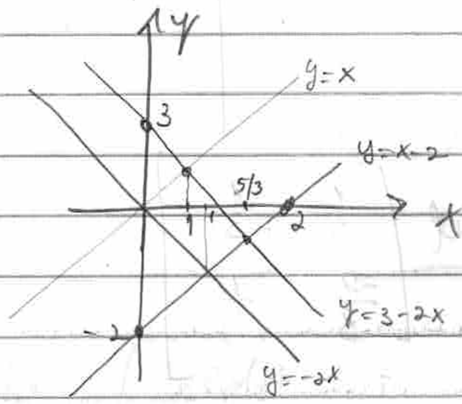
Lista 14 - Cálculo B - Soluções

1. $\displaystyle\iint_D (3x+4y)\, dA$



$$\begin{cases} x = \frac{1}{3}(u+v) \\ y = \frac{1}{3}(v-2u) \end{cases} \Leftrightarrow$$

$3x = u+v$

$3y = v-2u$

$\therefore\ 3x-3y = 3u$

$$\| u = x-y \|$$

$$\begin{rcases} y = 3-2x \\ y = x-2 \end{rcases} \quad 3-2x = x-2, \quad 5 = 3x, \quad x = \frac{5}{3}$$

$3x = \underbrace{u}+v = x-y+v$

$\therefore\ \| v = 2x+y \|$

Daí,

$$\begin{cases} y = x \Leftrightarrow x-y=0 \longrightarrow u=0 \\ y = x-2 \Leftrightarrow x-y=2 \longrightarrow u=2 \\ y=3-2x \Leftrightarrow 2x+y=3 \longrightarrow v=3 \\ y=-2x \Leftrightarrow 2x+y=0 \longrightarrow v=0 \end{cases}$$

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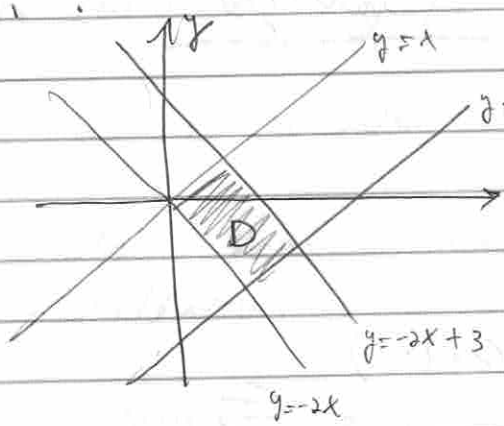
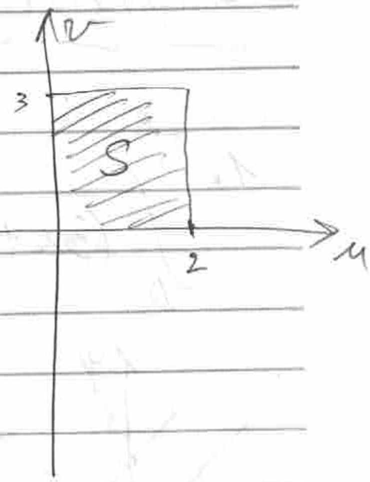


$x = \frac{1}{3}(u+v)$

$y = \frac{1}{3}(v-2u)$



$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \det \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \frac{1}{9} + \frac{2}{9} = \frac{3}{9} = \frac{1}{3}$$

$$\int_D (3x+4y)\, dA = \int_{u=0}^{2}\int_{v=0}^{3} \left(u+v + \frac{4}{3}(v-2u)\right)\frac{1}{3}\, du\, dv$$

$$= \int_{u=0}^{2}\int_{v=0}^{3} \left(-\frac{5}{9}u + \frac{7}{9}v\right) du\, dv$$

$$= \int_{u=0}^{2} -\frac{5}{9}u\, v + \frac{7}{9}\frac{v^2}{2}\Big]_{v=0}^{3} du$$

$$= \int_{u=0}^{2} -\frac{5}{9}u.3 + \frac{7}{\cancel{9}}\frac{\cancel{9}}{2}\, du$$

$$= \int_{u=0}^{2} \left(-\frac{15}{9}u + \frac{7}{2}\right) du$$

$$= -\frac{15}{9}\frac{u^2}{2} + \frac{7}{2}u \Big]_0^2$$

$$= -\frac{15}{9}\frac{\cancel{4}^{2}}{\cancel{2}} + \frac{7}{\cancel{2}}\cancel{2}$$

$$= -\frac{30}{9} + 7 = \frac{-30+63}{9}$$

$$= \frac{33}{9} = \frac{11}{3}$$

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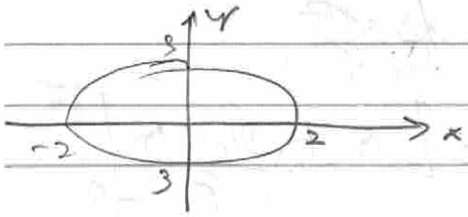
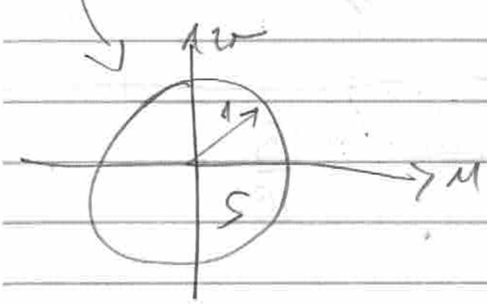
2. $\int_D x^2 dA$; $9x^2 + 4y^2 = 36 \Leftrightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1$



$$\left.\begin{array}{l} x = 2u \\ y = 3v \end{array}\right\} \quad \frac{x^2}{4} + \frac{y^2}{9} = 1 \Leftrightarrow \underbrace{u^2 + v^2 = 1}_{\text{circulo}}$$



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

$$\therefore \int_D x^2 dA = \int_S 4u^2 \frac{\partial(x,y)}{\partial(u,v)}\, du\, dv = \int_S 24u^2\, du\, dv$$

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$$= \int_S 24u^2 \, \underbrace{du\,dv} = \int_{r=0}^{1}\int_{\theta=0}^{2\pi} 24 r^2\cos^2\theta \, \underbrace{r\,dr\,d\theta}$$

$$\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = r$$

$$= \int_{\theta=0}^{2\pi}\left(\int_{r=0}^{1} 24\cos^2\theta \, r^3 dr\right) d\theta$$

$$= \int_{\theta=0}^{2\pi} 24\cos^2\theta \left.\frac{r^4}{4}\right]_0^1 d\theta$$

$$= \int_0^{2\pi} 6\,\underbrace{\cos^2\theta}\, d\theta$$

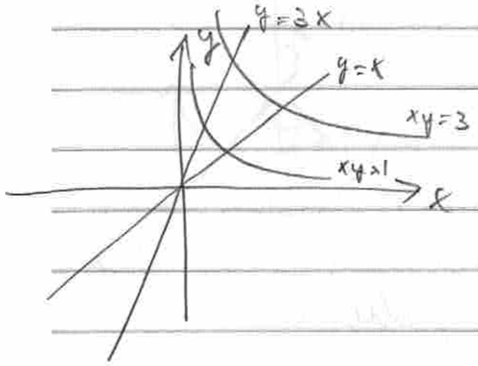
$$= \cancel{6}\int_0^{2\pi} \frac{1+\cos 2\theta}{\cancel{2}} \, d\theta$$

$$= 3\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= 3\left[2\pi\right]$$

$$= 6\pi \;//$$

3. $\int_D xy\, dA$

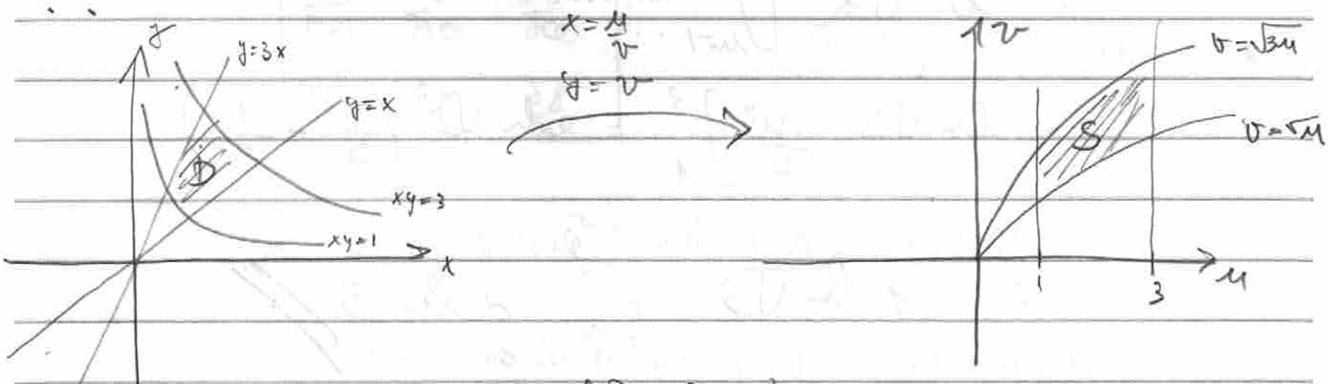


$x = \frac{u}{v}$, $y = v$ $\Big\}$ $xy = u$ $\therefore$ $xy = 1 \rightarrow u = 1$
$xy = 3 \rightarrow u = 3$

$y = x : v = \frac{u}{v}, \quad v^2 = u \rightarrow v = \sqrt{u}$

$y = 3x : v = \frac{3u}{v}, \quad v^2 = 3u \rightarrow v = \sqrt{3u}$

$\therefore$



$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{bmatrix} = \frac{1}{v}$$

$\therefore$

$$\int_D xy\, dA = \int_{u=1}^{3}\int_{v=\sqrt{u}}^{\sqrt{3u}} \frac{u}{v}\, v \underbrace{\frac{\partial(x,y)}{\partial(u,v)}}\, du\, dv$$

$$= \int_{u=1}^{3}\int_{v=\sqrt{u}}^{\sqrt{3u}} \frac{u}{v}\cancel{v}\, \frac{1}{\cancel{v}}\, du\, dv =$$

$$= \int_{u=1}^{3} \left(\int_{v=\sqrt{u}}^{\sqrt{3u}} \frac{u}{v} \, dv \right) du$$

$$= \int_{u=1}^{3} u \ln v \Big]_{v=\sqrt{u}}^{\sqrt{3u}} du$$

$$= \int_{u=1}^{3} u \left(\ln\sqrt{3u} - \ln\sqrt{u} \right) du$$

$$\int_{u=1}^{3} u \ln \frac{\sqrt{3u}}{\sqrt{u}} \, du$$

$$= \int_{u=1}^{3} u \ln\sqrt{3} \, du$$

$$= \ln\sqrt{3} \int_{u=1}^{3} u \, du$$

$$= \ln\sqrt{3} \, \frac{u^2}{2} \Big]_1^3 = \ln\sqrt{3} \left(\frac{9}{2} - \frac{1}{2} \right)$$

$$= 4 \ln\sqrt{3} = 2 \ln 3$$

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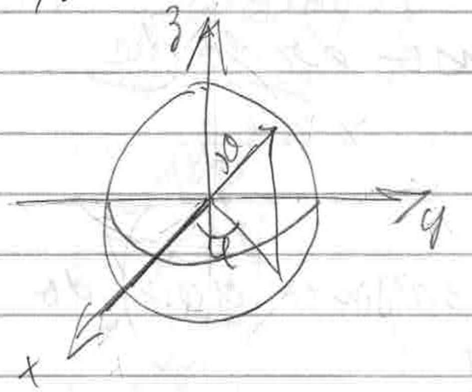
4.



$$\int_B 3\,dV = \int_B 3\,\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\,dr\,d\theta\,d\varphi$$

$$x \geq 0 \implies \underline{\underline{B:}} \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq \pi \\ -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} \end{cases}$$

$$\begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r \cos\theta \end{cases}$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \det \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \det \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\cos^2\theta\cos^2\varphi\sin\theta + r^2\sin^3\theta\sin^2\varphi$$

$$+ r^2\sin\theta\cos^2\theta\sin^2\varphi + r^2\sin^3\theta\cos^2\varphi$$

$$= r^2\cos^2\theta\sin\theta\,(\cos^2\varphi + \sin^2\varphi) + r^2\sin^3\theta\,(\sin^2\varphi + \cos^2\varphi)$$

$$= r^2\cos^2\theta\sin\theta + r^2\sin^3\theta$$

$$= r^2\sin\theta\,(\cos^2\theta + \sin^2\theta)$$

$$= r^2\sin\theta \;//$$

$$\therefore \int_B 3\,dV = \int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\varphi=-\frac{\pi}{2}}^{\frac{\pi}{2}} 3r^2 \sin\theta \, dr\, d\theta\, d\varphi$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{\pi}\left(\int_{\varphi=-\frac{\pi}{2}}^{\frac{\pi}{2}} 3r^2 \sin\theta \, d\varphi\right) d\theta\, dr$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{\pi} 3r^2 \sin\theta \, \pi \, d\theta\, dr$$

$$= \int_{r=0}^{1}\left(\int_{\theta=0}^{\pi} 3\pi r^2 \sin\theta \, d\theta\right) dr$$

$$= \int_{r=0}^{1} 3\pi r^2 (-\cos\theta)\Big]_0^{\pi} dr$$

$$= \int_{r=0}^{1} 3\pi r^2 (\underbrace{-\cos\pi}_{1} + \underbrace{\cos 0}_{1})\, dr$$

$$= \int_{r=0}^{1} 6\pi r^2 \, dr$$

$$= 6\pi \frac{r^3}{\cancel{3}}\Big]_0^1$$

$$= 2\pi \; //$$

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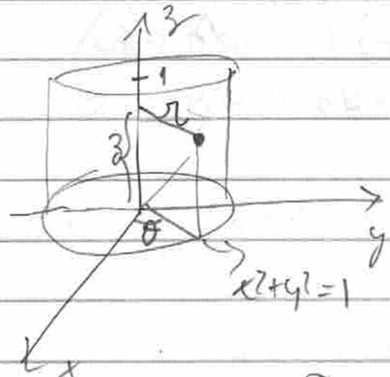
5. $\int_B x^2\,dx\,dy\,dz$



$$\begin{cases} x = r\cos\theta & ; \quad 0 \le \theta \le 2\pi \\ y = r\sin\theta & ; \quad 0 \le r \le 1 \\ z = z & ; \quad 0 \le z \le 1 \end{cases}$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \det \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

$\therefore$

$$\int_B x^2\,dx\,dy\,dz = \int r^2\cos^2\theta \,\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\,dr\,d\theta\,dz$$

$$= \int_{z=0}^{1}\int_{r=0}^{1}\int_{\theta=0}^{2\pi} r^2\cos^2\theta\, r\,dr\,d\theta\,dz$$

$$= \int_{z=0}^{1}\int_{r=0}^{1}\left(\int_{\theta=0}^{2\pi} r^3\cos^2\theta\,d\theta\right)dr\,dz$$

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$$= \int_{z=0}^{1} \int_{r=0}^{1} \int_{\theta=0}^{2\pi} r^3 \left(\frac{1+\cos 2\theta}{2} \right) d\theta \, dr \, dz$$

$$= \int_{z=0}^{1} \int_{r=0}^{1} \frac{r^3}{2} \left(\theta + \frac{\sin 2\theta}{2} \right) \Big]_0^{2\pi} dr \, dz$$

$$= \int_{z=0}^{1} \int_{r=0}^{1} \frac{r^3}{\cancel{2}} \, \cancel{2}\pi \, dr \, dz$$

$$= \int_{z=0}^{1} \pi \frac{r^4}{4} \Big]_0^1 dz$$

$$= \int_{z=0}^{1} \frac{\pi}{4} \, dz = \frac{\pi}{4}$$

13. $\displaystyle\int_D x^2\, dA$

$D:\ 9x^2 + 4y^2 = 36 \longrightarrow \dfrac{9x^2}{36} + \dfrac{4y^2}{36} = 1$

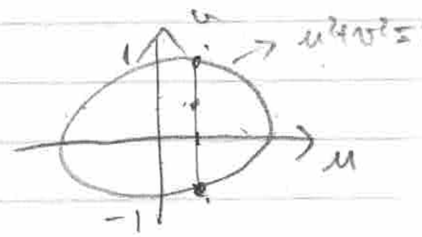
$(x,y) \to (u,v)\ ;\ \begin{cases} x = 2u \\ y = 3v \end{cases}$ $\quad \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

$\therefore\ u^2 + v^2 = 1$

$$\int_{D(x,y)} x^2\, dA = \int_{D(u,v)} (4u^2) \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| du\, dv$$

$$= \int_{D(u,v)} 4u^2 \left| \det \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \right| du\, dv$$

$$= \int_{D(u,v)} 4u^2 \cdot 6\ du\, dv$$



$$= 24 \int u^2\, du\, dv$$

$$= 24 \cdot \int_{u=-1}^{1} \left(\int_{v=-\sqrt{1-u^2}}^{\sqrt{1-u^2}} u^2\, dv \right) du$$

$D:\ -1 \le u \le 1$

$-\sqrt{1-u^2} \le v \le \sqrt{1-u^2}$

$$= 24 \int_{u=-1}^{1} u^2 v \Big]_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} du$$

$$= 24 \int_{-1}^{1} 2u^2\sqrt{1-u^2}\, du = 48 \int_{-1}^{1} u^2 \sqrt{1-u^2}\, du$$

$$\int u^2\sqrt{1-u^2}\,du$$

Seja $u = \sin\theta$

$du = \cos\theta\, d\theta$

$$\int u^2\sqrt{1-u^2}\,du = \int \sin^2\theta\,\underbrace{\sqrt{\cos^2\theta}}\,\cos\theta\,d\theta$$

$$= \int \sin^2\theta\,\underbrace{\cos\theta}\,\cos\theta\,d\theta$$

$$= \int \sin^2\theta\cos^2\theta\,d\theta$$

$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$$= \int (\sin\theta\cos\theta)^2\,d\theta$$

$\sin^2 2\theta = \dfrac{1-\cos 4\theta}{2}$

$$= \int \left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$

$$= \frac{1}{4}\int \sin^2 2\theta\,d\theta$$

$$= \frac{1}{4}\int \frac{1-\cos 4\theta}{2}\,d\theta$$

$$= \frac{1}{8}\int d\theta - \frac{1}{8}\int \cos 4\theta\,d\theta$$

$$= \frac{1}{8}\theta - \frac{1}{32}\sin 4\theta$$

$$= \frac{1}{8}\arcsin u - \frac{1}{32}\sin(4\arcsin u)$$

$$\int_{-1}^{1} u^2\sqrt{1-u^2}\,du = \frac{1}{8}\arcsin u\,\Big]_{-1}^{1} - \frac{1}{32}\sin(4\arcsin u)\Big|_{-1}^{1}$$

$$= \frac{1}{8}\arcsin 1 - \frac{1}{8}\arcsin(-1) -$$

$$- \frac{1}{32}\sin(4\arcsin 1) + \frac{1}{32}\sin(4\arcsin -1)$$

$$= \frac{1}{8}\frac{\pi}{2} - \frac{1}{8}\left(-\frac{\pi}{2}\right) - \frac{1}{32}\sin\cancel{\left(4\cdot\frac{\pi}{2}\right)}^{0} +$$

$$+ \frac{1}{32}\sin\cancel{\left(4\left(-\frac{\pi}{2}\right)\right)}^{0}$$

$$= \frac{\pi}{16} + \frac{\pi}{16} = \frac{\pi}{8}$$

$$\int_{D(x,y)} x^2\, dA = 48\int_{-1}^{1} u^2\sqrt{1-u^2}\, du$$

$$= 48\,\frac{\pi}{8}$$

$$= 6\pi$$