Cálculo C - Lista 4

Integrais de Linha

Calcule as integrais de linha a seguir

1. $\int_\gamma (9+8y^{\frac{1}{2}})\,ds$. γ é a curva parametrizada por $\vec{r}(t) = 2t^{\frac{3}{2}}\vec{i} + t^2\vec{j}$, $0 \le t \le 1$.

2. $\int_\gamma y\,ds$. γ é a curva parametrizada por $\vec{r}(t) = t\vec{i} + t^3\vec{j}$, $-1 \le t \le 0$.

3. $\int_\gamma 2xyz\ ds$. γ é a curva parametrizada por $\vec{r}(t) = e^t\vec{i} + e^{-t}\vec{j} + \sqrt{2}t\vec{k}$, $0 \le t \le 1$

4. $\int_\gamma (1+\frac{9}{4}z^{\frac{2}{3}})^{\frac{1}{4}}\ ds$. γ é a curva parametrizada por $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t^{\frac{3}{2}}\vec{k}$, $0 \le t \le \frac{20}{3}$.

5. $\int_\gamma (y+2z)\ ds$. γ é a trajetória triangular formada pelos segmentos de $(-1,0,0)$ a $(0,1,0)$, $(0,1,0)$ a $(0,0,1)$ e $(0,0,1)$ a $(-1,0,0)$.

Nos exercícios a seguir calcule $\int_\gamma \vec{F}\cdot d\vec{r}$ onde γ é a curva parametrizada por $\vec{r}(t)$.

6. $\vec{F} = z\vec{i} - y\vec{j} - x\vec{k}$. $\vec{r}(t) = 5\vec{i} - \sin t\vec{j} - \cos t\vec{k}$, $0 \le t \le \frac{\pi}{4}$.

7. $\vec{F} = y\vec{i} + xy\vec{j} + z^3\vec{k}$. $\vec{r} = \cos t\vec{i} + \sin t\vec{j} + 2t\vec{k}$, $0 \le t \le \frac{\pi}{2}$.

8. $\vec{F} = -z\vec{i} + x\vec{k}$. $\vec{r} = \cos(\pi - t)\vec{i} + \sin(\pi - t)\vec{k}$, $0 \le t \le \pi$.

9. $\vec{F} = 5e^{\sin \pi x}\vec{i} - 4e^{\cos \pi x}\vec{j}$. $\vec{r} = \frac{1}{2}\vec{i} + 2\vec{j} - \ln(\cosh t)\vec{k}$, $0 \le t \le \frac{\pi}{6}$.

Calcule as integrais de linha dadas na forma $\int F_1(x,y,z)\,dx + F_2(x,y,z)\,dy + F_3(x,y,z)\,dz$

10. $\int_\gamma ydx - xdy + xyz^2dz$. γ é a curva parametrizada por $\vec{r}(t) = e^{-t}\vec{i} + e^t\vec{j} + t\vec{k}$, $0 \le t \le 1$.

11. $\int_\gamma e^x dx + xydy + xyzdz$. γ é a curva parametrizada por $\vec{r}(t) = (2-t)\vec{i} + (2-t)\vec{j} + 2(2-t)\vec{k}$, $-1 \le t \le 1$. corrigir

12. $\int_\gamma xydx + (x+z)dy + z^2dz$. γ é a curva parametrizada por $\vec{r} = (t+1)\vec{i} + (t-1)\vec{j} + t^2\vec{k}$, $-1 \le t \le 2$.

13. $\int_\gamma \frac{1}{1+x^2}dx + \frac{2}{1+y^2}dy$. γ é a parte do círculo unitário passando pelos pontos $(1,0), (0,1)$.

14. $\int_\gamma x\ln(\frac{xz}{y})dx + \cos(\frac{\pi xy}{z})dy$. γ é a curva parametrizada por $\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$, $1 \le t \le 2$.

15. Calcule $\int_\gamma ydx + zdy + xdz$. γ é a curva parametrizada pelos segmentos de reta de $(0,0,0)$ a $(0,-5,0)$, e de $(0,-5,0)$ a $(0,0,1)$.

16. Seja γ_1 a curva parametrizada por $\vec{r}_1(t) = t\vec{i} + t\vec{j} + t\vec{k}$, $0 \le t \le \frac{1}{2}$ e γ_2 a curva parametrizada por $\vec{r}(t) = \sin t\vec{i} + \sin t\vec{j} + \sin t\vec{k}$, $0 \le t \le \frac{\pi}{6}$. Calcule
$$\int_{\gamma_1} (xy+z)\,ds \quad \text{e} \quad \int_{\gamma_2} (xy+z)\,ds$$
As respostas são iguais? Explique.

17. Seja uma curva parametrizada por $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $a \le t \le b$, tal que $x(t) = k$ é constante. Seja $M(x,y,z)$ uma função contínua. Calcule
$$\int_\gamma M(x,y,z)dx$$

Cálculo C - Lista 4

1. $$\int_\gamma (9+8y^{1/2})\, ds$$
$\varphi(x,y,z)$

$$\begin{cases} \gamma : \vec{r}(t) = 2t^{3/2}\,\hat{i} + t^2\,\hat{j} \\ 0 \leq t \leq 1 \end{cases}$$

$$\begin{cases} \vec{r}\,'(t) = 3t^{1/2}\,\hat{i} + 2t\,\hat{j} \\ |\vec{r}\,'(t)| = \sqrt{9t + 4t^2} \end{cases}$$

$x(t) = 2t^{3/2}$, $y(t) = t^2$

$\therefore \quad ds = |\vec{r}\,'(t)|\, dt$

$$\int_\gamma (9+8y^{1/2})\, ds =$$

$$= \int_0^1 \left(9 + 8(t^2)^{1/2}\right) \sqrt{9t+4t^2}\, dt$$

$$= \int_0^1 (9 + 8t) \sqrt{9t+4t^2}\, dt$$

$$= \frac{2}{3}(9t+4t^2)^{3/2} \Big|_0^1$$

$$= \frac{2}{3}(9+4)^{3/2}$$

$$= \frac{2}{3}(13)^{3/2} = \frac{26\sqrt{13}}{3}$$

2. $$\int_\gamma y\, ds$$

$$\gamma : \begin{cases} \vec{r}(t) = t\,\hat{i} + t^3\,\hat{j} \\ -1 \leq t \leq 0 \end{cases}$$

$$\begin{cases} \vec{r}\,'(t) = \hat{i} + 3t^2\,\hat{j} \\ |\vec{r}\,'(t)| = \sqrt{1+9t^4} \end{cases}$$

$x(t) = t$, $y(t) = t^3$

$$\int_\gamma y\, ds = \int_{-1}^0 t^3 \sqrt{1+9t^4}\, dt$$

$$= \frac{1}{36}\,\frac{2}{3}(1+9t^4)^{3/2} \Big|_{-1}^0$$

$$= \frac{1}{54}\left(1^{3/2} - 10^{3/2}\right)$$

$$= \underline{\frac{1}{54}\left(1 - 10^{3/2}\right)}$$

$u = 9t + 4t^2$
$du = (9+8t)\, dt$

3. $\displaystyle\int_\gamma 2xyz\,ds$

$$\gamma : \begin{cases} \vec{r}(t) = e^t\,\hat{i} + e^{-t}\,\hat{j} + \sqrt{2}t\,\hat{k} \\ 0 \leq t \leq 1 \end{cases}$$

$x(t) = e^t, \quad y(t) = e^{-t}, \quad z(t) = \sqrt{2}t$

$$\begin{cases} \vec{r}\,'(t) = e^t\,\hat{i} - e^{-t}\,\hat{j} + \sqrt{2}\,\hat{k} \\ |\vec{r}\,'(t)| = \sqrt{e^{2t} + e^{-2t} + 2} \\ \qquad = \sqrt{(e^t + e^{-t})^2} \\ \qquad = e^t + e^{-t} \end{cases}$$

$\therefore$

$$\int_\gamma 2xyz\,ds = \int_0^1 2\,e^t e^{-t}\,\sqrt{2}t \cdot (e^t + e^{-t})\,dt$$

1

$$= \int_0^1 \left(2\sqrt{2}t\,e^t + 2\sqrt{2}t\,e^{-t}\right)dt$$

Mas

$$\int t\,e^t\,dt = te^t - \int e^t\,dt$$

$u = t \to du = dt$

$dv = e^t dt \to v = e^t$

$$= te^t - e^t$$

$$= (t-1)\,e^t$$

$\therefore$

$$\int_0^1 2\sqrt{2}\,t\,e^t\,dt = 2\sqrt{2}\Big[(t-1)e^t\Big]_0^1$$

$$= 2\sqrt{2}\left(0 - (-1)e^0\right)$$

$$= 2\sqrt{2}$$

também

$$\int t\,e^{-t}\,dt = (-t-1)\,e^{-t}$$

$\therefore$

$$\int_0^1 2\sqrt{2}\,t\,e^{-t}\,dt = 2\sqrt{2}\,(-t-1)\,e^{-t}\Big|_0^1$$

$$= 2\sqrt{2}\left[(-1-1)\,e^{-1} - (-1)e^0\right]$$

$$= 2\sqrt{2}\left[-2e^{-1} + 1\right]$$

$$= 2\sqrt{2}\left(-\frac{2}{e} + 1\right)$$

$$= -\frac{4\sqrt{2}}{e} + 2\sqrt{2}$$

$\therefore$

$$\int_\gamma 2xyz\,ds = 2\sqrt{2} - \frac{4\sqrt{2}}{e} + 2\sqrt{2}$$

$$= 4\sqrt{2} - \frac{4\sqrt{2}}{e}$$

$$= 4\sqrt{2}\left(1 - \frac{1}{e}\right)$$

4. $\int_\gamma \left(1+\frac{9}{4} z^{2/3}\right)^{\frac{1}{4}} ds$

$$\gamma : \begin{cases} \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t^{3/2}\,\hat{k} \\ 0 \le t \le \frac{20}{3} \end{cases}$$

$x(t) = \cos t, \quad y(t) = \sin t, \quad z(t) = t^{3/2}$

$$\begin{cases} \vec{r}\,'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + \frac{3}{2} t^{1/2}\,\hat{k} \\ |\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + \frac{9}{4} t} \\ \qquad = \sqrt{1 + \frac{9}{4} t} \end{cases}$$

$$\int_\gamma \left(1+\frac{9}{4} z^{2/3}\right)^{\frac{1}{4}} ds =$$

$$= \int_0^{\frac{20}{3}} \left(1+\frac{9}{4}t\right)^{\frac{1}{4}} \sqrt{1+\frac{9}{4}t}\; dt$$

$\frac{1}{4}+\frac{1}{2} =$

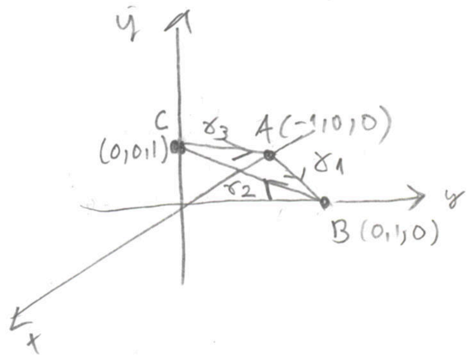
$$= \int_0^{\frac{20}{3}} \left(1+\frac{9}{4}t\right)^{\frac{3}{4}} dt$$

$$= \frac{\left(1+\frac{9}{4}t\right)^{\frac{7}{4}}}{\frac{7}{4}} \; \frac{4}{9} \Bigg]_0^{\frac{20}{3}}$$

$$= \frac{16}{63}\left(1+\frac{\cancel{9}^{3}}{\cancel{4}}\cdot\frac{\cancel{20}^{5}}{\cancel{3}}\right)^{\frac{7}{4}} - \frac{16}{63}$$

$$= \frac{16}{63}\left(16^{7/4} - 1\right) = \frac{16}{63}\left(2^7 - 1\right) = \frac{16}{63}(128-1) = \frac{16 \cdot 127}{63} = \boxed{\frac{2032}{63}}$$

5. $\int_\gamma (y+2z)\,ds$



$$\gamma_1 : \begin{cases} \vec{m}_1 = (1,1,0) \\ A = (-1,0,0) \\ \vec{r}_1(t) = (-1+t)\,\hat{i} + t\,\hat{j} \\ 0 \le t \le 1 \end{cases} \quad \underline{(1)}$$

$$\gamma_2 : \begin{cases} \vec{m}_2 = (0,-1,1) \\ B = (0,1,0) \\ \vec{r}_2(t) = (1-t)\,\hat{j} + t\,\hat{k} \\ 0 \le t \le 1 \;\rightarrow\; 1 \le t+1 \le 2 \end{cases}$$

$\tilde{t} = t+1$

$\vec{r}_2(\tilde{t}) = (2-\tilde{t})\,\hat{j} + (\tilde{t}-1)\,\hat{k}$

$1 \le \tilde{t} \le 2$ isto é:

$$\therefore \quad \underline{(2)} \begin{cases} \vec{r}_2(t) = (2-t)\,\hat{j} + (t-1)\,\hat{k} \\ 1 \le t \le 2 \end{cases}$$

5. Cont.

$$\gamma_3 : \begin{cases} \vec{n}_3 = (-1,0,-1) \\ C = (0,0,1) \\ \vec{r}_3(t) = -t\,\hat{i} + (1-t)\,\hat{k} \\ 0 \le t \le 1 \;\rightarrow\; 2 \le t+2 \le 3, \quad \tilde{t} = t+2 \end{cases}$$

$$\therefore \quad \vec{r}_3(\tilde{t}) = -(\tilde{t}-2)\,\hat{i} + (3-\tilde{t})\,\hat{k}$$

$$2 \le \tilde{t} \le 3$$

isto é

$$\overset{(3)}{=} \begin{cases} \vec{r}_3(t) = (2-t)\,\hat{i} + (3-t)\,\hat{k} \\ 2 \le t \le 3 \end{cases}$$

Daí,

$$\vec{r}'(t) = \begin{cases} \hat{i} + \hat{j}, & 0 \le t \le 1 \\ -\hat{j} + \hat{k}, & 1 \le t \le 2 \\ -\hat{i} - \hat{k} & 2 \le t \le 3 \end{cases}$$

$$\int_\gamma (y+2z)\,ds = \int_{\gamma_1} (y+2z)\,ds + \int_{\gamma_2} (y+2z)\,ds + \int_{\gamma_3} (y+2z)\,ds$$

Mas

$$\int_{\gamma_1} (y+2z)\,ds = \int_0^1 t\sqrt{2}\,dt = \sqrt{2}\,\frac{t^2}{2}\Big|_0^1 = \frac{\sqrt{2}}{2}$$

em $\vec{r}_1(t)$:

$$\begin{cases} y(t) = t \\ x(t) = -1+t \\ z(t) = 0 \end{cases}$$

$$|\vec{r}_1{}'(t)| = \sqrt{2}$$

$$\int_{\gamma_2} (y+2z)\,ds = \int_1^2 \left[(2-t) + 2(t-1)\right]\sqrt{2} = \sqrt{2}\left(-\frac{(2-t)^2}{2} + \frac{\cancel{2}(t-1)^2}{\cancel{2}}\right)\Big|_1^2 = \sqrt{2}\left(1 - \left(-\frac{(2-1)^2}{2}\right)\right) = \sqrt{2}\left(1+\frac{1}{2}\right) = \frac{3\sqrt{2}}{2}$$

em $\vec{r}_2(t)$:

$$\begin{cases} y(t) = 2-t \\ z(t) = t-1 \end{cases}$$

$$|\vec{r}_2{}'(t)| = \sqrt{2}$$

$$\int_{\gamma_3} (y+2z)\,ds = \int_2^3 2(3-t)\sqrt{2}\,dt = \cancel{2}\sqrt{2}\,(-)\frac{(3-t)^2}{\cancel{2}}\Big|_2^3 = -\sqrt{2}(\cancel{3-3})^2 + \sqrt{2} = \sqrt{2}$$

em $\vec{r}_3(t)$:

$x(t) = 2-t$
$y(t) = 0$
$z(t) = 3-t$

$$|\vec{r}_3{}'(t)| = \sqrt{2}$$

$$\therefore \int_\gamma (y+2z)\,ds = \frac{\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} + \sqrt{2} = \boxed{3\sqrt{2}}$$

6.

$\gamma: \begin{cases} \vec{r}(t) = 5\hat{i} - \sin t\,\hat{j} - \cos t\,\hat{k} \\ 0 \leq t \leq \frac{\pi}{4} \\ x(t) = 5 \\ y(t) = -\sin t \\ z(t) = -\cos t \end{cases}$

$\vec{F}(x,y,z) = z\hat{i} - y\hat{j} - x\hat{k}$

$\vec{F}$ tomada sobre a curva γ tem

$$\vec{F}(t) = \vec{F}(x(t), y(t), z(t))$$
$$= -\cos t\,\hat{i} + \sin t\,\hat{j} - 5\hat{k}$$

Daí

$$\int_\gamma \vec{F}\cdot d\vec{r} = \int_0^{\frac{\pi}{4}} \vec{F}\cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^{\frac{\pi}{4}} (-\cos t\,\hat{i} + \sin t\,\hat{j} - 5\hat{k})\cdot(-\cos t\,\hat{j} + \sin t\,\hat{k})\,dt$$

$$= \int_0^{\frac{\pi}{4}} (-\sin t \cos t - 5\sin t)\,dt$$

$$= -\frac{\sin^2 t}{2} + 5\cos t\Big]_0^{\frac{\pi}{4}}$$

$$= -\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)^2 + 5\frac{\sqrt{2}}{2} - (-0+5) = -\frac{1}{2}\,\frac{1}{2} + \frac{5\sqrt{2}}{2} - 5 = \boxed{\frac{5\sqrt{2}}{2} - \frac{21}{4}}$$

7.

$\gamma: \begin{cases} \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 2t\,\hat{k} \\ 0 \leq t \leq \frac{\pi}{2} \end{cases}$

$x(t) = \cos t$
$y(t) = \sin t$
$z(t) = 2t$

$\vec{F}(x,y,z) = y\hat{i} + xy\hat{j} + z^3\hat{k}$

$\vec{F}$ sobre γ assume a forma

$$\vec{F}(t) = \vec{F}(x(t), y(t), z(t))$$
$$= \sin t\,\hat{i} + \cos t \sin t\,\hat{j} + 8t^3\hat{k}$$

$$\int_\gamma \vec{F}\cdot d\vec{r} = \int_0^{\frac{\pi}{2}} (\sin t\,\hat{i} + \cos t \sin t\,\hat{j} + 8t^3\hat{k})\cdot(-\sin t\,\hat{i} + \cos t\,\hat{j} + 2\hat{k})\,dt$$

$$= \int_0^{\frac{\pi}{2}} (\underbrace{-\sin^2 t}_{(*)} + \underbrace{\cos^2 t \sin t}_{(**)} + \underbrace{16t^3}_{(***)})\,dt$$

Mas,

$$(*) = \int_0^{\frac{\pi}{2}} -\sin^2 t \, dt$$

$$= \int_0^{\frac{\pi}{2}} -\frac{1-\cos 2t}{2} \, dt$$

$$= -\frac{1}{2}\left(t - \frac{\sin 2t}{2}\right)\Big]_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{2}\left(\frac{\pi}{2}\right) = -\frac{\pi}{4}$$

$$(**) = \int_0^{\frac{\pi}{2}} \cos^2 t \sin t \, dt$$

$$= -\frac{\cos^3 t}{3}\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{3}$$

$$(***) = \int_0^{\frac{\pi}{2}} 16t^3 \, dt$$

$$= 16\frac{t^4}{4}\Big]_0^{\frac{\pi}{2}}$$

$$= 4\left(\frac{\pi}{2}\right)^4 = \frac{4\pi^4}{16} = \frac{\pi^4}{4}$$

$\therefore$

$$\int_\gamma \vec{F}\cdot d\vec{r} = \boxed{-\frac{\pi}{4} + \frac{1}{3} + \frac{\pi^4}{4}}$$

8.

$$\gamma \begin{cases} \vec{r}(t) = \cos(\pi - t)\,\hat{i} + \sin(\pi - t)\,\hat{k} \\ 0 \leq t \leq \pi \\ \therefore \; x(t) = \cos(\pi - t) \\ \quad y(t) = 0 \\ \quad z(t) = \sin(\pi - t) \end{cases}$$

$$\vec{F}(x,y,z) = -z\hat{i} + x\hat{k}$$

Sobre a curva γ:

$$\vec{F}(t) = -\sin(\pi - t)\,\hat{i} + \cos(\pi - t)\,\hat{k}$$

e

$$\int_\gamma \vec{F}\cdot d\vec{r} = \int_0^{\pi} \left(-\sin(\pi - t)\,\hat{i} + \cos(\pi - t)\,\hat{k}\right) \cdot \left(\sin(\pi - t)\,\hat{i} - \cos(\pi - t)\,\hat{k}\right) dt$$

$$= \int_0^{\pi} \left[-\sin^2(\pi - t) - \cos^2(\pi - t)\right] dt$$

$$= \int_0^{\pi} -dt = -\pi$$

9.

$$\gamma: \begin{cases} \vec{r}(t) = \frac{1}{2}\hat{i} + 2\hat{j} - \ln(\cosh t)\hat{k} \\ 0 \le t \le \frac{\pi}{6} \end{cases}$$

$x(t) = \frac{1}{2}$

$y(t) = 2$

$z(t) = -\ln(\cosh t)$

$$\vec{F}(x,y,z) = 5e^{\sin \pi x}\hat{i} - 4e^{\cos \pi x}\hat{j}$$

em γ temos

$$\vec{F}(t) \equiv \vec{F}(x(t), y(t), z(t)) =$$

$$= 5e^{\sin \pi \frac{1}{2}}\hat{i} - 4e^{\cos \pi \frac{1}{2}}\hat{j}$$

$$= 5e\hat{i} - 4\hat{j}$$

$$\vec{r}\,'(t) = -\frac{1}{\cosh t}\frac{d}{dt}\cosh t\,\hat{k}$$

$$= -\frac{1}{\cosh t}\frac{d}{dt}\left(\frac{e^t + e^{-t}}{2}\right)\hat{k}$$

$$= -\frac{1}{\cosh t}\,\frac{e^t - e^{-t}}{2}\hat{k}$$

$$= -\frac{e^t - e^{-t}}{e^t + e^{-t}}\hat{k}$$

$$\int_\gamma \vec{F}\cdot d\vec{r} = \int_0^{\frac{\pi}{6}} \vec{F}\cdot\vec{r}\,'\,dt$$

$$= \int_0^{\frac{\pi}{6}} (5e\hat{i} - 4\hat{j})\cdot\left(-\frac{e^t - e^{-t}}{e^t + e^{-t}}\hat{k}\right)dt$$

$$= \int_0^{\frac{\pi}{6}} 0\,dt = 0$$

10.

$$\gamma: \begin{cases} \vec{r}(t) = e^{-t}\hat{i} + e^{t}\hat{j} + t\hat{k} \\ 0 \le t \le 1 \end{cases}$$

$x(t) = e^{-t}$, $y(t) = e^{t}$, $z(t) = t$

Lembremos a definição

$$\int F_1\,dx + F_2\,dy + F_3\,dz \;:=\; \int\left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right)dt$$

Daí

$$\int_\gamma y\,dx - x\,dy + xyz^2\,dz =$$

$$= \int_\gamma \left(y x' - x y' + x y z^2 z'\right)dt$$

$$= \int_0^1 \left(e^t(-)e^{-t} - e^{-t}e^{t} + e^{-t}e^{t}t^2\right)dt$$

$$= \int_0^1 (-2 + t^2)\, dt$$

$$= -2t + \frac{t^3}{3} \Big]_0^1$$

$$= -2 + \frac{1}{3} = -\frac{5}{3}$$

$$= e^{-t} - \frac{t^3}{3} + t^4 \Big]_{-1}^{1}$$

$$= e^{-1} - \frac{1}{3} + \cancel{1} - \left(e + \frac{1}{3} \cancel{+1}\right)$$

$$= \boxed{e^{-1} - e - \frac{2}{3}}$$

11.

$$\gamma : \begin{cases} \vec{r}(t) = -t\hat{\imath} - t\hat{\jmath} - 2t\hat{k} \\ -1 \leq t \leq 1 \end{cases}$$

$x(t) = -t$

$y(t) = -t$

$z(t) = -2t$

$$\int_\gamma F_1\, dx + F_2\, dy + F_3\, dz$$

$$= \int_\gamma \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt$$

$$\int_\gamma \overbrace{e^x}^{F_1} dx + \overbrace{xy}^{F_2} dy + \overbrace{xyz}^{F_3} dz =$$

$$= \int_{-1}^{1} \left[e^{-t}(-1) + (-t)(-t)(-1) + (-t)(-t)(-2t)(-2) \right] dt$$

$$= \int_{-1}^{1} \left(-e^{-t} - t^2 + 4t^3\right) dt$$

12.

$$\gamma : \begin{cases} \vec{r}(t) = (t+1)\hat{\imath} + (t-1)\hat{\jmath} + t^2\hat{k} \\ -1 \leq t \leq 2 \end{cases}$$

$x(t) = t+1$, $y(t) = t-1$, $z(t) =$

$$\int_\gamma xy \underbrace{dx}_{dt} + (x+z)\underbrace{dy}_{dt} + z^2 \underbrace{dz}_{2t\,dt} =$$

$$= \int_{-1}^{2} \left((t+1)(t-1) + (t+1+t^2) + t^4 2t\right)$$

$$= \int_{-1}^{2} \left(t^2 \cancel{-1} + t + \cancel{1} + t^2 + 2t^5\right) d$$

$$= \int_{-1}^{2} \left(t + 2t^2 + 2t^5\right) dt$$

$$= \frac{t^2}{2} + \frac{2t^3}{3} + \frac{2t^6}{6} \Big]_{-1}^{2}$$

$$= \frac{4}{2} + \frac{2 \cdot 8}{3} + \frac{\cancel{2} \cdot 2^6}{\cancel{6}\,3}$$

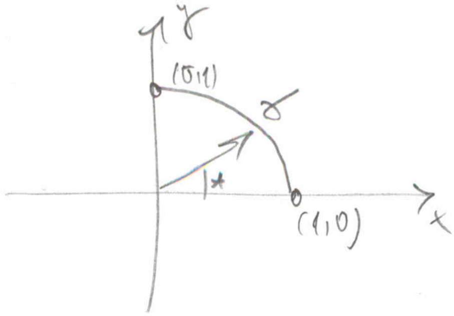
$$- \frac{1}{2} + \frac{2}{3} - \frac{\cancel{2}}{\cancel{6}} \frac{1}{3}$$

$$= 2 + \frac{16}{3} + \frac{64}{3} - \frac{1}{2} + \frac{1}{3} = 2 - \frac{1}{2} + \frac{81}{3} =$$

$$= \boxed{57/2}$$

13.

γ :



$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \\ 0 \le t \le \frac{\pi}{2} \end{cases}$$

$$\int_\gamma \frac{1}{1+x^2} \underbrace{dx}_{-\sin t\, dt} + \frac{2}{1+y^2} \underbrace{dy}_{\cos t\, dt}$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{1}{1+\cos^2 t}(-\sin t) + \frac{2}{1+\sin^2 t} \cos t \right] dt$$

$$= \int_0^{\frac{\pi}{2}} \Big(\underbrace{\frac{-\sin t}{1+\cos^2 t}}_{(*)} + \underbrace{\frac{2\cos t}{1+\sin^2 t}}_{(**)} \Big) dt$$

$$(*) = \int_0^{\frac{\pi}{2}} \frac{-\sin t}{1+\cos^2 t}\, dt$$

$u = \cos t \quad du = -\sin t\, dt$

$\therefore$

$$(*) = \int_1^0 \frac{du}{1+u^2} = \operatorname{arctg} x \Big]_1^0$$

$$= \underbrace{\operatorname{arctg} 0}_{} - \operatorname{arctg} 1$$

$$= 0 - \frac{\pi}{4}$$

$$= -\frac{\pi}{4}$$

$$(**) = \int_0^{\frac{\pi}{2}} \frac{2\cos t}{1+\sin^2 t}\, dt$$

$u = \sin t \to du = \cos t\, dt$

$$= \int_0^1 \frac{2\,du}{1+u^2}$$

$$= 2 \operatorname{arctg} u \Big]_0^1$$

$$= 2(\operatorname{arctg} 1 - \operatorname{arctg} 0)$$

$$= 2\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi}{2}$$

$\therefore$

$$\int_\gamma \frac{1}{1+x^2} dx + \frac{2}{1+y^2} dy = -\frac{\pi}{4} + \frac{\pi}{2}$$

$$= \boxed{\frac{\pi}{4}}$$

14.

$$\gamma : \begin{cases} \vec{r}(t) = t\,\hat{i} + t^2\,\hat{j} + t^3\,\hat{k} \\ 1 \leq t \leq 2 \end{cases}$$

$x(t) = t$, $y(t) = t^2$, $z(t) = t^3$

$$\int_\gamma x \ln\left(\frac{xz}{y}\right) \underbrace{dx}_{dt} + \cos\frac{\pi x y}{z} \underbrace{dy}_{2t\,dt}$$

$$= \int_1^2 \left(t \ln \frac{t\,t^3}{t^2} + \cos\left(\pi \frac{t\,t^2}{t^3}\right) 2t \right) dt$$

$$= \int_1^2 \left(t \underbrace{\ln t^2}_{2\ln t} + \underbrace{(\cos \pi)}_{-1} 2t \right) dt$$

$$= \int_1^2 \Big(\underbrace{2t \ln t}_{(*)} - \underbrace{2t}_{(**)} \Big) dt$$

$$(*) : \int_1^2 2t \ln t \, dt$$

$u = \ln t \rightarrow du = \frac{1}{t} dt$

$dv = 2t\,dt \rightarrow v = t^2$

$$= (\ln t)\, t^2 - \int t^2 \frac{1}{t} dt$$

$$= t^2 \ln t - \frac{t^2}{2}$$

$\therefore$

$$(*) = \int_1^2 2t \ln t = t^2 \ln t - \frac{t^2}{2} \Big]_1^2$$

$$= 4 \ln 2 - \frac{4}{2} - \left(0 - \frac{1}{2}\right)$$

$$= 4 \ln 2 - 2 + \frac{1}{2}$$

$$= 4 \ln 2 - \frac{3}{2}$$

$$(**) = \int_1^2 -2t \, dt$$

$$= -t^2 \Big]_1^2 = -4 - (-1)$$

$$= -3$$

$\therefore$

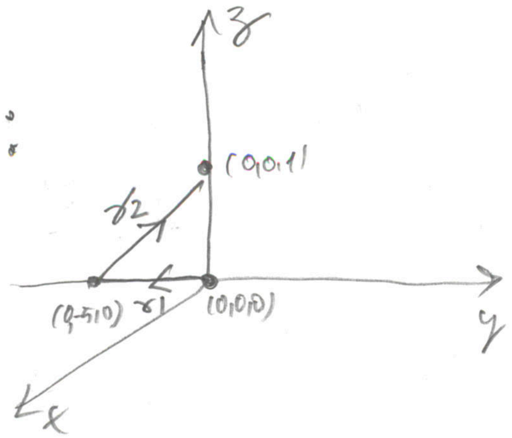
$$\int_\gamma \left(x \ln \frac{xz}{y}\, dx + \cos \frac{\pi x y}{z}\, dy \right) =$$

$$= 4 \ln 2 - \frac{3}{2} - 3$$

$$= \boxed{4 \ln 2 - \frac{9}{2}}$$

15.

γ :



$$\gamma_1 : \begin{cases} \vec{r}_1(t) = -5t\,\hat{j} \ ; \quad 0 \le t \le 1 \end{cases}$$

$$\gamma_2 \equiv \begin{cases} \hat{n} = (0,5,1) \\ P \equiv (0,-5,0) \\ \vec{r}_2(t) \equiv (-5+5t)\,\hat{j} + t\,\hat{k} \\ \quad 0 \le t \le 1 \\ \therefore \quad 1 \le t+1 \le 2 \\ \quad \tilde{t} = t+1 \end{cases}$$

$$\therefore \ \vec{r}_2(\tilde{t}) \equiv (-5+5(\tilde{t}-1))\,\hat{j} + (\tilde{t}-1)\,\hat{k}$$

$$= (-5+5\tilde{t}-5)\,\hat{j} + (\tilde{t}-1)\,\hat{k}$$

$$= (-10+5\tilde{t})\,\hat{j} + (\tilde{t}-1)\,\hat{k}$$

$$\begin{cases} \vec{r}_2(t) = (-10+5t)\,\hat{j} + (t-1)\,\hat{k} \\ 1 \le t \le 2 \end{cases}$$

$$\int_\gamma (y\,dx + z\,dy + x\,dz) =$$

$$= \underbrace{\int_{\gamma_1} y\,dx + z\,dy + x\,dz}_{(*)} + \underbrace{\int_{\gamma_2} y\,dx + z\,dy + x\,dz}_{(**)}$$

Cálculo de (*) :

$$\gamma_1 : \begin{cases} \vec{r}_1(t) = -5t\,\hat{j} \\ x(t) = 0 \ , \ y(t) = -5t \ , \ z(t) = 0 \\ 0 \le t \le 1 \end{cases}$$

$$\therefore \int_{\gamma_1} y\,\underbrace{dx}_{0} + \underbrace{z}_{0}\,dy + \underbrace{x}_{0}\,dz = \underline{\underline{0}}$$

Cálculo de (**)

$$\gamma_2 : \ \vec{r}_2(t) = (-10+5t)\,\hat{j} + (t-1)\,\hat{k}$$

$$\begin{cases} x(t) = -10+5t \\ y(t) = t-1 \\ 1 \le t \le 2 \end{cases}$$

$$\int_{\gamma_2} y\,dx + \cancel{z}^{0}\,dy + x\,\cancel{dz}^{0} =$$

$$= \int_1^2 (t-1)\,5\,dt = 5\,\frac{(t-1)^2}{2}\Big]_1^2$$

$$= \underline{\underline{5}}$$

$\therefore \int y\,dx + z\,dy$ ...

b)

$$\gamma_1 : \begin{cases} \bar{r}_1(t) = t\hat{i} + t\hat{j} + t\hat{k} \\ 0 \le t \le \frac{1}{2} \end{cases}$$

$$x(t) = y(t) = z(t) = t$$

$$\int_{\gamma_1} (xy + z)\, ds =$$

$$= \int_0^{1/2} (t^2 + t) \; |\bar{r}_1'(t)| \, dt$$

$$= \int_0^{1/2} (t^2 + t) \; \sqrt{3}\, dt$$

$$= \sqrt{3} \left(\frac{t^3}{3} + \frac{t^2}{2} \right) \Big|_0^{1/2}$$

$$= \sqrt{3} \left(\frac{1}{24} + \frac{1}{8/3} \right) = \frac{4\sqrt{3}}{24}$$

$$= \frac{\sqrt{3}}{6}$$

$$\gamma_2 : \begin{cases} \bar{r}_2(t) = \sin t\, \hat{i} + \sin t\, \hat{j} + \sin t\, \hat{k} \\ 0 \le t \le \frac{\pi}{6} \end{cases}$$

$$x(t) = y(t) = z(t) = \sin t$$

$$\int_{\gamma_2} (xy + z)\, ds =$$

$$= \int_0^{\frac{\pi}{6}} (\sin^2 t + \sin t) \; |\bar{r}'(t)| \, dt$$

$$= \int_0^{\frac{\pi}{6}} (\sin^2 t + \sin t) \sqrt{3 \cos^2 t}\, dt$$

$$= \int_0^{\frac{\pi}{6}} (\sin^2 t + \sin t) \sqrt{3} \cos t \, dt$$

$$= \int_0^{\frac{\pi}{6}} \sqrt{3} \sin^2 t \cos t \, dt + \int_0^{\frac{\pi}{6}} \sqrt{3} \sin t \cos t \, dt$$

$$= \sqrt{3}\, \frac{\sin^3 t}{3} \Big]_0^{\frac{\pi}{6}} + \sqrt{3}\, \frac{\sin^2 t}{2} \Big]_0^{\frac{\pi}{6}}$$

$$= \frac{\sqrt{3}}{3} \left(\sin\frac{\pi}{6}^3 - 0 \right) + \frac{\sqrt{3}}{2} \left(\sin\frac{\pi}{6}^2 - 0 \right)$$

$$= \frac{\sqrt{3}}{3} \, \frac{1}{8} + \frac{\sqrt{3}}{2} \, \frac{1}{4}$$

$$= \sqrt{3} \left(\frac{1}{24} + \frac{1}{8} \right)$$

$$= \sqrt{3} \, \frac{4}{24_6} = \frac{\sqrt{3}}{6}$$

16. Cont.

As duas integrais de linha coincidem pois $\vec{r}_1(t)$ e $\vec{r}_2(t)$ definem a mesma parametrização para a curva γ excetuando os extremos onde $\vec{r}_2'(0) = 0$.

---

17. $\gamma: \begin{cases} \vec{r}(t) = t\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k} \\ a \leq t \leq b \end{cases}$

$M(x,y,z)$ é contínua

$\int_\gamma M(x,y,z)\,dx$ pode ser vista como a integral de linha

$$\int_\gamma (M\hat{i} + 0\hat{j} + 0\hat{k}) \cdot d\vec{r} =$$

$$= \int_a^b (M\hat{i} + 0\hat{j} + 0\hat{k}) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b (M\hat{i} + 0\hat{j} + 0\hat{k}) \cdot (y'(t)\hat{j} + z'(t)\hat{k})$$

$$= 0$$

$$\boxed{\int_\gamma M(x,y,z)\,dx = 0}$$