Cálculo 3 - Prova 2

Nome:

1. Seja a curva γ dada pela interseção das superfícies $x = \sqrt[3]{z}$, $z = y^3$ e tal que se afasta da origem no primeiro octante.
   (i) Obtenha uma parametrização de γ. 0.5
   (ii) Determine $\vec{T}$ e $\vec{N}$. 0.5

2. Mostrar que

$$\vec{\nabla} \times (\varphi \vec{\nabla} \varphi) = 0$$

1.0

3. Calcule a integral de linha

$$\int_\gamma (x + y) z \, ds$$

onde γ é a curva $y = x$, $z = 1 + y^2$ e orientada de $(-1, -1, 2)$ à $(1, 1, 2)$. 1.5

4. Mostre que a integral de linha independe do caminho

1.5

$$\int_\gamma \frac{1}{(x-3)^2(y+5)} dx + \frac{1}{(x-3)(y+5)^2} dy + \frac{1}{z+4} dz$$

$\gamma = x = y = z$
de $(0,0,0)$ a $(2,2,2)$

e use o teorema fundamental da integral de linha para avaliar a integral de linha.

5. Usando o teorema de Green calcule a integral de linha

$$\int_\gamma (xy^2 + 2x) dx + (x^2 y + y + x^2) dy$$

1.0

onde γ é a curva orientada no sentido anti-horário que é borda da região delimitada por $y^2 - x^2 = 4$, $x = 0, x = 3$.

1. $\begin{cases} x = \sqrt[3]{z} \\ z = y^3 \end{cases}$

se afastando da origem no primeiro octante.

i) $\boxed{\vec{r}(t) = (t, t, t^3) \quad ; \quad t \in \mathbb{R}}$ 0.5

ii) $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

$\vec{r}\,'(t) = (1, 1, 3t^2)$

$|\vec{r}\,'(t)| = \sqrt{1 + 1 + 9t^4} = \sqrt{2 + 9t^4}$

$\vec{T}(t) = \dfrac{1}{\sqrt{2+9t^4}}(1, 1, 3t^2)$

0.5

$$\boxed{\vec{T}(t) = \left(\frac{1}{\sqrt{2+9t^4}}, \frac{1}{\sqrt{2+9t^4}}, \frac{3t^2}{\sqrt{2+9t^4}} \right)}$$

2. $\vec{\nabla} \times (\phi \, \vec{\nabla} \varphi) =$

$$= \cancel{\det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \phi \partial_x \varphi & \phi \partial_y \varphi & \phi \partial_z \\ & & \end{bmatrix}}$$

$$= \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \phi \partial_x \varphi & \phi \partial_y \varphi & \phi \partial_z \varphi \end{bmatrix}$$

$$= \hat{i} \left[\partial_y (\phi \partial_z \varphi) - \partial_z (\phi \partial_y \varphi) \right]$$

$$+ \hat{j} \left[\partial_z (\phi \partial_x \varphi) - \partial_x (\phi \partial_z \varphi) \right]$$

$$+ \hat{k} \left[\partial_x (\phi \partial_y \varphi) - \partial_y (\phi \partial_x \varphi) \right]$$

$$= \hat{i} \left[\cancel{\partial_y \phi \partial_z \varphi} + \cancel{\phi \partial_y \partial_z \varphi} - \cancel{\partial_z \phi \partial_y \varphi} - \cancel{\phi \partial_z \partial_y \varphi} \right]$$

$$+ \hat{j} \left[\cancel{\partial_z \phi \partial_x \varphi} + \cancel{\phi \partial_z \partial_x \varphi} - \cancel{\partial_x \phi \partial_z \varphi} - \cancel{\phi \partial_x \partial_z \varphi} \right]$$

$$+ \hat{k} \left[\cancel{\partial_x \phi \partial_y \varphi} + \cancel{\phi \partial_x \partial_y \varphi} - \cancel{\partial_y \phi \partial_x \varphi} - \cancel{\phi \partial_y \partial_x \varphi} \right]$$

$$= 0$$

1-0

3.

$$\gamma : \begin{cases} y = x \\ z = 1 + y \end{cases}$$

$$(-1,-1,0) \to (1,1,2)$$

$$\vec{r}(t) = (t, t, 1+t) \quad ; \quad -1 \leq t \leq 1 \quad \checkmark 0.5$$

$$\vec{r}\,'(t) = (1, 1, 1) \longrightarrow |\vec{r}\,'(t)| = \sqrt{3}$$

$$\int_\gamma (x+y)\, z \, ds = \int_{-1}^{1} (t+t)(1+t)\sqrt{3}\, dt$$

$$= \sqrt{3} \int_{-1}^{1} 2t(1+t)\, dt \quad \checkmark 0.5$$

$$= 2\sqrt{3} \int_{-1}^{1} (t + t^2)\, dt$$

$$= 2\sqrt{3} \left(\frac{t^2}{2} + \frac{t^3}{3} \right) \Big|_{t=-1}^{1}$$

$$= 2\sqrt{3} \left(\frac{1}{2} + \frac{1}{3} - \left(\frac{1}{2} - \frac{1}{3} \right) \right)$$

$$= 2\sqrt{3} \left(\cancel{\frac{1}{2}} + \frac{1}{3} - \cancel{\frac{1}{2}} + \frac{1}{3} \right)$$

$$= 2\sqrt{3}\ \frac{2}{3} = \frac{4\sqrt{3}}{3} \;//\quad 0.5$$

4.

$$\int_\gamma \frac{1}{(x-3)^2(y+5)}\,dx + \frac{1}{(x-3)(y+5)^2}\,dy + \frac{1}{(z+4)}\,dz$$

$$\gamma : \begin{cases} x = y = z \\ (0,0,0) \text{ a } (2,2,2) \end{cases}$$

$$\vec{E} = \nabla\varphi$$

$$\begin{cases} \partial_x \varphi = \dfrac{1}{(x-3)^2(y+5)} & (1) \\[2ex] \partial_y \varphi = \dfrac{1}{(x-3)(y+5)^2} & (2) \\[2ex] \partial_z \varphi = \dfrac{1}{z+4} & (3) \end{cases}$$

$$(1) : \varphi = \int \frac{1}{(x-3)^2(y+5)}\,dx + h(y,z)$$

$$\varphi(x,y,z) = -\frac{1}{(x-3)(y+5)} + h(y,z)$$

0.75

$$\partial_y \varphi = \frac{1}{(x-3)(y+5)^2}$$

$$\partial_y \left(-\frac{1}{(x-3)(y+5)} + h(y,z) \right) = \frac{1}{(x-3)(y+5)^2}$$

$$\therefore \cancel{\frac{1}{(x-3)(y+5)^2}} + \partial_y h(x,y) = \cancel{\frac{1}{(x-3)(y+5)^2}}$$

$$\partial_y h(y,z) = 0$$

$$\therefore \underline{h = h(z)}$$

$$\therefore \varphi(x,y,z) = -\frac{1}{(x-3)(y+5)} + h(z) \quad \underline{0.25}$$

(3) $$\partial_z \varphi = \frac{1}{z+4}$$

$$\partial_z \left(-\frac{1}{(x-3)(y+5)} + h(z) \right) = \frac{1}{z+4}$$

$$\partial_z h(z) = \frac{1}{z+4}$$

$$h(z) = \int \frac{1}{z+4}\, dz = \ln|z+4|$$

$$\boxed{\varphi(x,y,z) = -\frac{1}{(x-3)(y+5)} + \ln|z+4| + C}$$

0.35

Daí,

$$\int_\gamma \frac{1}{(x-3)^2(y+5)}\,dx + \frac{1}{(x-3)(y+5)^2}\,dy + \frac{1}{(z+4)}\,dz$$

$$= \varphi(2,2,2) - \varphi(0,0,0)$$

$$= -\frac{1}{-1\cdot 7} + \ln|6| + C$$

$$-\left(\frac{1}{15} + \ln|4| + C\right)$$

$$= \frac{1}{7} + \ln|6| + \cancel{C} - \frac{1}{15} - \ln|4| - \cancel{C}$$

$$= \frac{15-7}{105} + \ln\left|\frac{6}{4}\right|$$

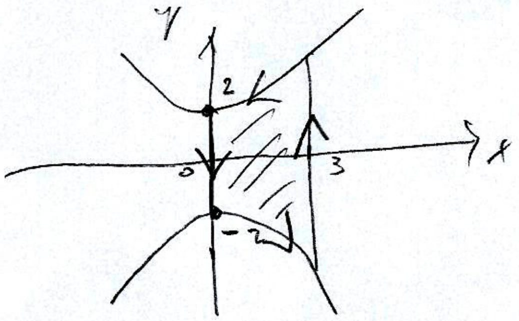
$$= \frac{8}{105} + \ln\frac{3}{2}$$

0,5

Ej:

$$\int_\gamma (xy^2+2x)\,dx + (x^2y+y+x^2)\,dy =$$

$$= \iint_D \Big[\partial_x (x^2y+y+x^2) - \partial_y (xy^2+2x) \Big]\, dA$$



$$= \int_D (2\cancel{x}y + 2x - 2\cancel{x}y)\, dA$$

$$= \int_D 2x\, dA = \int_{x=0}^{3} \left(\int_{y=-\sqrt{4+x^2}}^{\sqrt{4+x^2}} 2x\, dy \right) dx$$

0.5

$$D : \begin{cases} 0 \le x \le 3 \\ -\sqrt{4+x^2} \le y \le \sqrt{4+x^2} \end{cases}$$

$$= \int_{x=0}^{3} 2xy \Big]_{y=-\sqrt{4+x^2}}^{\sqrt{4+x^2}} dx$$

$$= \int_{x=0}^{3} 4x\sqrt{4+x^2}\, dx$$

$$= 2\cdot\frac{2}{3}(4+x^2)^{3/2}\Big]_{x=0}^{3}$$

$$= \frac{4}{3}(4+9)^{3/2} - \frac{4}{3}4^{3/2} = \frac{4}{3}\left(13^{3/2} - 4^{3/2}\right)$$

0.5