\nabla f)$

   (g) $(\nabla f) \times (\nabla \vec{F})$ $\vec{\nabla} \times \vec{F}$

   (h) $\nabla \cdot (\nabla \times (\nabla f))$

   (i) $\nabla \times (\nabla \cdot (\nabla f))$ $\neq 0$

Mostrar que

4. $\nabla \cdot (\nabla \times \vec{F}) = 0$ [o rotacional de um campo vetorial é solenoidal]

5. $\nabla \times (\nabla\varphi) = 0$ [o gradiente de um campo escalar é irrotacional]

6. $\nabla \cdot (f\vec{F}) = f\nabla \cdot \vec{F} + (\nabla f) \cdot \vec{F}$

7. $\nabla \cdot (\vec{F} \times \vec{G}) = (\nabla \times \vec{F}) \cdot \vec{G} - \vec{F} \cdot (\nabla \times \vec{G})$

8. $\nabla \times (f\vec{F}) = f\nabla \times \vec{F} + (\nabla f) \times \vec{F}$

9. $\nabla(fg) = f\nabla g + g\nabla f$

10. Sejam $f(x,y,z)$ e $g(x,y,z)$ funções com derivadas parciais de segunda ordem contínua. Mostre que $\nabla f \times \nabla g$ é solenoidal.

11. Seja $\vec{F}(x,y,z) = F_1(y,z)\vec{i} + F_2(x,z)\vec{j} + F_3(x,y)\vec{k}$. Mostre que $\vec{F}$ é solenoidal.

12. (a) Encontre constantes a, b, c de modo que o campo vetorial $\vec{F} = (x^2+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ seja irrotacional.
    (b) Se $\vec{F}$ é irrotacional, encontre um campo escalar $\varphi(x,y,z)$ tal que $\nabla\varphi = \vec{F}$.

13. Seja $\vec{v} = \vec{\omega} \times \vec{r}$ onde $\vec{\omega}$ é um vetor constante, e $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. Mostre que $\vec{\omega} = \frac{1}{2}(\nabla \times \vec{v})$.

14. Mostre que se a função $f(x,y,z)$ satisfaz a equação de Laplace
$$\nabla^2 f \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$
então ∇f é campo irrotacional e solenoidal.

15. Seja $\vec{F} = F_1\vec{i} + F_2\vec{j}$. Vimos que
$$\vec{F} = \nabla\varphi \Rightarrow \nabla \times \vec{F} = 0$$
Será que
$$\nabla \times \vec{F} = 0 \Rightarrow \vec{F} = \nabla\varphi?$$
Afim de responder esta questão considere, o campo vetorial
$$\vec{F}(x,y) = \frac{y}{x^2+y^2}\vec{i} - \frac{x}{x^2+y^2}\vec{j}$$
definido em $D = \{(x,y) \in R^2 \mid 0 < x^2+y^2 < 1\}$
(i) Mostre que
$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \text{ em } D$$
(ii) Mostre que $\vec{F}$ não se escreve como o gradiente de uma função escalar definida em D.

Omitir

Lista 3

1.

a) $\vec{F}(x,y,z) = x\hat{\imath} + y\hat{\jmath}$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 1 + 1 = 2$$

$$\vec{\nabla}\times\vec{F} = \left(\cancel{\partial_y F_3}^{0} - \cancel{\partial_z F_2}^{0}\right)\hat{\imath} + \left(\cancel{\partial_z F_1}^{0} - \cancel{\partial_x F_3}^{0}\right)\hat{\jmath} + \left(\cancel{\partial_x F_2}^{0} - \cancel{\partial_y F_1}^{0}\right)\hat{k}$$

$$= 0$$

$\therefore$ $\boxed{\vec{\nabla}\cdot\vec{F} = 2 \quad , \quad \vec{\nabla}\times\vec{F} = \vec{0}}$

b) $\vec{F}(x,y,z) = y\hat{\imath} + z\hat{\jmath} + x\hat{k}$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 0 + 0 + 0$$

$$= 0$$

$$\vec{\nabla}\times\vec{F} = \left(\cancel{\partial_y F_3}^{0} - \overbrace{\partial_z F_2}^{1}\right)\hat{\imath} + \left(\cancel{\partial_z F_1}^{0} - \overbrace{\partial_x F_3}^{1}\right)\hat{\jmath} + \left(\cancel{\partial_x F_2}^{0} - \overbrace{\partial_y F_1}^{1}\right)\hat{k}$$

$\boxed{\vec{\nabla}\cdot\vec{F} = 0 \quad , \quad \vec{\nabla}\times\vec{F} = -\hat{\imath} - \hat{\jmath} - \hat{k}}$

c. $\vec{F}(x,y,z) = x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 2x + 2y + 2z$$

$$\vec{\nabla}\times\vec{F} = \left(\cancel{\partial_y F_3}^{0} - \cancel{\partial_z F_2}^{0}\right)\hat{\imath} + \left(\cancel{\partial_z F_1}^{0} - \cancel{\partial_x F_3}^{0}\right)\hat{\jmath} + \left(\cancel{\partial_x F_2}^{0} - \cancel{\partial_y F_1}^{0}\right)\hat{k}$$

$$= 0$$

$\therefore$ $\boxed{\begin{array}{l}\vec{\nabla}\cdot\vec{F} = 2(x+y+z) \\ \vec{\nabla}\times\vec{F} = \vec{0}\end{array}}$

d.

$$\vec{F}(x,y,z) = -\frac{x}{z}\,\hat{i} - \frac{y}{z}\,\hat{j} + \frac{1}{z}\,\hat{k}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(-\frac{x}{z}\right) + \frac{\partial}{\partial y}\left(-\frac{y}{z}\right) + \frac{\partial}{\partial z}\left(\frac{1}{z}\right)$$

$$= -\frac{1}{z} - \frac{1}{z} - \frac{1}{z^2}$$

$$= -\frac{2}{z} - \frac{1}{z^2}$$

$$\vec{\nabla}\times\vec{F} = (\cancelto{0}{\partial_y F_3} - \partial_z F_y)\,\hat{i} + (\partial_z F_1 - \cancelto{0}{\partial_x F_3})\,\hat{j} + (\cancelto{0}{\partial_x F_2} - \cancelto{0}{\partial_y F_x})\,\hat{k}$$

$$= -\left(\frac{y}{z^2}\right)\hat{i} + \frac{x}{z^2}\,\hat{j}$$

$\therefore$

$$\boxed{\vec{\nabla}\cdot\vec{F} = -\frac{2}{z} - \frac{1}{z^2} \qquad \vec{\nabla}\times\vec{F} = -\frac{y}{z^2}\,\hat{i} + \frac{x}{z^2}\,\hat{j}}$$

e.

$$\vec{F}(x,y,z) = e^x\cos y\,\hat{i} + e^x\sin y\,\hat{j} + z\,\hat{k}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(e^x\cos y) + \frac{\partial}{\partial y}(e^x\sin y) + \frac{\partial}{\partial z} z$$

$$= e^x\cos y + e^x\cos y + 1$$

$$\vec{\nabla}\times\vec{F} = \left[\cancelto{0}{\frac{\partial}{\partial y} z} - \cancelto{0}{\frac{\partial}{\partial z}(e^x\sin y)}\right]\hat{i} + \left[\cancelto{0}{\frac{\partial}{\partial z}(e^x\cos y)} - \cancelto{0}{\frac{\partial}{\partial x} z}\right]\hat{j} + \left[\frac{\partial}{\partial x}(e^x\sin y) - \frac{\partial}{\partial y}(e^x\cos y)\right]\hat{k}$$

$$= \left[e^x\sin y + e^x\sin y\right]\hat{k}$$

$$= 2e^x\sin y\,\hat{k}$$

$\therefore$

$$\boxed{\vec{\nabla}\cdot\vec{F} = 2e^x\cos y + 1 \qquad \vec{\nabla}\times\vec{F} = 2e^x\sin y\,\hat{k}}$$

2.

a) $$\left.\begin{array}{l} \text{Dom } \vec{F} = \mathbb{R}^2 \\ \text{Se } \dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y} \end{array}\right\} \Rightarrow \vec{F} = \nabla\phi$$

$$\vec{F} = e^y \hat{i} + (xe^y + y)\hat{j}$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(xe^y + y) = e^y$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y} e^y = e^y$$

Aqui $$\left.\begin{array}{l} \dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y} \\ \text{Dom } \vec{F} = \mathbb{R}^2 \end{array}\right\} \Rightarrow$$

$$\vec{F} = \nabla\phi$$

$$\therefore \left\{\begin{array}{l} e^y = \dfrac{\partial \phi}{\partial x} \\ xe^y + y = \dfrac{\partial \phi}{\partial y} \end{array}\right.$$

$$e^y = \frac{\partial \phi}{\partial x} \rightarrow \phi = e^y x + C(y)$$

$$\frac{\partial \phi}{\partial y} = \cancel{e^y x} + \frac{dC(y)}{dy} = \cancel{xe^y} + y$$

$$\frac{dC(y)}{dy} = y \Rightarrow C(y) = \frac{y^2}{2} + K$$

$$\therefore \boxed{\vec{F} = \nabla\left(e^y x + \frac{y^2}{2} + K\right)}$$

b. $$\vec{F}(x,y) = \sin xy\, \hat{i} + \cos xy\, \hat{j}$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(\cos xy) = -y \sin xy$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(\sin xy) = x \cos xy$$

Aqui, $\dfrac{\partial F_2}{\partial x} \neq \dfrac{\partial F_1}{\partial y}$

$$\therefore \nexists\, \phi \text{ tq. } \vec{F} = \nabla\phi$$

c. $$\vec{F}(x,y,z) = \overbrace{2xyz}^{F_1}\,\hat{i} + \overbrace{x^2 z}^{F_2}\,\hat{j} + \overbrace{(x^2y+1)}^{F_3}\,\hat{k}$$

Temos

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} :$$

$$x^2 = x^2$$

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$$

$$2xy = 2xy$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

$$2xz = 2xz$$

Temos que

$$\vec{F} = \nabla \phi .$$

Agora :

$$\frac{\partial \phi}{\partial x} = F_1 = 2xyz$$

$$\therefore \quad \phi = x^2yz + C(y,z)$$

$$\frac{\partial \phi}{\partial y} = \cancel{x^2z} + \frac{\partial}{\partial y}C \equiv \cancel{x^2z}$$

$$\frac{\partial C}{\partial y} = 0 \Rightarrow C = C(z)$$

$$\therefore \quad \phi = x^2yz + C(z)$$

$$\frac{\partial \phi}{\partial z} = \cancel{x^2y} + \frac{dC}{dz} = \cancel{xy} + 1$$

$$\frac{dC}{dz} = 1 \Rightarrow C = z + K$$

$$\therefore \quad \boxed{\phi(x,y,z) = x^2yz + z + K.}$$

$$\boxed{\vec{F} = \nabla \phi}$$

d. $\vec{F}(x,y,z) = xz\,\hat{i} + yz\,\hat{j} + xz\,\hat{k}$

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

$y = 0 \Rightarrow$ não é verificada

$$\therefore \quad \boxed{\vec{F} \neq \nabla \phi}$$

e. $\vec{F}(x,y,z) = (y^2+x^2)\,\hat{i} + (z^2+y^2)\,\hat{j} + (x^2+z^2)\,\hat{k}$

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

$2z = 0 \Rightarrow$ não é verificada

$$\therefore \quad \boxed{\vec{F} \neq \nabla \phi}$$

3.

a) $\nabla(fg)$: campo vetorial

b) $\nabla \vec{F}$ não tem sentido

c) $\nabla \times (\nabla f)$: campo vetorial

d) $\nabla(\nabla \cdot \vec{F})$: campo vet.

e) $\nabla \times (\nabla \times \vec{F})$: campo vet.

f) $\vec{\nabla} \cdot (\nabla f)$: campo escalar

g) $(\nabla f) \times (\nabla \times \vec{F})$: campo vetorial

h) $\vec{\nabla} \cdot (\vec{\nabla} \times (\nabla f))$: campo escalar

i) $\nabla \times (\vec{\nabla} \cdot (\nabla f))$: não tem sentido

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4)

$$\vec{\nabla} \cdot (\vec{\nabla} \times F) =$$

$$= \vec{\nabla} \cdot \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k} \right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$= \cancel{\frac{\partial^2 F_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 F_2}{\partial x \partial z}} + \cancel{\frac{\partial^2 F_1}{\partial y \partial z}} - \cancel{\frac{\partial^2 F_3}{\partial y \partial x}} + \cancel{\frac{\partial^2 F_2}{\partial z \partial x}} - \cancel{\frac{\partial^2 F_1}{\partial z \partial y}}$$

$$= 0$$

$$\therefore \boxed{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0}$$

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5) $\nabla \times (\nabla \varphi) = 0$.

De fato:

$$\nabla \times \left(\frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} \right)$$

$$= \left(\frac{\partial}{\partial y}\frac{\partial \varphi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial \varphi}{\partial y} \right) \hat{i} + \left(\frac{\partial}{\partial z}\frac{\partial \varphi}{\partial x} - \frac{\partial}{\partial x}\frac{\partial \varphi}{\partial z} \right) \hat{j} + \left(\frac{\partial}{\partial x}\frac{\partial \varphi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial \varphi}{\partial x} \right) \hat{k}$$

6.

$$\nabla \cdot (f\vec{F}) = \partial_x(fF_1) + $$

$$+ \partial_y(fF_2) + \partial_z(fF_3)$$

$$= \partial_x f\, F_1 + f\,\partial_x F_1$$

$$+ \partial_y f\, F_2 + f\,\partial_y F_2$$

$$+ \partial_z f\, F_3 + f\,\partial_z F_3$$

$$= \nabla f \cdot \vec{F} + f\,\vec{\nabla}\cdot\vec{F}$$

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7. $\vec{\nabla}\cdot(\vec{F}\times\vec{G}) =$

10.

$$\nabla f \times \nabla g =$$

$$= \left(\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}\right) \times$$

$$\times \left(\frac{\partial g}{\partial x}\hat{\imath} + \frac{\partial g}{\partial y}\hat{\jmath} + \frac{\partial g}{\partial z}\hat{k}\right)$$

$$= \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}\hat{k} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial z}\hat{\jmath}$$

$$- \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\hat{k} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial z}\hat{\imath}$$

$$+ \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\hat{\jmath} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\hat{\imath}$$

$$= \left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right)\hat{\imath}$$

$$+ \left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial z}\right)\hat{\jmath}$$

$$+ \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)\hat{k}$$

Agora

$$\vec{\nabla} \cdot (\nabla f \times \nabla g) =$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right) +$$

$$+ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial z}\right)$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right) =$$

$$= \cancel{\frac{\partial^2 f}{\partial x \partial y}\frac{\partial g}{\partial z}} + \frac{\partial f}{\partial y}\cancel{\frac{\partial^2 g}{\partial x \partial z}}$$

$$- \cancel{\frac{\partial^2 f}{\partial x \partial z}\frac{\partial g}{\partial y}} - \frac{\partial f}{\partial z}\cancel{\frac{\partial^2 g}{\partial x \partial y}}$$

$$+ \cancel{\frac{\partial^2 f}{\partial y \partial z}\frac{\partial g}{\partial x}} + \frac{\partial f}{\partial z}\cancel{\frac{\partial^2 g}{\partial y \partial x}}$$

$$- \cancel{\frac{\partial^2 f}{\partial y \partial x}\frac{\partial g}{\partial z}} - \frac{\partial f}{\partial x}\cancel{\frac{\partial^2 g}{\partial y \partial z}}$$

$$+ \cancel{\frac{\partial^2 f}{\partial z \partial x}\frac{\partial g}{\partial y}} + \frac{\partial f}{\partial x}\cancel{\frac{\partial^2 g}{\partial z \partial y}}$$

$$- \cancel{\frac{\partial^2 f}{\partial z \partial y}\frac{\partial g}{\partial x}} - \frac{\partial f}{\partial y}\cancel{\frac{\partial^2 g}{\partial z \partial x}}$$

$$= 0$$

$$\therefore \quad \vec{\nabla} \cdot (\nabla f \times \nabla g) = 0$$

$$\therefore \quad \boxed{\nabla f \times \nabla g \text{ é solenoidal}}$$

11. $\vec{F}(x,y,z) = F_1(y,z)\,\hat{i} + F_2(x,z)\,\hat{j} + F_3(x,y)\,\hat{k}$

$$\vec{\nabla}\cdot\vec{F} = \cancel{\partial_x F_1(y,z)}^{\,0} + \cancel{\partial_y F_2(x,z)}^{\,0} + \cancel{\partial_z F_3(x,y)}^{\,0}$$

$$= 0$$

$\therefore$ $\vec{F}$ é solenoidal

12. a) $\vec{F} = (x^2+2y+az)\,\hat{i} + (bx-3y-z)\,\hat{j} + (4x+cy+2z)\,\hat{k}$

$$\vec{\nabla}\times\vec{F} = \left[\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right]\hat{i}$$

123
231
312

$$+\left[\frac{\partial}{\partial z}(x^2+2y+az) - \frac{\partial}{\partial x}(4x+cy+2z)\right]\hat{j}$$

$$+\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x^2+2y+az)\right]\hat{k}$$

$$\vec{\nabla}\times\vec{F} = (c+1)\,\hat{i} + (a-4)\,\hat{j} + (b-2)\,\hat{k}$$

$$\vec{\nabla}\times\vec{F} = 0 \Rightarrow \boxed{\begin{array}{l} c=-1 \\ a=4 \\ b=2 \end{array}}$$

13. $\begin{cases} \vec{v} = \vec{\omega}\times\vec{r} \;;\; \vec{\omega} = \text{cte} \\ \vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} \end{cases}$

$$\vec{\nabla}\times\vec{v} = \vec{\nabla}\times(\vec{\omega}\times\vec{r})$$

$$= \vec{\nabla}\times\Big((\omega_2 z - \omega_3 y)\,\hat{i} + (\omega_3 x - z\omega_1)\,\hat{j} + (\omega_1 y - \omega_2 x)\,\hat{k}\Big)$$

$$= \big[\partial_y(\omega_1 y - \omega_2 x) - \partial_z(\omega_3 x - z\omega_1)\big]\,\hat{i}$$

$$+ \big[\partial_z(\omega_2 z - \omega_3 y) - \partial_x(\omega_1 y - \omega_2 x)\big]\,\hat{j}$$

$$+ \big[\partial_x(\omega_3 x - z\omega_1) - \partial_y(\omega_2 z - \omega_3 y)\big]\,\hat{k}$$

$$= [\omega_1 - (-\omega_1)]\,\hat{i} + [\omega_2 - (-\omega_2)]\,\hat{j} + [\omega_3 - (-\omega_3)]\,\hat{k}$$

13: Cont.

$$\nabla \times \vec{v} = 2\omega_1 \hat{i} + 2\omega_2 \hat{j} + 2\omega_3 \hat{k}$$
$$= 2\vec{\omega}$$
$$\therefore \boxed{\vec{\omega} = \frac{1}{2} \vec{\nabla} \times \vec{v}}$$

14.

$$f(x,y,z)$$
$$(*)\quad \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$// \nabla \times \nabla f = \left(\cancel{\frac{\partial}{\partial y}\frac{\partial f}{\partial z}} - \cancel{\frac{\partial}{\partial z}\frac{\partial f}{\partial y}}\right)\hat{i}$$
$$+ \left(\cancel{\frac{\partial}{\partial z}\frac{\partial f}{\partial x}} - \cancel{\frac{\partial}{\partial x}\frac{\partial f}{\partial z}}\right)\hat{j}$$
$$+ \left(\cancel{\frac{\partial}{\partial x}\frac{\partial f}{\partial y}} - \cancel{\frac{\partial}{\partial y}\frac{\partial f}{\partial x}}\right)$$
$$= 0 //$$

$$\vec{\nabla} \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$
$$= 0 \quad \text{do } (*)$$

$$\therefore // \vec{\nabla} \cdot \nabla f = 0 //$$

15.

$$\vec{F}(x,y) = \frac{y}{x^2+y^2}\hat{i} - \frac{x}{x^2+y^2}\hat{j}$$

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 < x^2+y^2 < 1\}$$

i)
$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$
$$= \frac{1}{x^2+y^2} - \frac{y\,2y}{(x^2+y^2)^2}$$
$$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$
$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right)$$
$$= \frac{-1}{x^2+y^2} + \frac{x\,2x}{(x^2+y^2)^2}$$
$$= \frac{-(x^2+y^2)+2x^2}{x^2+y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \text{ em } D$$

15 - cont -

$$\vec{F} = \nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath}$$

$$\Rightarrow \frac{\partial\varphi}{\partial x} = \frac{y}{x^2+y^2}$$

$$\therefore \varphi = \operatorname{arctg}\frac{x}{y} + C(y)$$

$$\frac{d}{dx}\operatorname{arctg} x = \frac{1}{1+x^2}$$

$$\frac{1}{a}\frac{d}{dx}\operatorname{arctg}\frac{x}{a} = \frac{1}{x^2+a^2}$$

$$\frac{\partial\varphi}{\partial y} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(-\frac{x}{y^2}\right) + C'(y)$$

$$\cancel{\frac{-x}{x^2+y^2}} = \cancel{\frac{-x}{x^2+y^2}} + C'(y)$$

$$C'(y) = 0$$

$$\Rightarrow C(y) = cte = K$$

$$\boxed{\varphi = \operatorname{arctg}\frac{x}{y} + K}$$

No entanto φ não está definida para $y=0$

isto é,

$$\vec{F} \neq \nabla\varphi \text{ em } \underline{D}$$

Cálculo C — Lista 3 Resposta

1. a) $\vec{\nabla}\cdot\vec{F} = 2$

$\vec{\nabla}\times\vec{F} = \vec{0}$

b) $\vec{\nabla}\cdot\vec{F} = 0$

$\vec{\nabla}\times\vec{F} = -\hat{i} - \hat{j} - \hat{k}$

c) $\vec{\nabla}\cdot\vec{F} = 2(x+y+z)$

$\vec{\nabla}\times\vec{F} = \vec{0}$

d) $\vec{\nabla}\cdot\vec{F} = -\frac{2}{z} - \frac{1}{z^2}$

$\vec{\nabla}\times\vec{F} = -\frac{y}{z^2}\hat{i} + \frac{x}{z^2}\hat{j}$

---

2)

a) $\vec{F} = \nabla\left(e^{4x} + \frac{y^2}{2} + k\right)$

b) Não existe ϕ

c) $\vec{F} = \nabla(x^2yz + z + k)$

d) Não existe ϕ

e) Não existe ϕ

3. a) Campo vetorial

b) Não tem sentido

c) Campo vetorial

d) Campo vetorial

e) Campo vetorial

f) Campo escalar

g) Campo vetorial

h) Campo escalar

i) Não tem sentido

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12. $c = -1$, $a = 4$, $b = 2$