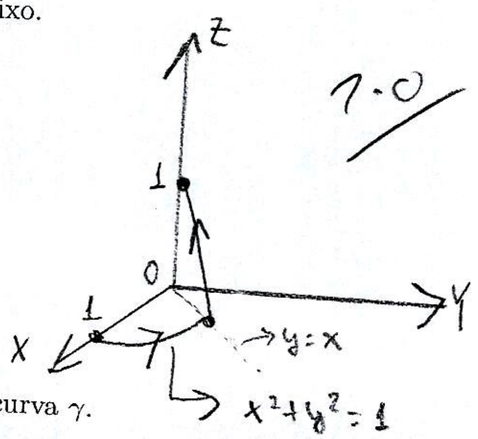
Cálculo 3 - Prova 2

Nome:

1. Seja γ a curva mostrada na figura abaixo.



1.0

(i) Dê uma parametrização $\vec{r}(t)$ para a curva γ.
(ii) Calcule $\vec{r}'(t)$.

2. Mostrar que

$$\vec{\nabla}\left(\frac{\varphi}{\chi}\right) = \frac{\chi\vec{\nabla}\varphi - \varphi\vec{\nabla}\chi}{\chi^2}$$

onde φ e χ são campos escalares.

1,0

3. Usando a definição de integral de linha calcule

$$\int_\gamma x dx + yzdy + x^2 dz$$

1,0

onde γ é a curva $y = x$, $z = x^2$ e orientada de $(-1,-1,1)$ a $(2,2,4)$.

4. Mostre que a integral de linha independe do caminho

γ: $x = y = z$
de $(0,0,0)$ a $(2,2,2)$

$$\int_\gamma 3x^2yzdx + x^3zdy + (x^3y - 4z)dz$$

1,0

e use o teorema fundamental da integral de linha para avaliar a integral de linha.

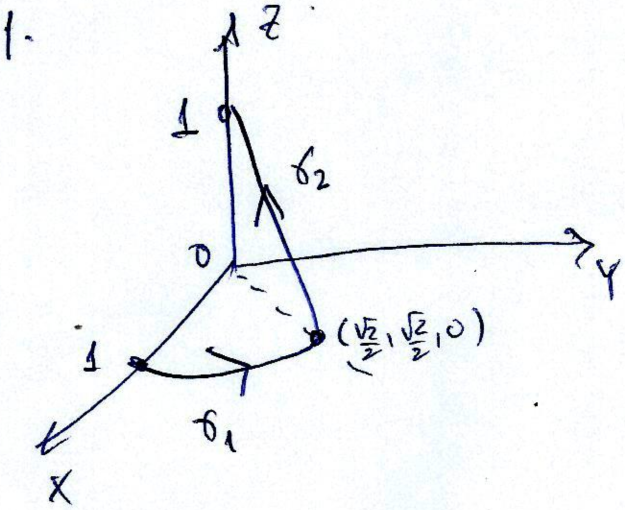
5. Usando o teorema de Green calcule a integral de linha

$$\int_\gamma (xy^2 + 2x)dx + (x^2y + y + x^2)dy$$

1,0

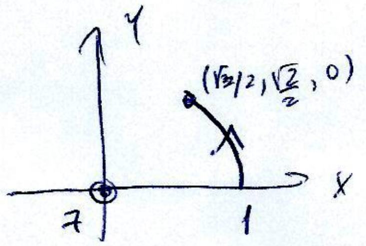
onde γ é a curva orientada no sentido anti-horário que é borda da região delimitada por $y^2 - x^2 = 4$, $x = 0, x = 3$.

1.



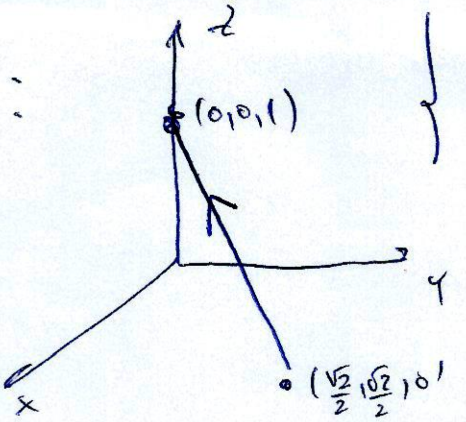
$$\gamma = \gamma_1 \cup \gamma_2$$

γ_1 :



$$\vec{r}_1(t) = (\cos t, \operatorname{sen} t, 0), \quad 0 \le t \le \frac{\pi}{4}$$

γ_2 :



$$\vec{r}_2(t) = \left(\frac{\sqrt2}{2} - \frac{\sqrt2}{2}t, \frac{\sqrt2}{2} - \frac{\sqrt2}{2}t, t\right)$$

$$0 \le t \le 1$$

$$\vec{v} = \left(-\frac{\sqrt2}{2}, -\frac{\sqrt2}{2}, 1\right)$$

Unindo γ_1 a γ_2 : $\vec{r}_2(t) = \left(\frac{\sqrt2}{2} - \frac{\sqrt2}{2}t, \frac{\sqrt2}{2} - \frac{\sqrt2}{2}t, t\right), 0 \le t \le 1$

$$\lambda = t + \frac{\pi}{4} \rightarrow t = \lambda - \frac{\pi}{4}$$

$$\cancel{\vec{r}(\lambda) = \vec{r}_2\left(t + \frac{\pi}{4}\right)}$$

$$\vec{r}_2(t) = \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}t,\ \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}t,\ t\right), \quad 0 \le t \le 1$$

$$\Downarrow$$

$$\frac{\pi}{4} \le \underbrace{t + \frac{\pi}{4}}_{\lambda} \le 1 + \frac{\pi}{4}$$

$$\lambda = t + \frac{\pi}{4} \therefore t = \lambda - \frac{\pi}{4}$$

Defin.

$$\cancel{\vec{r}_2^{\,*}(\lambda) = \vec{r}_2\left(\lambda - \frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(\lambda - \frac{\pi}{4}\right.\right.}$$

$$\vec{r}_2^{\,*}(\lambda) = \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(\lambda - \frac{\pi}{4}\right),\ \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(\lambda - \frac{\pi}{4}\right),\ \lambda - \frac{\pi}{4}\right)$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2}\lambda,\ \frac{\sqrt{2}}{2} + \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2}\lambda,\ \lambda - \frac{\pi}{4}\right)$$

$$\therefore \boxed{\vec{r}_2(t) = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2}t,\ \frac{\sqrt{2}}{2} + \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2}t,\ t - \frac{\pi}{4}\right);\ \frac{\pi}{4} \le t \le 1 + \frac{\pi}{4}}$$

Daí

$$\boxed{\vec{r}(t) = \begin{cases} (\cos t, \operatorname{sen} t, 0); & 0 \le t \le \frac{\pi}{4} \\ \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2}t,\ \frac{\sqrt{2}}{2} + \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2}t,\ t - \frac{\pi}{4}\right); & \frac{\pi}{4} \le t \le 1 + \frac{\pi}{4} \end{cases}}$$

~~1,0~~

$$\vec{r}\,'(t) = (-\sin t, \cos t, 0) \quad ; \quad 0 \leq t < \frac{\pi}{4}$$

$$\vec{r}\,'(t) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1\right) \quad ; \quad \frac{\pi}{4} < t < 1+\frac{\pi}{4}$$

$t = \frac{\pi}{4}$:

$$\lim_{t \to \frac{\pi}{4}^-} \frac{\vec{r}(t) - \vec{r}(\frac{\pi}{4})}{t - \frac{\pi}{4}} = \lim_{t \to \frac{\pi}{4}^-} \frac{(\cos t, \sin t, 0) - (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)}{t - \frac{\pi}{4}}$$

$$= \lim_{t \to \frac{\pi}{4}^-} \left(\frac{\cos t - \frac{\sqrt{2}}{2}}{t - \frac{\pi}{4}}, \frac{\sin t - \frac{\sqrt{2}}{2}}{t - \frac{\pi}{4}}, 0 \right)$$

$$= \left(\lim_{t \to \frac{\pi}{4}^-} \frac{-\sin t}{1}, \lim_{t \to \frac{\pi}{4}^-} \frac{\cos t}{1}, 0 \right)$$

$$= \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right)$$

$$\lim_{t \to \frac{\pi}{4}^+} \frac{\vec{r}(t) - \vec{r}(\frac{\pi}{4})}{t - \frac{\pi}{4}} = \cancel{\lim_{t \to \frac{\pi}{4}^+}}$$

$$= \lim_{t \to \frac{\pi}{4}^+} \frac{\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2}t, \frac{\sqrt{2}}{2} + \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2}t, t - \frac{\pi}{4}\right) - \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)}{t - \frac{\pi}{4}}$$

$$= \lim_{t \to \frac{\pi}{4}^+} \left(\frac{\frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2}t}{t - \frac{\pi}{4}}, \frac{\frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2}t}{t - \frac{\pi}{4}}, \frac{t - \frac{\pi}{4}}{t - \frac{\pi}{4}} \right) = \lim_{t \to \frac{\pi}{4}^+} \left(\frac{\frac{\sqrt{2}}{2}\left(\frac{\pi}{4} - t\right)}{t - \frac{\pi}{4}}, \frac{\frac{\sqrt{2}}{2}\left(\frac{\pi}{4} - t\right)}{t - \frac{\pi}{4}}, 1 \right)$$

$$= \lim_{t \to \frac{\pi}{4}^+} \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 \right) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 \right)$$

$\therefore$ $\vec{r}'\left(\frac{\pi}{4}\right)$ não existe

$$\therefore \vec{r}'(t) = \begin{cases} (-\sin t, \cos t, 0) & \text{se } 0 < t < \frac{\pi}{4} \quad 0.25 \\ \nexists & t = \frac{\pi}{4} \quad 0.5 \\ \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, t\right) & \text{se } \frac{\pi}{4} < t < 1 + \frac{\pi}{4} \end{cases}$$

0.25

2.

$$\vec{\nabla}\left(\frac{\varphi}{\chi}\right) = \frac{\chi\vec{\nabla}\varphi - \varphi\vec{\nabla}\chi}{\chi^2}$$

$$\vec{\nabla}\left(\frac{\varphi}{\chi}\right) = \left(\partial_x\left(\frac{\varphi}{\chi}\right), \partial_y\left(\frac{\varphi}{\chi}\right), \partial_z\left(\frac{\varphi}{\chi}\right)\right)$$

Mq

$$\left.\begin{aligned}
\partial_x\left(\frac{\varphi}{\chi}\right) &= \frac{\partial_x\varphi\,\chi - \varphi\,\partial_x\chi}{\chi^2} \\
\partial_y\left(\frac{\varphi}{\chi}\right) &= \frac{\partial_y\varphi\,\chi - \varphi\,\partial_y\chi}{\chi^2} \\
\partial_z\left(\frac{\varphi}{\chi}\right) &= \frac{\partial_z\varphi\,\chi - \varphi\,\partial_z\chi}{\chi^2}
\end{aligned}\right\} \quad 0.5 \;//$$

$\therefore$

$$\vec{\nabla}\left(\frac{\varphi}{\chi}\right) = \frac{1}{\chi^2}\left(\underline{\partial_x\varphi\,\chi} - \varphi\,\partial_x\chi, \underline{\partial_y\varphi\,\chi} - \varphi\,\partial_y\chi, \underline{\partial_z\varphi\,\chi} - \varphi\,\partial_z\chi\right)$$

$$= \frac{1}{\chi^2}\left\{\chi\left(\partial_x\varphi, \partial_y\varphi, \partial_z\varphi\right) - \varphi\left(\partial_x\chi, \partial_y\chi, \partial_z\chi\right)\right\}$$

$$= \frac{1}{\chi^2}\left\{\chi\,\vec{\nabla}\varphi - \varphi\,\vec{\nabla}\chi\right\}$$

$\therefore$ $$\boxed{\vec{\nabla}\left(\frac{\varphi}{\chi}\right) = \frac{\chi\vec{\nabla}\varphi - \varphi\vec{\nabla}\chi}{\chi^2}}$$ 0.5 //

3. $\int_\gamma x\,dx + yz\,dy + x^2\,dz$

γ : $y = x$, $z = x^2$

$(-1,-1,1) \longrightarrow (2,2,4)$

$\vec{r}(t) = (t, t, t^2)$; $-1 \leq t \leq 2$ 0,5

$$\begin{cases} x(t) = t, & \to x'(t) = 1 \\ y(t) = t & \to y'(t) = 1 \\ z(t) = t^2 & \to z'(t) = 2t \end{cases}$$

$$\int_\gamma x\,dx + yz\,dy + x^2\,dz = \int_{-1}^{2} \left[x(t)\,x'(t) + y(t)\,z(t)\,y'(t) + x^2(t)\,z'(t) \right] dt$$

$$= \int_{-1}^{2} \left[t + t\,t^2 + t^2\,2t \right] dt$$

0,5

$$= \int_{-1}^{2} (t + 3t^3)\,dt = \left(\frac{t^2}{2} + \frac{3t^4}{4} \right) \Big|_{t=-1}^{2}.$$

$= \cancel{\frac{4}{2} + 8 - \left(\frac{1}{2} - 1\right) = 10 - \frac{1}{2} + 1 = 10 + \frac{1}{2} = \frac{21}{2}}$

$\cancel{10,21}$

$$= \frac{4}{2} + 3 \cdot \frac{16}{4} - \left(\frac{1}{2} + \frac{3}{4} \right)$$

$$= 2 + 12 - \frac{5}{4}$$

$$= 14 - \frac{5}{4} = \frac{51}{4}$$ 0

4. $\int_\gamma 3x^2yz\,dx + x^3z\,dy + (x^3y - 4z)\,dz$

$\bar{E} = (3x^2yz,\ x^3z,\ x^3y - 4z)$

$$\begin{cases} \partial_x \varphi = E_1 \\ \partial_y \varphi = E_2 \\ \partial_z \varphi = E_3 \end{cases}$$

$\partial_x \varphi = 3x^2yz$

$\varphi = \int 3x^2yz\,dx$

$\varphi(x,y,z) = x^3yz + h(y,z)$ 0.25

$\partial_y \varphi = x^3z$

$\partial_y (x^3yz + h(y,z)) = x^3z$

$\cancel{x^3z} + \partial_y h(y,z) = \cancel{x^3z}$

$\partial_y h(y,z) = 0 \quad \therefore \quad h = h(z)$

$\therefore \varphi(x,y,z) = x^3yz + h(z)$ 0.25

$$\partial_z \varphi = x^3 y - 4z$$

$$\partial_z \left(x^3 y z + h(z) \right) = x^3 y - 4z$$

$$\cancel{x^3 y} + \partial_z h(z) = \cancel{x^3 y} - 4z$$

$$\partial_z h(z) = -4z$$

$$h(z) = -2z^2 + C$$

$$\therefore \boxed{\varphi(x,y,z) = x^3 y z - 2z^2 + C}$$

0,1

$$\int_\gamma 3x^2 y z \, dx + x^3 z \, dy + (x^3 y - 4z) \, dz =$$

$$= \varphi(2,2,2) - \varphi(0,0,0)$$

$$= 8 \cdot 2 \cdot 2 - 2 \cdot 4 + C - C$$

$$= 32 - 8 = 24$$

0,1

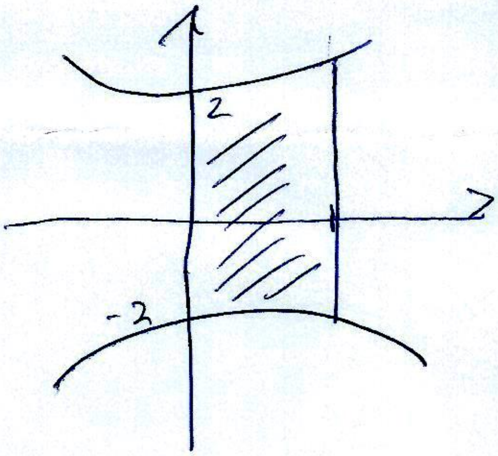
5. $\int_\gamma (xy^2+2x)\,dx + (x^2y+y+x^2)\,dy =$

$= \cancel{\iint_D \partial_y(xy^2+2x) - \partial_x}$

$= \int_D \left\{ \partial_x (x^2y+y+x^2) - \partial_y (xy^2+2x) \right\} dA$

$= \int_D \left\{ \cancel{2xy} + 2x - \cancel{2xy} \right\} dA$

$= \int_D 2x \, dA$ 0.5



$D: \begin{cases} 0 \le x \le 3 \\ -\sqrt{4+x^2} \le y \le \sqrt{4+x^2} \end{cases}$

$= \int_{x=0}^{3} \left(\int_{y=-\sqrt{4+x^2}}^{\sqrt{4+x^2}} 2x \, dy \right) dx$

$$= \int_{x=0}^{3} 2xy \Big]_{y=-\sqrt{4+x^2}}^{\sqrt{4+x^2}} dx$$

$$= \int_{x=0}^{3} 4x\sqrt{4+x^2}\, dx$$

$$= \frac{4}{3}(4+x^2)^{3/2} \Big]_{x=0}^{3} = \frac{4}{3}(4+9)^{3/2} - \frac{4}{3}4^{3/2}$$

$$= \frac{4}{3}\left(13^{3/2} - 8\right)$$

0.5