Cálculo III - Prova 1

1. Calcule $\int_D xy\, dA$ onde D é a região do plano limitada pelas curvas
$y = 3\sqrt{x}$, $y = \sqrt{x}$, $y = x^2$ e $y = 2x^2$.
Faça uma mudança de variáveis $(x,y) \to (u,v)$ de modo que a região D se transforme em um retângulo.

2. Calcule $\int_B z\, dV$ onde B é a região sólida limitada superiormente por $x^2+y^2+z^2 = 1$ e inferiormente por $z = \sqrt{x^2+y^2}$.

3. Calcule $\int_B \frac{1}{x^2+y^2}\, dV$ onde B é a região sólida no primeiro octante limitada inferiormente por $z = 0$, superiormente por $z = x^2+y^2$ e lateralmente por $y = 2x$, $x^2+y^2 = 1$ e $x = 0$. Use coordenadas cilíndricas.

4. Calcule $\int_B \frac{1}{x^2+y^2+z^2}\, dV$ onde B é a região sólida limitada superiormente por $z = \sqrt{2-x^2-y^2}$ e inferiormente por $z = \sqrt{x^2+y^2}$. Use coordenadas esféricas.

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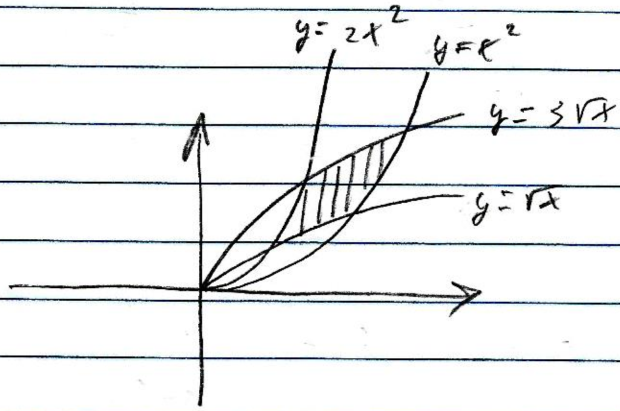
$I = \iint_D xy \, dA$

$\begin{cases} y = 3\sqrt{x} \\ y = \sqrt{x} \\ y = x^2 \\ y = 2x^2 \end{cases}$



$\rightarrow u = \dfrac{y}{\sqrt{x}}, \quad \begin{cases} y = \sqrt{x} \rightarrow u = 1 \\ y = 3\sqrt{x} \rightarrow u = 3 \end{cases}$

$\rightarrow v = \dfrac{y}{x^2}, \quad \begin{cases} y = x^2 \rightarrow v = 1 \\ y = 2x^2 \rightarrow v = 2 \end{cases}$

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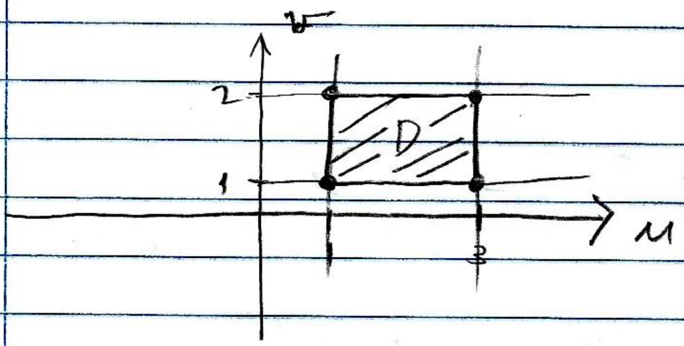

$u = \dfrac{y}{\sqrt{x}} \rightarrow u^2 = \dfrac{y^2}{x} \Rightarrow u^2 = \dfrac{v^2 \cancel{x^4}^{\,3}}{\cancel{x}}$

$u^2 = v^2 x^3$

$v = \dfrac{y}{x^2} \rightarrow y = vx^2$

$\therefore \quad \left\| x = \left(\dfrac{u}{v}\right)^{2/3} \right\|$

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$$y = v x^2 = v \left(\frac{u}{v}\right)^{4/3}$$

$$= \frac{v\, u^{4/3}}{v^{4/3}} = v^{-\frac{1}{3}} u^{\frac{4}{3}}$$

$$\therefore \left\| y = \frac{u^{4/3}}{v^{1/3}} \right\|$$

$$x = \left(\frac{u}{v}\right)^{2/3} \longrightarrow \begin{cases} \dfrac{\partial x}{\partial u} = \dfrac{2}{3} \dfrac{u^{-1/3}}{v^{2/3}} \\ \dfrac{\partial x}{\partial v} = u^{2/3} \left(-\dfrac{2}{3}\right) v^{-5/3} \\ \qquad = -\dfrac{2}{3} \dfrac{u^{2/3}}{v^{5/3}} \end{cases}$$

$$y = \frac{u^{4/3}}{v^{1/3}} \longrightarrow \begin{cases} \dfrac{\partial y}{\partial u} = \dfrac{4}{3} \dfrac{u^{1/3}}{v^{1/3}} \\ \dfrac{\partial y}{\partial v} = -\dfrac{1}{3} \dfrac{u^{4/3}}{v^{4/3}} \end{cases}$$

$$\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} \frac{2}{3}\frac{u^{-1/3}}{v^{2/3}} & -\frac{2}{3}\frac{u^{2/3}}{v^{5/3}} \\ \frac{4}{3}\frac{u^{1/3}}{v^{4/3}} & -\frac{1}{3}\frac{u^{4/3}}{v^{4/3}} \end{pmatrix}$$

$$= -\frac{2}{9}\frac{u}{v^2} + \frac{8}{9}\frac{u}{v^2}$$

$$= \frac{6}{9}\frac{u}{v^2} = \frac{2}{3}\frac{u}{v^2}$$

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Daí

$$\int_D xy\, dA = \int_{D(u,v)} x(u,v)\, y(u,v) \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right| du\,dv$$

$$= \int_{D(u,v)} \frac{u^{2/3}}{v^{2/3}}\frac{u^{4/3}}{v^{1/3}}\frac{2}{3}\frac{u}{v^2}\, du\,dv$$

$$= \int_{v=1}^{2}\left(\int_{u=1}^{3} \frac{2}{3}\frac{u^3}{v^3}\, du\right) dv$$

$$= \int_{v=1}^{2} \frac{2}{3}\frac{u^4}{4v^3}\Big]_{u=1}^{3} dv$$

$$= \int_{v=1}^{2} \frac{2}{3}\left(\frac{81}{4v^3} - \frac{1}{4v^3}\right) dv$$

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$$= \int_{v=1}^{2} \frac{2}{3}\,\frac{20}{v^3}\,dv$$

$$= \int_{v=1}^{2} \frac{40}{3}\,\frac{1}{v^3}\,dv$$

$$= \frac{40}{3}\left(\frac{-1}{2}\right) v^{-2} \Big]_{v=1}^{2}$$

$$= -\frac{20}{3}\,\frac{1}{v^2} \Big|_{v=1}^{2}$$

$$= -\frac{20}{3}\left(\frac{1}{4} - 1\right)$$

$$= -\frac{20}{\cancel{3}}\left(\frac{\cancel{-3}}{4}\right)$$

$$= 5$$

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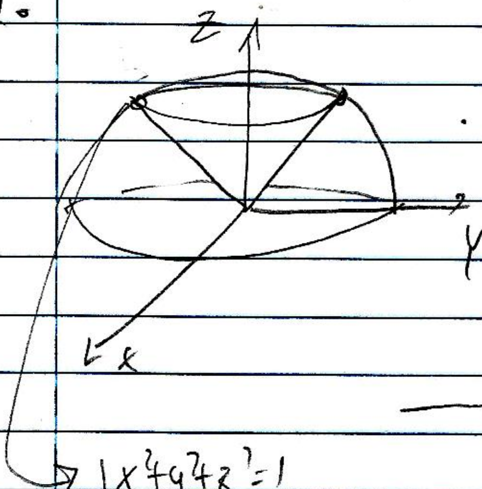
2.



$$B: \begin{cases} \sqrt{x^2+y^2} \le z \le \sqrt{1-x^2-y^2} \\ (x,y) \in D \;:\; x^2+y^2 = \frac{1}{2} \end{cases}$$

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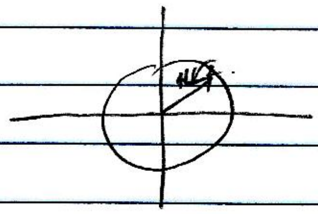

$$\begin{cases} x^2+y^2+z^2=1 \\ z^2=x^2+y^2 \end{cases}$$

$\therefore \; 2x^2+2y^2=1$

$x^2+y^2=\frac{1}{2}$

$$\int_B z\,dx\,dy\,dz = \int_{D(x,y)} \left(\int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} z\,dz \right) dx\,dy$$

$$= \int_{D(x,y)} \frac{z^2}{2} \Big|_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dx\,dy$$

$$= \int_{D(x,y)} \left[\frac{1}{2}(1-x^2-y^2) - \frac{1}{2}(x^2+y^2) \right] dx\,dy$$

$$= \int_{D(x,y)} \left(\frac{1}{2} - x^2 - y^2 \right) dx\,dy$$

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$x = r\cos\theta$
$y = r\sin\theta$; $D(r,\theta)$: $\begin{cases} 0 \le r \le \frac{1}{\sqrt{2}} \\ 0 \le \theta \le 2\pi \end{cases}$

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$$= \int_{\theta=0}^{2\pi} \left(\int_{r=0}^{1/\sqrt{2}} \left(\frac{1}{2} - r^2\right) r \, dr \right) d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1/\sqrt{2}} \left(\frac{1}{2} r - r^3\right) dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\frac{1}{2} \frac{r^2}{2} - \frac{r^4}{4} \right) \Bigg|_{r=0}^{1/\sqrt{2}} d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\frac{1}{4} \frac{1}{2} - \frac{1}{16} \right) d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{1}{16} \, d\theta = \frac{1}{16} \theta \Bigg]_{\theta=0}^{2\pi} = \frac{2\pi}{16} = \frac{\pi}{8}$$

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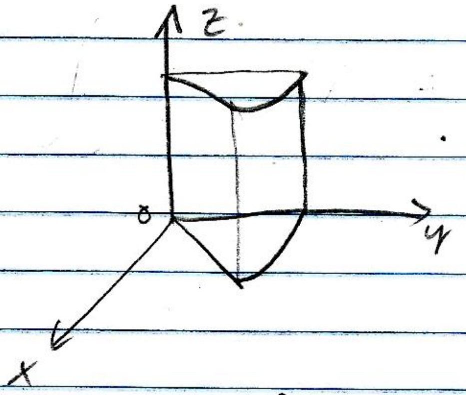
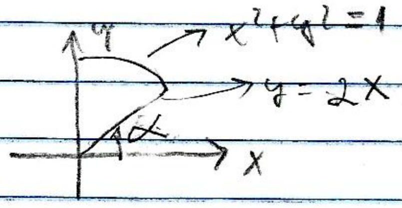
3.



$$B: \begin{cases} 0 \le z \le \rho^2 & 0{,}5 \\ \alpha \le \theta \le \frac{\pi}{2} & 0{,}25 \\ 0 \le \rho \le 1 & 0{,}25 \end{cases}$$

$z = x^2 + y^2$



$tg\,\alpha = \frac{2}{1} \Rightarrow \alpha = \text{arc}\,tg\,2$

$$\int_{B(\rho,\theta,z)} \frac{1}{x^2+y^2}\, dx\,dy\,dz = \qquad = \qquad x^2+y^2\ \rho\ z$$

$\rho\rho$

$$= \int_{\theta=\alpha}^{\frac{\pi}{2}} \left(\int_{\rho=0}^{1} \left(\int_{z=0}^{\rho^2} \frac{1}{\rho^2}\, \rho\, dz \right) d\rho \right) d\theta$$

$\frac{1}{1+\rho^2}$

$$= \int_{\theta=\alpha}^{\frac{\pi}{2}} \int_{\rho=0}^{1} \frac{1}{\rho^2}\, \rho\, z \Big]_0^{\rho^2} d\rho\, d\theta$$

$$= \int_{\theta=\alpha}^{\frac{\pi}{2}} \int_{\rho=0}^{1} \frac{1}{\rho^2} \rho^3\, d\rho\, d\theta = \int_{\theta=\alpha}^{\frac{\pi}{2}} \left(\int_{\rho=0}^{1} \rho\, d\rho \right) d\theta$$

$$= \int_{\theta=\alpha}^{\frac{\pi}{2}} \frac{\rho^2}{2}\Big|_{\rho=0}^{1} d\theta = \frac{1}{2}\theta \Big]_{\alpha}^{\frac{\pi}{2}} = \frac{1}{2}\left(\frac{\pi}{2} - \text{arctg}\sqrt{2}\right)$$

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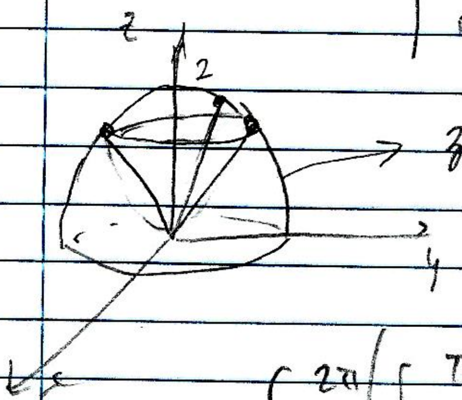
4. $$\int_{B(x,y,z)} \frac{1}{x^2+y^2+z^2}\, dx\,dy\,dz =$$

$$= \int_{B(r,\theta,\varphi)} \frac{1}{\cancel{r^2}} \cancel{r^2} \sin\theta \, dr\, d\theta\, d\varphi$$

$$B(r,\theta,\varphi) : \begin{cases} 0 \le r \le \sqrt{2} & \checkmark 0{,}75 \\ 0 \le \theta \le \pi/4 & \checkmark 0{,}5 \\ 0 \le \varphi \le 2\pi & 0{,}25 \end{cases}$$



$$= \int_{\varphi=0}^{2\pi} \left(\int_{\theta=0}^{\pi/4} \int_{r=0}^{\sqrt{2}} 1 \quad \sin\theta \, dr \right) d\theta \Bigg) d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{4}} r \sin\theta \Big]_{r=0}^{\sqrt{2}} d\theta\, d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi/4} \sqrt{2} \sin\theta \, d\theta\, d\varphi$$

$$= \int_{\varphi=0}^{2\pi} -\sqrt{2}\cos\theta \Big]_0^{\pi/4} d\varphi = \int_{\varphi=0}^{2\pi} \left(-\sqrt{2}\frac{\sqrt{2}}{2} + \sqrt{2}\right) d\varphi$$

$$= \int_{\varphi=0}^{2\pi} (\sqrt{2}-1)\, d\varphi = (\sqrt{2}-1)\, 2\pi$$ //

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