Subst. Trigonométrica

/ /

Ex. (pg. 550)

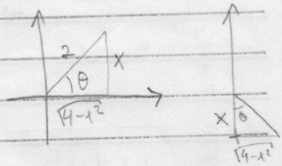
1. $\int \frac{dx}{x^2\sqrt{4-x^2}}$; $\sqrt{a^2-x^2} \longrightarrow x = a\sin\theta$

$\begin{cases} x = 2\sin\theta \ , \ -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \theta = \sin^{-1}\frac{x}{2} \end{cases}$

$dx = 2\cos\theta \, d\theta$

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta \, d\theta}{4\sin^2\theta \sqrt{4-4\sin^2\theta}} =$$

$$= \int \frac{\cancel{2\cos\theta} \, d\theta}{4\sin^2\theta \, \cancel{2\cos\theta}}$$



$$= \frac{1}{4}\int \operatorname{cosec}^2\theta \, d\theta$$

$$= -\frac{1}{4}\operatorname{cotg}\theta + C$$

$\operatorname{cotg}\theta = \frac{\sqrt{4-x^2}}{x}$

$$= -\frac{1}{4}\,\frac{\sqrt{4-x^2}}{x} + C$$

$$\boxed{\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{\sqrt{4-x^2}}{4x} + C}$$

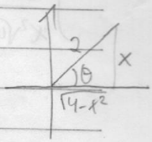
2. $\int \frac{\sqrt{4-x^2}}{x^2}\,dx$ $\qquad \sqrt{a^2-x^2} \longrightarrow x = a\sin\theta$

$$\begin{cases} x = 2\sin\theta & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \theta = \sin^{-1}\frac{x}{2} \end{cases}$$

$dx = 2\cos\theta\,d\theta$

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx = \int \frac{\sqrt{4(1-\sin^2\theta)}}{4\sin^2\theta}\,2\cos\theta\,d\theta$$

$$= \int \frac{\cancel{2}\cos\theta\ \cancel{2}\cos\theta}{\cancel{4}\sin^2\theta}\,d\theta$$



$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \cot^2\theta\,d\theta$$

$$= \int (\operatorname{cosec}^2\theta - 1)\,d\theta$$

$$= -\cot\theta - \theta + C$$

$$\boxed{\int \frac{\sqrt{4-x^2}}{x^2}\,dx = -\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\frac{x}{2} + C}$$

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3. $\int \frac{dx}{x\sqrt{x^2+4}} =$ $\qquad \sqrt{x^2+a^2} \longrightarrow x = a\,tg\theta$

$$\begin{cases} x = 2\,tg\theta \,, & -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta = tg^{-1}\frac{x}{2} \end{cases}$$

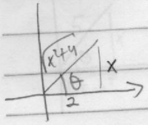
$dx = 2\sec^2\theta\, d\theta$

Daí,

$$\int \frac{dx}{x\sqrt{x^2+4}} = \int \frac{\cancel{2}\sec^2\theta\, d\theta}{2\,tg\theta\, \cancel{2\sec\theta}}$$

$$= \frac{1}{2}\int \frac{\sec\theta}{tg\theta}\, d\theta$$

$tg\theta = \frac{x}{2}$



$$= \frac{1}{2}\int \frac{1}{\cancel{\cos\theta}} \frac{1}{\frac{\sin\theta}{\cancel{\cos\theta}}}\, d\theta$$

$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{x}{\sqrt{x^2+4}}} = \frac{\sqrt{x^2+4}}{x}$

$cotg\,\theta = \frac{2}{x}$

$$= \frac{1}{2}\int \operatorname{cosec}\theta\, d\theta$$

$$= \frac{1}{2}\ln|\operatorname{cosec}\theta - cotg\,\theta| + C$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}}{x} - \frac{2}{x}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}-2}{x}\right| + C$$

$\therefore$

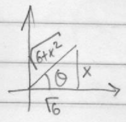
$$\boxed{\int \frac{dx}{x\sqrt{x^2+4}} = \frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}-2}{x}\right| + C}$$

4. $\int \frac{x^2\,dx}{\sqrt{x^2+6}}$ $\qquad \sqrt{x^2+a^2} \longrightarrow x = a\,tg\,\theta$

$dx = \sqrt{6}\sec^2\theta\,d\theta$ $\qquad \begin{cases} x = \sqrt{6}\,tg\,\theta \\ \theta = tg^{-1}\frac{x}{\sqrt{6}} \end{cases}$ $\qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$\int \frac{x^2\,dx}{\sqrt{x^2+6}} = \int \frac{6\,tg^2\theta\,\sqrt{6}\,\sec^2\theta\,d\theta}{\sqrt{6(1+tg^2\theta)}}$$

$$= \int \frac{6\,tg^2\theta\,\sqrt{6}\,\sec^2\theta\,d\theta}{\sqrt{6}\,\cancel{\sec\theta}}$$



$$= \int 6\,tg^2\theta\,\sec\theta\,d\theta$$

$tg\,\theta = \frac{x}{\sqrt{6}}$

$$= 6\int(\sec^2\theta - 1)\sec\theta\,d\theta$$

$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{\sqrt{6}}{\sqrt{6+x^2}}} = \frac{\sqrt{6+x^2}}{\sqrt{6}}$

$$= 6\int \sec^3\theta\,d\theta - 6\int \sec\theta\,d\theta$$

$$= 6\left[\tfrac{1}{2}\sec\theta\,tg\,\theta + \tfrac{1}{2}\ln|\sec\theta + tg\,\theta|\right] - 6\ln|\sec\theta + tg\,\theta| + C$$

$$= 3\sec\theta\,tg\,\theta + 3\ln|\sec\theta + tg\,\theta| - 6\ln|\sec\theta + tg\,\theta| + C$$

$$= 3\sec\theta\,tg\,\theta - 3\ln|\sec\theta + tg\,\theta| + C$$

$$= 3\,\frac{\sqrt{6+x^2}}{\sqrt{6}}\,\frac{x}{\sqrt{6}} - 3\ln\left|\frac{\sqrt{6+x^2}}{\sqrt{6}} + \frac{x}{\sqrt{6}}\right| + C$$

$\therefore$

$$\boxed{\int \frac{x^2\,dx}{\sqrt{x^2+6}} = \frac{x}{2}\sqrt{6+x^2} - 3\ln\left|\sqrt{6+x^2} + x\right| + C'}$$

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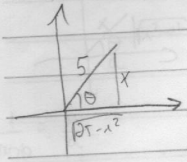
5. $\displaystyle\int \frac{dx}{x\sqrt{25-x^2}}$ $\qquad \sqrt{a^2-x^2} \longrightarrow x = a\sin\theta$

$$dx = 5\cos\theta\, d\theta \qquad \begin{cases} x = 5\sin\theta \\ \theta = \sin^{-1}\dfrac{x}{5} \end{cases} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\therefore \int \frac{dx}{x\sqrt{25-x^2}} = \int \frac{\cancel{5}\cos\theta\, d\theta}{\cancel{5}\sin\theta\sqrt{25(1-\sin^2\theta)}}$$



$$= \int \frac{\cancel{\cos\theta}\, d\theta}{\sin\theta\, 5\cancel{\cos\theta}}$$

$$= \frac{1}{5}\int \operatorname{cosec}\theta\, d\theta$$

$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta} = \dfrac{5}{x}$

$\operatorname{cotg}\theta = \dfrac{\sqrt{25-x^2}}{x}$

$$= \frac{1}{5}\ln\left|\operatorname{cosec}\theta - \operatorname{cotg}\theta\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{5}{x} - \frac{\sqrt{25-x^2}}{x}\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{5-\sqrt{25-x^2}}{x}\right| + C$$

$$\therefore \boxed{\int \frac{dx}{x\sqrt{25-x^2}} = \frac{1}{5}\ln\left|\frac{5-\sqrt{25-x^2}}{x}\right| + C}$$

6. $\int \sqrt{1-u^2}\, du$

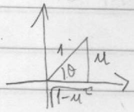
$$\begin{cases} u = 1 \sin\theta, & -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ \theta = \sin^{-1} u \end{cases}$$

$du = \cos\theta \, d\theta$

$\therefore$

$$\int \sqrt{1-u^2}\, du = \int \underbrace{\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta =$$

$$= \int \cos\theta \cos\theta \, d\theta$$



$$= \int \cos^2\theta \, d\theta = \int \frac{1+\cos 2\theta}{2} d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$\sin\theta = u$

$\cos\theta = \sqrt{1-u^2}$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$$

$$= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$$

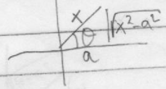
$$\boxed{\int \sqrt{1-u^2}\, du = \frac{1}{2}\sin^{-1} u + \frac{1}{2} u \sqrt{1-u^2} + C}$$

7. $\int \frac{dx}{\sqrt{x^2-a^2}}$ $\quad \begin{cases} x = a\sec\theta \\ \theta = \sec^{-1}\frac{x}{a} \end{cases}$; $\theta \in (0,\frac{\pi}{2}) \cup (\pi,\frac{3\pi}{2})$

$dx = a\sec\theta\, tg\theta\, d\theta$

$$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{\cancel{a}\sec\theta\, tg\theta\, d\theta}{\cancel{a}\sqrt{\sec^2\theta - 1}} = \int \frac{\sec\theta\, \cancel{tg\theta}\, d\theta}{\cancel{tg\theta}}$$

$$= \ln|\sec\theta + tg\theta| + C$$



$$\therefore = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C$$

$\begin{cases} \sec\theta = \frac{1}{\cos\theta} = \frac{x}{a} \\ tg\theta = \sqrt{x^2-a^2}/a \end{cases}$

$$\boxed{\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C}$$

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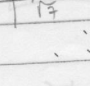
8. $\int \frac{dw}{w^2\sqrt{w^2-7}}$ $\quad \begin{cases} w = \sqrt{7}\sec\theta \\ \sec\theta = \frac{w}{\sqrt{7}} \end{cases}$ $\quad \theta \in [0,\frac{\pi}{2}) \cup [\pi,\frac{3\pi}{2})$

$dw = \sqrt{7}\sec\theta\, tg\theta\, d\theta$

$$\int \frac{dw}{w^2\sqrt{w^2-7}} = \int \frac{\cancel{\sqrt{7}}\,\cancel{\sec\theta}\,\cancel{tg\theta}\, d\theta}{7\sec^2\theta\, \cancel{\sqrt{7}}\,\cancel{tg\theta}} = \frac{1}{7}\int \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{1}{7}\int \cos\theta\, d\theta$$



$$= \frac{1}{7}\sin\theta + C$$

$\sec\theta = \frac{w}{\sqrt{7}}$
$\downarrow$
$\cos\theta = \frac{\sqrt{7}}{w}$

$$\therefore \boxed{\int \frac{dw}{w^2\sqrt{w^2-7}} = \frac{1}{7}\frac{\sqrt{w^2-7}}{w} + C}$$

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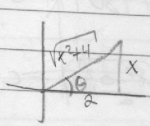
9. $\displaystyle\int \frac{x^2dx}{(x^2+4)^2}$ $\quad\left|\begin{array}{l} x = 2\,tg\,\theta \,, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta = tg^{-1}\frac{x}{2} \end{array}\right.$

$dx = 2\sec^2\theta\, d\theta$

$\therefore$

$$\int \frac{x^2dx}{(x^2+4)^2} = \int \frac{4\,tg^2\theta \; 2\sec^2\theta\, d\theta}{(4\,tg^2\theta+4)^2}$$



$$= \int \frac{8\,tg^2\theta\, \sec^2\theta}{16\,(1+tg^2\theta)^2}\, d\theta$$

$$= \frac{1}{2}\int \frac{tg^2\theta\,\sec^2\theta}{(\sec^2\theta)^2}\, d\theta$$

$$= \frac{1}{2}\int \frac{tg^2\theta\,\cancel{\sec^2\theta}}{\sec^{\cancel{4}\,2}\theta}\, d\theta$$

$$= \frac{1}{2}\int \frac{\sin^2\theta}{\cancel{\cos^2\theta}}\,\cancel{\cos^2\theta}\, d\theta$$

$$= \frac{1}{2}\int \sin^2\theta\, d\theta$$

$$= \frac{1}{2}\int \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right] + C$$

$$= \frac{1}{4}\left[\theta - \sin\theta\cos\theta\right] + C$$

$$= \frac{1}{4}\left[tg^{-1}\frac{x}{2} - \frac{x}{\sqrt{x^2+4}}\,\frac{2}{\sqrt{x^2+4}}\right] + C$$

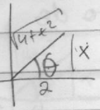
$$= \frac{1}{4}\,tg^{-1}\frac{x}{2} - \frac{x}{2(x^2+4)} + C \quad //$$

10. $\displaystyle\int \frac{dx}{(4+x^2)^{3/2}}$ $\quad \left\{ \begin{array}{l} x = 2\,tg\,\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta = tg^{-1} \frac{x}{2} \end{array} \right.$

$dx = 2 \sec^2\theta \, d\theta$

$$\int \frac{dx}{(4+x^2)^{3/2}} = \int \frac{2\sec^2\theta \, d\theta}{(4 + 4\,tg^2\theta)^{3/2}}$$

$$= \int \frac{2\sec^2\theta \, d\theta}{4^{3/2}(1+tg^2\theta)^{3/2}}$$



$$= \int \frac{2\sec^2\theta \, d\theta}{2^3\,(\sec^2\theta)^{3/2}}$$

$$= \frac{1}{4}\int \frac{\sec^2\theta}{\sec^3\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{1}{4}\int \cos\theta \, d\theta$$

$$= \frac{1}{4} \sin\theta + C$$

$$\boxed{\int \frac{dx}{(4+x^2)^{3/2}} = \frac{1}{4}\frac{x}{\sqrt{4+x^2}} + C}$$

11. $\int \frac{dx}{(4x^2-9)^{3/2}} =$

$$= \int \frac{dx}{[4(x^2-\frac{9}{4})]^{3/2}} = \int \frac{dx}{4^{3/2}(x^2-\frac{9}{4})^{3/2}}$$

$$= \frac{1}{8} \int \frac{dx}{(x^2-\frac{9}{4})^{3/2}},$$

$x = \frac{3}{2}\sec\theta$; $[0,\frac{\pi}{2}) \cup [\pi,\frac{3\pi}{2}) \ni \theta$

$\theta = \sec^{-1}\frac{2x}{3}$

$dx = \frac{3}{2}\sec\theta \, tg\theta \, d\theta$

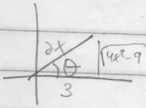
$$= \frac{1}{8} \int \frac{\frac{3}{2}\sec\theta \, tg\theta \, d\theta}{(\frac{9}{4} tg^2\theta)^{3/2}}$$

$$= \frac{1}{8} \int \frac{\frac{3}{2}\sec\theta \, \cancel{tg\theta} \, d\theta}{(\frac{9}{4})^{3/2} tg^{\cancel{3}}\theta}$$

$\sec\theta = \frac{2x}{3}$

$\downarrow$

$\cos\theta = \frac{3}{2x}$



$$= \frac{1}{8} \times \frac{3}{2} \frac{1}{(\frac{9}{4})^{3/2}} \int \frac{\sec\theta}{tg^2\theta} \, d\theta$$

$\csc\theta = \frac{2x}{\sqrt{4x^2-9}}$

$$= \frac{1}{\cancel{8}} \cdot \frac{3}{2} \left(\frac{\cancel{2}}{3}\right)^3 \int \frac{1}{\cancel{\cos\theta}} \frac{1}{\frac{\sin^2\theta}{\cos^{\cancel{2}}\theta}} \, d\theta$$

$$= \frac{1}{18} \int \cos\theta \, (\sin\theta)^{-2} \, d\theta$$

$$= \frac{1}{18} (-) \sin\theta^{-1} + C$$

$$= -\frac{1}{18} \csc\theta + C$$

$$= -\frac{1}{18} \frac{2x}{\sqrt{4x^2-9}} + C$$

$$\boxed{\int \frac{dx}{(4x^2-9)^{3/2}} = -\frac{x}{9\sqrt{4x^2-9}} + C}$$

$$\int_a^b f(x)\,dx = F\Big|_a^b = F(b) - F(a)$$

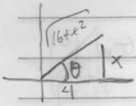
12. $\displaystyle\int \frac{dx}{x^4\sqrt{16+x^2}}$

$$\begin{cases} x = 4\,tg\,\theta & -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta = tg^{-1}\frac{x}{4} \end{cases}$$

$$dx = 4\sec^2\theta\,d\theta$$

$$\int \frac{dx}{x^4\sqrt{16+x^2}} = \int \frac{4\sec^2\theta\,d\theta}{4^4\,tg^4\theta\,\sqrt{16+16\,tg^2\theta}}$$

$$= \frac{1}{4^3}\int \frac{\sec^2\theta\,d\theta}{tg^4\theta\;4\cancel{\sec\theta}}$$



$$= \frac{1}{4^4}\int \frac{\sec\theta\,d\theta}{tg^4\theta}$$

$cosec\,\theta = \frac{1}{\sin\theta} = \frac{\sqrt{16+x^2}}{x}$

$$= \frac{1}{4^4}\int \frac{1}{\cos\theta}\,\frac{1}{\frac{\sin^4\theta}{\cos^4\theta}}\,d\theta$$

$$= \frac{1}{4^4}\int \frac{\cos^3\theta}{\sin^4\theta}\,d\theta = \frac{1}{4^4}\int \cos\theta\,\frac{\cos^2\theta\,d\theta}{\sin^4\theta}$$

$$= \frac{1}{4^4}\int \cos\theta\,\frac{(1-\sin^2\theta)}{\sin^4\theta}\,d\theta$$

$$\begin{array}{r} 256 \\ 3 \\ \hline 768 \end{array}$$

$$= \frac{1}{4^4}\int \cos\theta\,\sin^{-4}\theta\,d\theta - \frac{1}{4^4}\int \cos\theta\,(\sin\theta)^{-2}\,d\theta$$

$$= \frac{1}{4^4}\,\frac{\sin^{-3}\theta}{-3} - \frac{1}{4^4}\,\frac{(\sin\theta)^{-1}}{-1} + C$$

$$= -\frac{1}{768}\,cosec^3\theta + \frac{1}{256}\,cosec\,\theta + C$$

$$\boxed{\int \frac{dx}{x^4\sqrt{16+x^2}} = -\frac{1}{768}\,\frac{(16+x^2)^{3/2}}{x^3} + \frac{1}{256}\,\frac{\sqrt{16+x^2}}{x} + C}$$

13. $\displaystyle\int \frac{2\,dt}{t\sqrt{t^4+25}}$

$t^2 = u \longrightarrow du = 2t\,dt \Leftrightarrow dt = \frac{1}{2t}\,du = \frac{1}{2\sqrt{u}}\,du$

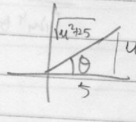
$$= \int \frac{\cancel{2}\,\frac{1}{2\sqrt{u}}\,du}{\sqrt{u}\,\sqrt{u^2+25}}$$

$$= \int \frac{du}{u\sqrt{u^2+25}}$$

$\begin{cases} u = 5\,tg\,\theta \,, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta = tg^{-1}\frac{u}{5} \\ du = 5\sec^2\theta\,d\theta \end{cases}$

$$= \int \frac{\cancel{5}\sec^2\theta\,d\theta}{\cancel{5}\,tg\,\theta\,\sqrt{25\,tg^2\theta+25}}$$

$$= \int \frac{\sec^2\theta\,d\theta}{tg\,\theta\;5\cancel{\sec\theta}} = \frac{1}{5}\int \frac{\sec\theta}{tg\,\theta}\,d\theta$$



$$= \frac{1}{5}\int \frac{1}{\cancel{\cos\theta}}\,\frac{1}{\frac{\sin\theta}{\cancel{\cos\theta}}}\,d\theta$$

$$= \frac{1}{5}\int \operatorname{cossec}\theta\,d\theta$$

$\begin{cases} \operatorname{cossec}\theta = \frac{1}{\sin\theta} = \frac{\sqrt{u^2+25}}{u} \\ \operatorname{cotg}\theta = \frac{5}{u} \end{cases}$

$$= \frac{1}{5}\ln\left|\operatorname{cossec}\theta - \operatorname{cotg}\theta\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{\sqrt{u^2+25}}{u} - \frac{5}{u}\right| + C$$

$(u = t^2)$

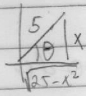
$$\therefore\; = \frac{1}{5}\ln\left|\underbrace{\frac{\sqrt{t^4+25}}{t^2} - \frac{5}{t^2}}_{>0\ \forall t}\right| + C$$

$$\boxed{\int \frac{2\,dt}{t\sqrt{t^4+25}} = \frac{1}{5}\ln\left(\frac{\sqrt{t^4+25}-5}{t^2}\right) + C}$$

14. $\displaystyle\int \frac{x^3\,dx}{(25-x^2)^2}$ $\quad \left| \begin{array}{l} x = 5\,\text{sin}\,\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta = \text{sin}^{-1}\frac{x}{5} \\ dx = 5\cos\theta\, d\theta \end{array} \right.$

$$\int \frac{x^3\,dx}{(25-x^2)^2} = \int \frac{5^3\,\text{sin}^3\theta\;5\cos\theta\,d\theta}{(25\,(1-\text{sin}^2\theta)\,)^2}$$

$$= \int \frac{\cancel{5^4}}{\cancel{5^4}}\,\frac{\text{sin}^3\theta\,\cos\theta}{(\cos^2\theta\,)^2}\,d\theta$$



$$= \int \text{sin}^3\theta\,\frac{\cos\theta}{\cos^4\theta}\,d\theta$$

$\text{tg}\,\theta = \dfrac{x}{\sqrt{25-x^2}}$

$$= \int \frac{\text{sin}^3\theta}{\cos^3\theta}\,d\theta = \int (\text{tg}\,\theta)^3\,d\theta$$

$$= \int \text{tg}\,\theta\;\text{tg}^2\theta\,d\theta$$

$$= \int \text{tg}\,\theta\;(\sec^2\theta - 1)\,d\theta$$

$$= \int \text{tg}\,\theta\,\sec^2\theta\,d\theta - \int \text{tg}\,\theta\,d\theta$$

$$= \frac{\text{tg}^2\theta}{2} - \ln|\sec\theta| + C$$

$$= \frac{\text{tg}^2\theta}{2} + \ln\cos\theta + C$$

$$\boxed{\int \frac{x^3\,dx}{(25-x^2)^2} = \frac{x^2}{2(25-x^2)} + \ln\frac{\sqrt{25-x^2}}{5} + C}$$

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15. $\displaystyle\int \frac{dx}{\sqrt{4x+x^2}}$

$$4x+x^2 = x^2+4x+4-4 = (x+2)^2-4$$

$$= \int \frac{dx}{\sqrt{(x+2)^2-4}}$$

$u = x+2 \longrightarrow du = dx$

$$= \int \frac{du}{\sqrt{u^2-4}}$$

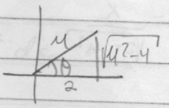
$u = 2\sec\theta \quad (0,\frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$

$\theta = \sec^{-1}\frac{u}{2}$

$du = 2\sec\theta\, tg\theta\, d\theta$

$$= \int \frac{2\sec\theta\, tg\theta\, d\theta}{\sqrt{4(\sec^2\theta-1)}}$$

$$= \int \frac{\sec\theta\, \cancel{tg\theta}}{\cancel{tg\theta}}\, d\theta$$



$\sec\theta = \frac{u}{2}$

$\cos\theta = \frac{2}{u}$

$tg\theta = \frac{\sqrt{u^2-4}}{2}$

$$= \ln|\sec\theta + tg\theta| + C$$

$$= \ln\left|\frac{u}{2} + \frac{\sqrt{u^2-4}}{2}\right| + C$$

$$= \ln\left|\frac{x+2}{2} + \frac{\sqrt{x^2+4x}}{2}\right| + C$$

$$\int \frac{dx}{\sqrt{4x+x^2}} = \ln\left|\frac{x+2+\sqrt{x^2+4x}}{2}\right| + C$$

$$\boxed{\int \frac{dx}{\sqrt{4x+x^2}} = \ln|x+2+\sqrt{x^2+4x}| + C'}$$

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16. $\int \frac{dx}{\sqrt{4x-x^2}}$ $\qquad 4x - x^2 = -(x^2-4x)$

$= -(x^2-4x+4)+4$

$= -(x-2)^2+4$

$\therefore$

$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}}$ $\qquad$ Seja $\begin{cases} u = x-2 \\ du = dx \end{cases}$

$= \int \frac{du}{\sqrt{4-u^2}}$ $\qquad \begin{cases} u = 2\sin\theta, & -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta = \sin^{-1}\frac{u}{2} \end{cases}$

$= \int \frac{2\cos\theta\, d\theta}{2\sqrt{1-\sin^2\theta}}$ $\qquad du = 2\cos\theta\, d\theta$

$= \int \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}}$

$= \theta + C$

$= \sin^{-1}\frac{u}{2} + C$

$\therefore$

$$\boxed{\int \frac{dx}{\sqrt{4x-x^2}} = \sin^{-1}\left(\frac{x-2}{2}\right) + C}$$

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17. $\displaystyle\int \frac{dx}{(5-4x-x^2)^{3/2}}$ $\qquad 5-4x-x^2 = 9-4-4x-x^2$

$\qquad\qquad\qquad = 9-(x+2)^2$

$$= \int \frac{dx}{(9-(x+2)^2)^{3/2}}$$

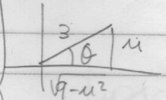
$u = x+2 \rightarrow du = dx$

$$= \int \frac{du}{(9-u^2)^{3/2}}$$

$\begin{cases} u = 3\sin\theta & -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta = \sin^{-1}\frac{u}{3} \\ du = 3\cos\theta\, d\theta \end{cases}$

$$= \int \frac{3\cos\theta\, d\theta}{(9(1-\sin^2\theta))^{3/2}}$$

$$= \int \frac{3\cos\theta\, d\theta}{3^3 \cos^3\theta}$$



$$= \frac{1}{9}\int \frac{1}{\cos^2\theta}\, d\theta$$

$\operatorname{tg}\theta = \dfrac{u}{\sqrt{9-u^2}}$

$$= \frac{1}{9}\int \sec^2\theta\, d\theta$$

$$= \frac{1}{9}\operatorname{tg}\theta + C$$

$$= \frac{1}{9}\frac{u}{\sqrt{9-u^2}} + C$$

$u = x+2$

$$= \frac{1}{9}\frac{x+2}{\sqrt{9-(x+2)^2}} + C$$

$\therefore$

$$\boxed{\int \frac{dx}{(5-4x-x^2)^{3/2}} = \frac{x+2}{9\sqrt{5-4x-x^2}} + C}$$

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18. $\displaystyle\int \frac{dx}{x\sqrt{x^4-4}} =$ $\left\{ x^2 = t \rightarrow 2x\,dx = dt \right.$

$dx = \frac{1}{2}\frac{dt}{\sqrt{t}}$

$= \displaystyle\int \frac{\frac{1}{2}\frac{dt}{\sqrt{t}}}{\sqrt{t}\sqrt{t^2-4}}$

$= \displaystyle\frac{1}{2}\int \frac{dt}{t\sqrt{t^2-4}}$ $\left\{ \begin{array}{l} t = 2\sec\theta \quad , [0,\frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2}) \\ \theta = \sec^{-1}\frac{t}{2} \\ dt = 2\sec\theta\, tg\theta\, d\theta \end{array} \right.$

$= \displaystyle\frac{1}{2}\int \frac{\cancel{2\sec\theta}\, tg\theta\, d\theta}{\cancel{2\sec\theta}\sqrt{4(\sec^2\theta-1)}}$

$= \displaystyle\frac{1}{2}\int \frac{\cancel{tg\theta}\, d\theta}{2\,\cancel{tg\theta}}$

$= \frac{1}{4}\theta + C$

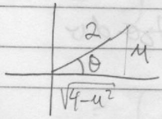
$= \frac{1}{4}\sec^{-1}\frac{t}{2}$

$$\boxed{\int \frac{dx}{x\sqrt{x^4-4}} = \frac{1}{4}\sec^{-1}\frac{x^2}{2} + C}$$

19. $\displaystyle\int \frac{sec^2 x \, dx}{(4-tg^2x)^{3/2}}$

Seja $u = tg\,x \rightarrow du = sec^2 x\, dx$

$\therefore \displaystyle\int \frac{sec^2 x}{(4-tg^2x)^{3/2}}\,dx = \int \frac{du}{(4-u^2)^{3/2}}$



$\left. \begin{array}{l} u = 2\sin\theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta = \sin^{-1}\frac{u}{2} \\ du = 2\cos\theta\, d\theta \end{array} \right.$

$tg\,\theta = \dfrac{u}{\sqrt{4-u^2}}$

$= \displaystyle\int \frac{2\cos\theta\, d\theta}{(4(1-\sin^2\theta))^{3/2}}$

$= \displaystyle\int \frac{2\cos\theta\, d\theta}{2^3 \cos^3\theta} = \frac{1}{4}\int \frac{1}{\cos^2\theta}\, d\theta$

$= \displaystyle\frac{1}{4}\int sec^2\theta\, d\theta$

$= \frac{1}{4}\, tg\,\theta + C$

$= \dfrac{1}{4}\dfrac{u}{\sqrt{4-u^2}} + C$

$u = tg\,x$

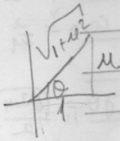
$\boxed{\displaystyle\int \frac{sec^2 x}{(4-tg^2x)^{3/2}}\,dx = \frac{1}{4}\frac{tg\,x}{\sqrt{4-tg^2x}} + C}$

20. $\displaystyle\int \frac{e^{-x}\,dx}{(9e^{-2x}+1)^{3/2}}$ $\quad \left\{ \begin{array}{l} u = 3e^{-x} \\ du = -3e^{-x}\,dx \end{array} \right.$

$$= \int \frac{-\frac{1}{3}\,du}{(u^2+1)^{3/2}}$$

$$= -\frac{1}{3}\int \frac{du}{(u^2+1)^{3/2}} \qquad \left\{ \begin{array}{l} u = tg\,\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta = tg^{-1} u \\ du = \sec^2\theta\, d\theta \end{array} \right.$$

$$= -\frac{1}{3}\int \frac{\sec^2\theta\, d\theta}{(tg^2\theta+1)^{3/2}}$$



$$= -\frac{1}{3}\int \frac{\cancel{\sec^2\theta}}{\sec^{\cancel{3}}\theta}\, d\theta = -\frac{1}{3}\int \cos\theta\, d\theta$$

$\sin\theta = \dfrac{u}{\sqrt{1+u^2}}$

$$= -\frac{1}{3}\sin\theta + C$$

$$= -\frac{1}{3}\frac{u}{\sqrt{1+u^2}} + C$$

$$= -\frac{1}{\cancel{3}}\frac{\cancel{3}e^{-x}}{\sqrt{1+9e^{-2x}}} + C$$

$\therefore$

$$\boxed{\int \frac{e^{-x}\,dx}{(9e^{-2x}+1)^{3/2}} = -\frac{e^{-x}}{\sqrt{1+9e^{-2x}}} + C}$$

21. $\int \frac{\ln^3 \omega \, d\omega}{\omega \sqrt{\ln^2 \omega - 4}}$

$$\begin{cases} u = \ln \omega \\ du = \frac{1}{\omega}\, d\omega \end{cases}$$

$$= \int \frac{u^3 \, du}{\sqrt{u^2 - 4}}$$

$$\begin{cases} u = 2 \sec\theta & (0, \tfrac{\pi}{2}) \cup (\pi, \tfrac{3\pi}{2}) \\ \theta = \sec^{-1} \frac{u}{2} \\ du = 2 \sec\theta \, tg\theta \, d\theta \end{cases}$$

$$= \int \frac{8 \sec^3\theta \, \cancel{2} \sec\theta \, tg\theta \, d\theta}{\sqrt{\cancel{4}(\underbrace{\sec^2\theta - 1}_{tg^2\theta})}}$$

$$= 8 \int \frac{\sec^4\theta \, \cancel{tg\theta}}{\cancel{tg\theta}} \, d\theta$$

$$= 8 \int \sec^4\theta \, d\theta$$

$$= 8 \int \sec^2\theta \, \underline{\sec^2\theta} \, d\theta$$

$$= 8 \int \sec^2\theta \, (1 + tg^2\theta) \, d\theta$$

$$= 8 \int \sec^2\theta \, d\theta + 8 \int \sec^2\theta \, tg^2\theta \, d\theta$$

$$= 8 \, tg\theta + \frac{8}{3} tg^3\theta + C$$

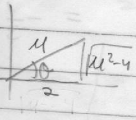
$$= 8 \frac{\sqrt{u^2-4}}{2} + \frac{8}{3} \left(\frac{\sqrt{u^2-4}}{2} \right)^3 + C$$

$$= 4 \sqrt{u^2-4} + \frac{1}{3} (u^2-4)^{3/2} + C$$

$$= 4 \sqrt{\ln^2\omega - 4} + \frac{1}{3} (\ln^2\omega - 4)^{3/2} + C$$

$$= \sqrt{\ln^2\omega - 4} \left[4 + \frac{\ln^2\omega - 4}{3} \right] + C$$

$$= \sqrt{\ln^2\omega - 4} \left(\frac{8 + \ln^2\omega}{3} \right) + C \quad //$$



$\sec\theta = \frac{u}{2}$

$\cos\theta = \frac{2}{u}$

$tg\theta = \frac{\sqrt{u^2-4}}{2}$

22. $$\int \frac{dz}{(z^2-6z+18)^{3/2}}$$

$$z^2-6z+18 = (z-3)^2+9$$

$$= \int \frac{dz}{[(z-3)^2+9]^{3/2}}$$

$$\begin{cases} u = z-3 \\ du = dz \end{cases}$$

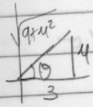
$$= \int \frac{du}{(u^2+9)^{3/2}}$$

$$\begin{cases} u = 3\,tg\,\theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \theta = tg^{-1}\frac{u}{3} \\ du = 3\sec^2\theta\, d\theta \end{cases}$$

$$= \int \frac{3\sec^2\theta\, d\theta}{[9(tg^2\theta+1)]^{3/2}}$$

$$= \int \frac{\cancel{3}\cancel{\sec^2\theta}\, d\theta}{3^{3^2}\sec^3\theta} = \frac{1}{9}\int \cos\theta\, d\theta$$

$$= \frac{1}{9}\sin\theta + C$$



$tg\,\theta = \frac{u}{3}$

$\sin\theta = \frac{u}{\sqrt{9+u^2}}$

$$= \frac{1}{9}\frac{u}{\sqrt{9+u^2}} + C$$

$u = z-3$

$$= \frac{1}{9}\frac{(z-3)}{\sqrt{9+(z-3)^2}} + C$$

$\therefore$

$$\boxed{\int \frac{dz}{(z^2-6z+18)^{3/2}} = \frac{1}{9}\frac{z-3}{\sqrt{z^2-6z+18}} + C}$$

thomas clarkson

1785

23. $\displaystyle\int \frac{e^t\,dt}{(e^{2t}+8e^t+7)^{3/2}} = \int \frac{e^t\,dt}{(e^{2t}+8e^t+16-9)^{3/2}}$

$\displaystyle = \int \frac{e^t\,dt}{((e^t+4)^2-9)^{3/2}}$

$\begin{cases} u = e^t+4 \\ du = e^t\,dt \end{cases}$

$\displaystyle = \int \frac{du}{(u^2-9)^{3/2}}$

$\begin{cases} u = 3\sec\theta \; ; \; (0,\frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2}) \\ \theta = \sec^{-1}\frac{u}{3} \\ du = 3\sec\theta\,\mathrm{tg}\,\theta\,d\theta \end{cases}$

$\displaystyle = \int \frac{3\sec\theta\,\mathrm{tg}\,\theta\,d\theta}{(9(\sec^2\theta-1))^{3/2}}$

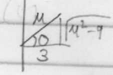
$\displaystyle = \int \frac{3\sec\theta\,\mathrm{tg}\,\theta\,d\theta}{3^3\,\mathrm{tg}^3\theta}$

$\displaystyle = \frac{1}{9}\int \frac{\sec\theta}{\mathrm{tg}^2\theta}\,d\theta = \frac{1}{9}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\,d\theta$

$\sec\theta = \frac{u}{3}$

$\downarrow$

$\cos\theta = 3/u$



$\displaystyle = \frac{1}{9}\int \cos\theta\,\sin^{-2}\theta\,d\theta$

$\sin\theta = \frac{\sqrt{u^2-9}}{u}$

$\downarrow$

$\operatorname{cosec}\theta = \frac{u}{\sqrt{u^2-9}}$

$\displaystyle = \frac{1}{9}\,\frac{(\sin\theta)^{-1}}{-1} + C$

$\displaystyle = -\frac{1}{9}\operatorname{cosec}\theta + C$

$\therefore$

$\displaystyle = -\frac{1}{9}\,\frac{u}{\sqrt{u^2-9}} + C$

$$\boxed{\int \frac{e^t\,dt}{(e^{2t}+8e^t+7)^{3/2}} = -\frac{1}{9}\cdot\frac{e^t+4}{\sqrt{e^{2t}+8e^t+7}} + C}$$

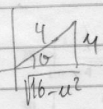
24. $$\int \frac{\sqrt{16-e^{2x}}}{e^x}\,dx \quad ; \quad u = e^x, \quad du = e^x\,dx$$

$$= \int \frac{\sqrt{16-u^2}}{u}\,\frac{du}{u}$$

$$= \int \frac{\sqrt{16-u^2}}{u^2}\,du \quad ; \quad \begin{cases} u = 4\sin\theta \ ; \ -\frac{\pi}{2}<\theta<\frac{\pi}{2} \\ \theta = \sin^{-1}\frac{u}{4} \\ du = 4\cos\theta\,d\theta \end{cases}$$

$$= \int \frac{\sqrt{16(1-\sin^2\theta)}\;4\cos\theta\,d\theta}{16\sin^2\theta}$$

$$= \int \frac{4\cos\theta\;4\cos\theta\,d\theta}{\cancel{16}\sin^2\theta}$$



$$= \int \cot^2\theta\,d\theta$$

$$= \int (\csc^2\theta - 1)\,d\theta$$

$$= \int \csc^2\theta\,d\theta - \int d\theta$$

$$= -\cot\theta - \theta + C$$

$$= -\frac{\sqrt{16-u^2}}{u} - \sin^{-1}\frac{u}{4} + C \quad ; \quad u = e^x$$

$$\therefore \boxed{\int \frac{\sqrt{16-e^{2x}}}{e^x}\,dx = -\frac{\sqrt{16-e^{2x}}}{e^x} - \sin^{-1}\frac{e^x}{4} + C}$$