Cálculo 3 - Prova 3

Nome:

1. Calcule usando a definição

2.0

$$\int_\sigma x^2 z \, dS$$

onde σ é parte do cilindro $x^2 + z^2 = 1$ situado entre os planos $y = -1$ e $y = 2$ e acima do plano xy.

2. Calcule

2.5

$$\int_\sigma \vec{F} \cdot d\vec{S}$$

onde $\vec{F} = (x, y, 0)$ e σ é a superfície dada por $y^2 = x^2 + z^2$ acima do plano xy e situada entre os planos $y = 0$ e $y = 1$ e orientada com vetor normal $\hat{n}$ tal que $\hat{n} \cdot \hat{k} > 0$.

3. Usando o teorema de Stokes, calcule

2.0

$$\int_\gamma \vec{F} \cdot d\vec{r}$$

onde $\vec{F} = (0, 0, yz)$ e γ é a borda da parte do cone $z = \sqrt{x^2 + y^2}$ situado no primeiro octante e cortado pelos planos $x = 0$, $y = 0$, $z = 2$ e $z = 3$. Assuma que γ é orientada de modo que a parte da curva que está no plano $z = 2$ é percorrida no sentido anti-horário, enquanto que a parte de γ que está no plano $z = 3$ é percorrida no sentido horário.

4. Usando o teorema de Gauss, calcule

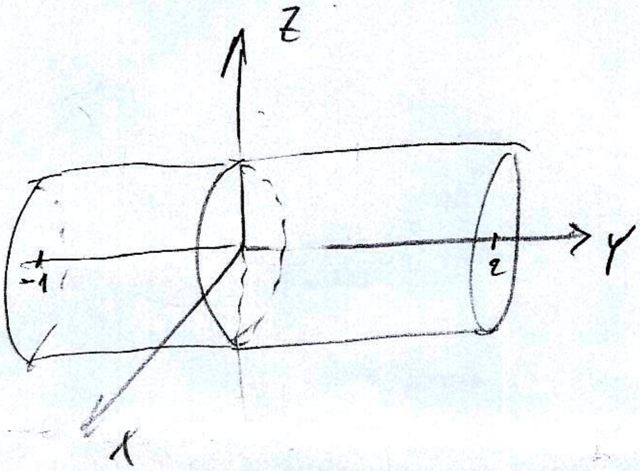
2.0

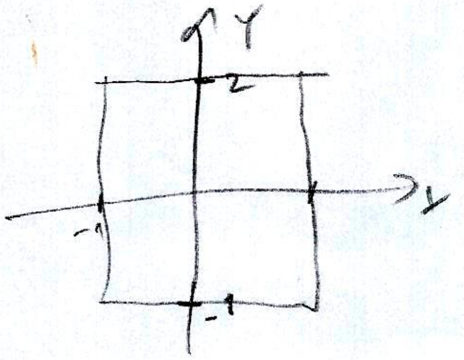
$$\int_\sigma \vec{F} \cdot d\vec{S}$$

onde $\vec{F} = x^2\hat{\imath} + y\hat{\jmath} - 2z^2\hat{k}$, e a superfície σ é borda da região B situada acima do plano $z = 0$, abaixo do plano $z = x$ e limitada lateralmente pela superfície $y^2 = 2 - x$

1. $\int_\sigma x^2 z \, dS$

σ: parte do cilindro $x^2+z^2=1$ entre os planos $y=-1$ e $y=2$ e acima do plano xy.





$$\sigma: \begin{cases} x = x \\ y = y \\ z = f(x,y) = \sqrt{1-x^2} \quad \underline{\underline{0.5}} \longrightarrow \dfrac{\partial f}{\partial x} = \dfrac{-x}{\sqrt{1-x^2}}, \quad \dfrac{\partial f}{\partial y} = 0 \\ (x,y) \in D : \begin{cases} -1 \le x \le 1 \\ -1 \le y \le 2 \end{cases} \quad \underline{\underline{0.25}} \end{cases}$$

$$dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\,dy = \sqrt{1 + \frac{x^2}{1-x^2}}\, dx\,dy$$

$$= \frac{1}{\sqrt{1-x^2}}\, dx\,dy \quad \underline{\underline{0.25}}$$

$$\int_{\sigma} x^2 z \, dS = \int_{D(x,y)} x^2 \cancel{\sqrt{1-x^2}} \frac{1}{\cancel{\sqrt{1-x^2}}} \, dx \, dy$$

$$= \int_{y=-1}^{2} \left(\int_{x=-1}^{1} x^2 \, dx \right) dy$$

$$= \int_{y=-1}^{2} \frac{x^3}{3} \Big]_{x=-1}^{1} dy$$

$$= \int_{y=-1}^{2} \left[\frac{1}{3} - \left(\frac{-1}{3}\right) \right] dy$$

$$= \int_{y=-1}^{2} \frac{2}{3} \, dy = \frac{2}{3} y \Big]_{y=-1}^{2}$$

$$= \frac{4}{3} - \left(-\frac{2}{3}\right) = \frac{4}{3} + \frac{2}{3} = 2$$

1.0

2:

$$\int_\sigma F\cdot d\vec{S}$$

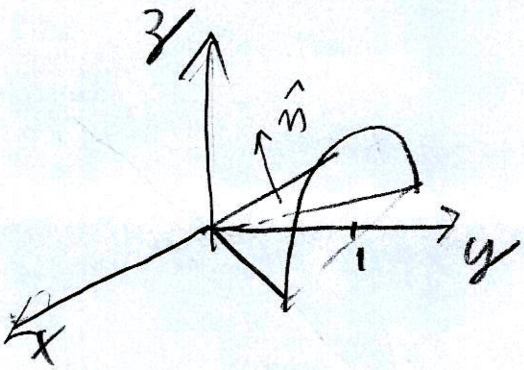
$F=(x,y,0)$

σ : $y^2=x^2+z^2$

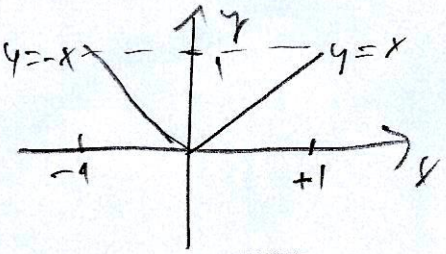
acima do plano xy

entre os planos $y=0$ e $y=1$.

$\hat{n}\cdot\hat{k}>0$



$$\sigma:\begin{cases} z=\sqrt{y^2-x^2} \quad \underline{0.25} \\ x=x \\ y=y \\ (x,y)\in D : \begin{cases} 0\le y\le 1 \\ -y\le x\le y \end{cases} \quad \underline{0.5} \end{cases}$$



$z=f(x,y)=\sqrt{y^2-x^2}$

$$\begin{cases} \dfrac{\partial f}{\partial x}=\dfrac{-x}{\sqrt{y^2-x^2}} \\ \dfrac{\partial f}{\partial y}=\dfrac{y}{\sqrt{y^2-x^2}} \end{cases}$$

$$d\vec{S}=\left(\pm\frac{\partial f}{\partial x},\pm\frac{\partial f}{\partial y},\pm 1\right)dx\,dy$$

$$=\left(-\frac{\partial f}{\partial x},-\frac{\partial f}{\partial y},1\right)dx\,dy$$

$$//\ d\vec{S}=\left(\frac{x}{\sqrt{y^2-x^2}},\frac{-y}{\sqrt{y^2-x^2}},1\right)dx\,dy\ //$$

$\underline{0.25}$

$$\int_S \vec{F}\cdot d\vec{S} = \int_{D(x,y)} (x,y,0)\cdot\left(\frac{x}{\sqrt{y^2-x^2}}, \frac{-y}{\sqrt{y^2-x^2}}, 1\right) dx\,dy$$

$$= \int_{D(x,y)} \left(\frac{x^2}{\sqrt{y^2-x^2}} - \frac{y^2}{\sqrt{y^2-x^2}}\right) dx\,dy$$

$$= \int_{D(x,y)} \frac{x^2-y^2}{\sqrt{y^2-x^2}}\, dx\,dy$$

$$= \int_{D(x,y)} -\frac{(y^2-x^2)}{\sqrt{y^2-x^2}}\, dx\,dy$$

$$= \int_{y=0}^{1}\left(\int_{x=-y}^{y} -\sqrt{y^2-x^2}\, dx\right) dy \qquad \overset{0.5}{=\!=}$$

$$\int \sqrt{y^2-x^2}\, dx = \int \sqrt{y^2-y^2\sin^2\theta}\; y\cos\theta\, d\theta$$

$x = y\sin\theta$

$dx = y\cos\theta\, d\theta$

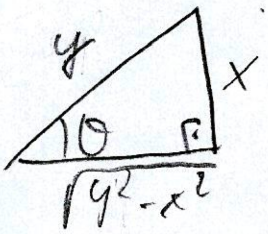
$$= \int y\cos\theta\, y\cos\theta\, d\theta$$

$$= \int y^2\cos^2\theta\, d\theta$$

$$= \int y^2\, \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{y^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)$$

$$= \frac{y^2}{2}\left(\theta + \sin\theta\cos\theta\right)$$



$$\begin{cases} \sin\theta = \dfrac{x}{y} \\ \cos\theta = \dfrac{\sqrt{y^2-x^2}}{x} \end{cases}$$

$$\therefore \int \sqrt{y^2-x^2}\,dx = \frac{y^2}{2}\left(\arcsin\frac{x}{y} + \frac{x}{y}\frac{\sqrt{y^2-x^2}}{x}\right)$$

$$= \frac{y^2}{2}\left(\arcsin\frac{x}{y} + \frac{\sqrt{y^2-x^2}}{y}\right) \quad 0.5$$

Dai

$$\int_{x=-y}^{y} -\sqrt{y^2-x^2}\,dx = -\frac{y^2}{2}\left(\arcsin\frac{x}{y} + \frac{\sqrt{y^2-x^2}}{y}\right)\Bigg|_{x=-y}^{y}$$

$$= -\frac{y^2}{2}\left(\arcsin 1 - \arcsin(-1)\right)$$

$$= -\frac{y^2}{2}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

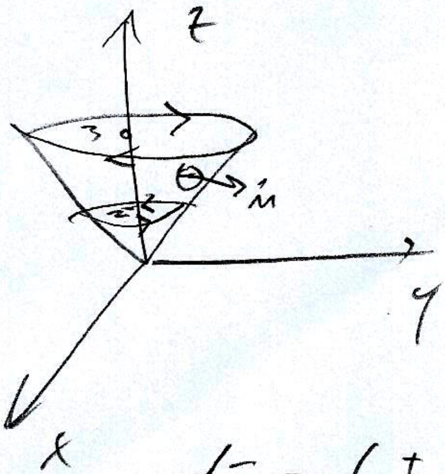
$$= -\frac{y^2\pi}{2}$$

$$\therefore \int_C \vec{F}\cdot d\vec{s} = \int_{y=0}^{1} -\frac{\pi}{2}y^2\,dy = -\frac{\pi}{2}\frac{y^3}{3}\Bigg]_{y=0}^{1}$$

$$= -\frac{\pi}{6} \quad 0.5$$

3.

$\bar{F} = (0, 0, yz)$



$$\sigma : \begin{cases} x = x \\ y = y \\ z = \sqrt{x^2+y^2} \\ (x,y) \in D : 4 \le x^2+y^2 \le 9 \\ x \geq 0,\ y \geq 0 \end{cases}$$

0.25

0.5

$$d\bar{S} = \left(\pm \frac{\partial f}{\partial x}, \pm \frac{\partial f}{\partial y}, \mp 1 \right) dx\,dy$$

$$= \left(\pm \frac{x}{\sqrt{x^2+y^2}}, \pm \frac{y}{\sqrt{x^2+y^2}}, \mp 1 \right) dx\,dy$$

$$\boxed{d\bar{S} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right) dx\,dy}$$

0.25

$$\bar{\nabla} \times \bar{F} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & yz \end{bmatrix}$$

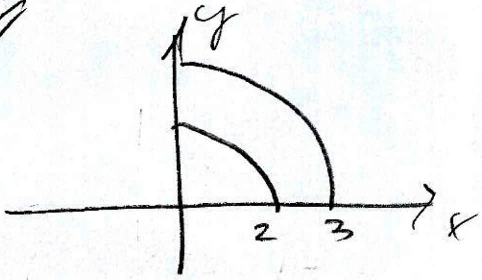
$$= (z, 0, 0)$$

$$\therefore \int_\gamma \bar{F} \cdot d\bar{r} = \int_\sigma (z, 0, 0) \cdot \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right) dx\,dy$$

$$= \iint_{D(x,y)} \frac{x^3}{\sqrt{x^2+y^2}}\, dx\, dy$$

$$= \iint_{D(x,y)} \frac{x\,\cancel{(x^2+y^2)}}{\sqrt{\cancel{x^2+y^2}}}\, dx\, dy$$

$$= \iint_{D(x,y)} x\, dx\, dy$$ 0.5



$$= \int_{\theta=0}^{\frac{\pi}{2}} \left(\int_{r=2}^{3} r\cos\theta \; r\, dr \right) d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left(\int_{r=2}^{3} r^2\cos\theta\, dr \right) d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \frac{r^3}{3}\cos\theta \Big]_{r=2}^{3} d\theta$$

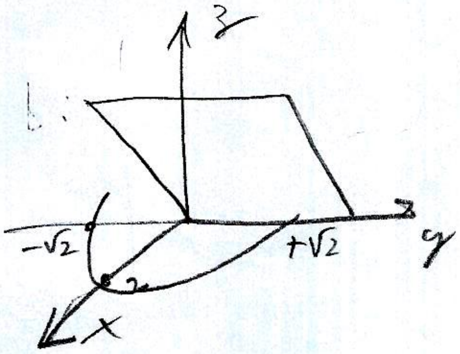
$$= \int_{\theta=0}^{\frac{\pi}{2}} \left(\frac{27}{3}-\frac{8}{3}\right)\cos\theta\, d\theta = \int_{\theta=0}^{\frac{\pi}{2}} \frac{19}{3}\cos\theta\, d\theta$$

$$= +\frac{19}{3}\sin\theta \Big]_0^{\pi/2} = +\frac{19}{3}$$ 0.5

4. $\displaystyle\int_\sigma \vec{F}\cdot d\vec{S} = \int_B \vec{\nabla}\cdot\vec{F}\, dV$

$\vec{F} = (x^2, y, -2z^2)$

B : região acima do plano $z=0$, abaixo do plano $z=x$ e lateralmente por $y^2 = 2-x$.



$\vec{\nabla}\cdot\vec{F} = 2x + 1 - 4z$

$\therefore \displaystyle\int_\sigma \vec{F}\cdot d\vec{S} = \int_B (2x+1-4z)\, dV$ B: $0 \le z \le x$ 0.25

0.25 $\begin{cases} -\sqrt{2} \le y \le \sqrt{2} \\ 0 \le x \le 2-y^2 \end{cases}$

$$= \int_{y=-\sqrt{2}}^{\sqrt{2}} \int_{x=0}^{2-y^2} \int_{z=0}^{x} (2x+1-4z)\, dz\, dx\, dy$$

$$= \int_{y=-\sqrt{2}}^{\sqrt{2}} \int_{x=0}^{2-y^2} 2xz + z - \frac{\cancel{4}^{\,2} z^2}{\cancel{2}} \Big]_{z=0}^{x} dx\, dy$$

$$= \int_{y=-\sqrt{2}}^{\sqrt{2}} \int_{x=0}^{2-y^2} (\cancel{2x^2} + x - \cancel{2x^2})\, dx\, dy$$

$$= \int_{y=-\sqrt{2}}^{\sqrt{2}} \int_{x=0}^{2-y^2} x \, dx \, dy \qquad 0.5$$

$$= \int_{y=-\sqrt{2}}^{\sqrt{2}} \frac{x^2}{2} \Big]_{x=0}^{2-y^2} dy$$

$$= \int_{y=-\sqrt{2}}^{\sqrt{2}} \frac{(2-y^2)^2}{2} \, dy \qquad 0.5$$

$$= \int_{y=-\sqrt{2}}^{\sqrt{2}} \frac{4-4y^2+y^4}{2} \, dy$$

$$= \int_{y=-\sqrt{2}}^{\sqrt{2}} \left(2 - 2y^2 + \frac{1}{2} y^4\right) dy$$

$$= 2y - \frac{2y^3}{3} + \frac{1}{2} \frac{y^5}{5} \Big|_{y=-\sqrt{2}}^{\sqrt{2}}$$

$$= 2\sqrt{2} - \frac{2}{3} \cdot 2\sqrt{2} + \frac{1}{10} 4 \cdot \sqrt{2}$$

$$= \left(-2\sqrt{2} + \frac{2}{3} \cdot 2\sqrt{2} - \frac{1}{10} 4 \cdot \sqrt{2}\right)$$

$$4 - \frac{8}{3} + \frac{4}{5} =$$

$$= 4\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$= 4 \frac{(15-10+3)}{15}$$

$$= \frac{32}{15} =$$

$$= \underline{2\sqrt{2}} - \frac{4}{3}\sqrt{2} + \frac{2}{5}\sqrt{2} + \underline{2\sqrt{2}} - \frac{4}{3}\sqrt{2} + \frac{2}{5}\sqrt{2}$$

$$= 4\sqrt{2} - \frac{8}{3}\sqrt{2} + \frac{4}{5}\sqrt{2} = \frac{32\sqrt{2}}{15}. \qquad 0.5$$

$$= \int_{y=-\sqrt{2}}^{\sqrt{2}} \int_{x=0}^{2-y^2} x \, dx \, dy \quad \underline{0.5}$$

$$= \int_{y=-\sqrt{2}}^{\sqrt{2}} \frac{x^2}{2} \Big]_{x=0}^{2-y^2} dy$$

$$= \int_{y=-\sqrt{2}}^{\sqrt{2}} \frac{(2-y^2)^2}{2} dy \quad \underline{\underline{0.5}}$$

$$= \int_{y=-\sqrt{2}}^{\sqrt{2}} \frac{4-4y^2+y^4}{2} dy$$

$$= \int_{y=-\sqrt{2}}^{\sqrt{2}} \left(2 - 2y^2 + \frac{1}{2} y^4\right) dy$$

$$= 2y - \frac{2y^3}{3} + \frac{1}{2}\frac{y^5}{5} \Big|_{y=-\sqrt{2}}^{\sqrt{2}}$$

$$4 - \frac{8}{3} + \frac{4}{5} =$$

$$= 2\sqrt{2} - \frac{2}{3}.2\sqrt{2} + \frac{1}{10} 4.\sqrt{2}$$

$$= 4\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$= \frac{4(15-10+3)}{15}$$

$$- \left(-2\sqrt{2} + \frac{2}{3}.2\sqrt{2} - \frac{1}{10} 4.\sqrt{2}\right)$$

$$= \frac{32}{15}$$

$$= \underline{2\sqrt{2}} - \frac{4}{3}\sqrt{2} + \frac{2}{5}\sqrt{2} + \underline{2\sqrt{2}} - \frac{4}{3}\sqrt{2} + \frac{2}{5}\sqrt{2}$$

$$= 4\sqrt{2} - \frac{8}{3}\sqrt{2} + \frac{4}{5}\sqrt{2} = \frac{32\sqrt{2}}{15}. \quad \underline{0.5}$$