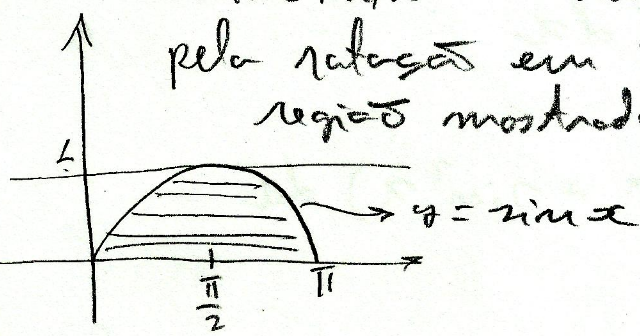
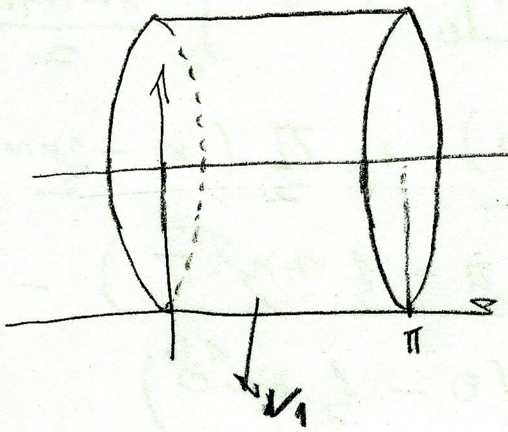
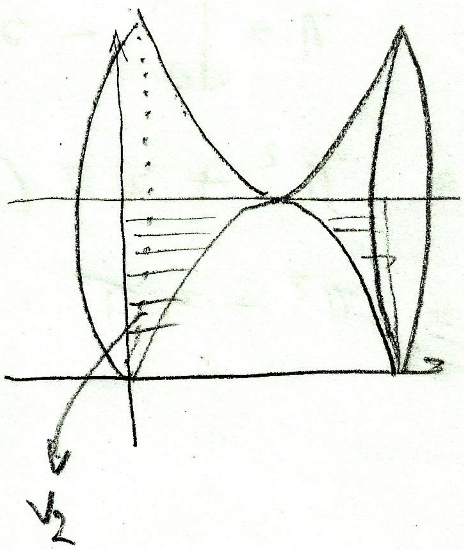
## #9. Determine o volume do sólido obtido pela rotação em torno da reta $y=1$ da região mostrada na figura.




### 1ª Solução



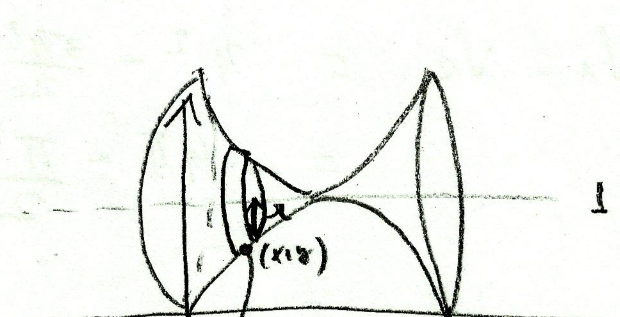




$V = V_1 - V_2$ (Volume procurado)

Temos

$$V_1 = \underbrace{\pi \cdot 1^2}_{\text{Área da base}} \cdot \underbrace{\pi}_{\text{altura}} = \pi^2$$

### $V_2$




$r = 1 - y(x)$

$$dV = \pi r^2 dx = \pi(1 - y(x))^2 dx$$

$$V_2 = \int_0^{\pi} \pi (1 - \sin x)^2 \, dx$$

$$= \int_0^{\pi} \pi (1 - 2 \sin x + \sin^2 x) \, dx$$

$$= \int_0^{\pi} \pi \, dx - 2\pi \int_0^{\pi} \sin x \, dx + \pi \int_0^{\pi} \sin^2 x \, dx$$

$$= \pi x \Big]_0^{\pi} - 2\pi (-\cos x) \Big]_0^{\pi} + \pi \int_0^{\pi} \frac{1 - \cos 2x}{2} \, dx$$

$$= \pi^2 + 2\pi (\underbrace{\cos \pi}_{-1} - \underbrace{\cos 0}_{1}) + \frac{\pi}{2} \left( x - \frac{1}{2} \sin 2x \right) \Big]_0^{\pi}$$

$$= \pi^2 - 4\pi + \frac{\pi}{2} \left( \pi - \frac{1}{2} \cancel{\sin 2\pi}^{\,0} \right) - \frac{\pi}{2} \left( 0 - \frac{1}{2} \cancel{\sin 0}^{\,0} \right)$$

$$= \pi^2 - 4\pi + \frac{\pi^2}{2}$$

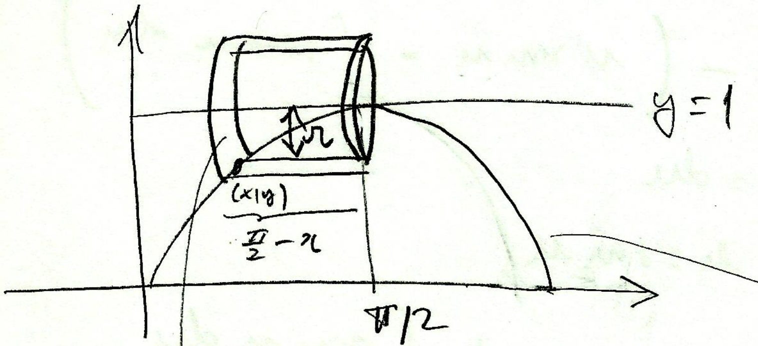
$$= \frac{3\pi^2}{2} - 4\pi$$

Daí

$$V = V_1 - V_2 = \pi^2 - \frac{3\pi^2}{2} + 4\pi$$

$$= 4\pi - \frac{\pi^2}{2} \;//$$

2ª Solução




$r = 1 - y$

$y = \sin x$

$\therefore\ x = \arcsin y$

$$dV_1 = 2\pi r \left(\frac{\pi}{2} - x\right) dy$$

$$= 2\pi(1-y)\left(\frac{\pi}{2} - x(y)\right) dy$$

$$V_1 = \int_0^1 2\pi(1-y)\left(\frac{\pi}{2} - \arcsin y\right) dy$$

Cálculo da integral indefinida:

$$2\pi \int (1-y)\left(\frac{\pi}{2} - \arcsin y\right) dy$$

Seja $u = \arcsin y \ \therefore\ \sin u = y$

Daí

$$2\pi \int (1-y)\left(\frac{\pi}{2} - \arcsin y\right) \underline{dy} =$$

$$= 2\pi \int (1 - \sin u)\left(\frac{\pi}{2} - u\right) \underline{\cos u\, du}$$

$$= 2\pi \int \left(\frac{\pi}{2} - u - \frac{\pi}{2}\sin u + u \sin u\right)\cos u\, du$$

$$= 2\pi \left\{ \int \frac{\pi}{2} \cos u\, du - \int u \cos u\, du - \frac{\pi}{2}\int \sin u \cos u\, du + \int u \sin u \cos u\, du \right]$$

$$\rightarrow \int \frac{\pi}{2} \cos u \, du = \frac{\pi}{2} \sin u = \frac{\pi}{2} y$$

$$\rightarrow -\int u \cos u \, du = -\left( u \sin u - \int \sin u \, du \right)$$

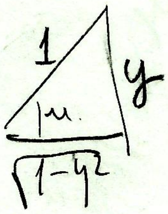
$w = u \rightarrow dw = du$

$dh = \cos u \, du \rightarrow h = \sin u$

$$= -u \sin u + \int \sin u \, du$$

$$= -u \sin u - \cos u$$

$$= -(\arcsin y)\, y - \sqrt{1-y^2}$$




$$\rightarrow \int u \sin u \cos u \, du = \int u \, \frac{1}{2} \sin 2u \, du$$

$k = 2u$

$$= \int \frac{k}{2} \, \frac{1}{2} \sin k \, \frac{1}{2} dk$$

$$= \frac{1}{8} \int k \sin k \, dk$$

$w = k \rightarrow dw = dk$

$dh = \sin k \, dk \rightarrow h = -\cos k$

$$= \frac{1}{8} \left( -k \cos k + \int \cos k \, dk \right)$$

$$= \frac{1}{8} \left( -k \cos k + \sin k \right)$$

$$= \frac{1}{8} \left( -2u \cos 2u + \sin 2u \right)$$

$$= -\frac{1}{4}u\left(\cos^2 u - \sin^2 u\right) + \frac{1}{4}\sin u \cos u$$

$$= -\frac{1}{4}\arcsin y\left(1-y^2-y^2\right) + \frac{1}{4}y\sqrt{1-y^2}$$

$$= -\frac{1}{4}\arcsin y\left(1-2y^2\right) + \frac{1}{4}y\sqrt{1-y^2}$$

$$\rightarrow -\frac{\pi}{2}\int \sin u \cos u \, du =$$

$$= -\frac{\pi}{4}\int \sin 2u \, du$$

$$= -\frac{\pi}{4}\,\frac{-1}{2}\cos 2u$$

$$= \frac{\pi}{8}\left(\cos^2 u - \sin^2 u\right)$$

$$= \frac{\pi}{8}\left(1-y^2-y^2\right) = \frac{\pi}{8}\left(1-2y^2\right)$$

Daí

$$2\pi\int_0^1 (1-y)\left(\frac{\pi}{2}-\arcsin y\right)dy =$$

$$= 2\pi\left\{\frac{\pi}{2}y - y\arcsin y - \sqrt{1-y^2} - \frac{1}{4}\arcsin y\,(1-2y^2) + \frac{1}{4}y\sqrt{1-y^2} + \frac{\pi}{8}(1-2y^2)\right\}\Bigg]_{y=0}^{1}$$

$$= 2\pi\left\{\frac{\pi}{2} - \overbrace{\arcsin 1}^{\pi/2} - \cancelto{0}{\sqrt{1-1}} - \frac{1}{4}\overbrace{\arcsin 1}^{\pi/2}(1-2) + \frac{1}{4}1\cancelto{0}{\sqrt{1-1}} + \right.$$

$$+ \frac{\pi}{8}(1-2)$$

$$\left. - \left(-\sqrt{1} - \frac{1}{4}\cancelto{0}{\arcsin 0} + \frac{\pi}{8}\right)\right\}$$

$$= 2\pi\left(\cancel{\frac{\pi}{2}} - \cancel{\frac{\pi}{2}} + \cancel{\frac{1}{4}\frac{\pi}{2}} - \cancel{\frac{\pi}{8}} + 1 - \frac{\pi}{8}\right)$$

$$V_1 = 2\pi\left(1 - \frac{\pi}{8}\right)$$

Mas o volume $V_1$ corresponde a metade do sólido, logo o volume procurado é

$$V = 2V_1$$

$$= 2\ 2\pi\left(1-\frac{\pi}{8}\right)$$

$$= 4\pi\left(1-\frac{\pi}{8}\right)$$

$$= 4\pi - \frac{\pi^2}{2}$$