Como $\operatorname{tg}\theta = x - 2$ e $-\frac{1}{2}\pi < 0 < \frac{1}{2}\pi$, $\theta = \operatorname{tg}^{-1}(x-2)$. Encontramos sen θ e cos θ das Figuras 1 (se $x \geqslant 2$) e 2 (se $x < 2$). Em ambos os casos

$$\operatorname{sen}\theta = \frac{x-2}{\sqrt{x^2-4x+5}} \qquad \cos\theta = \frac{1}{\sqrt{x^2-4x+5}}$$

Assim,

$$\int \frac{dx}{[(x-2)^2+1]^2} = \frac{1}{2}\operatorname{tg}^{-1}(x-2) + \frac{1}{2}\cdot\frac{x-2}{\sqrt{x^2-4x+5}}\cdot\frac{1}{\sqrt{x^2-4x+5}} + C_1$$

$$\int \frac{dx}{[(x-2)^2+1]^2} = \frac{1}{2}\operatorname{tg}^{-1}(x-2) + \frac{x-2}{2(x^2-4x+5)} + C_1 \qquad (8)$$

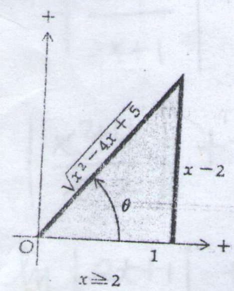


FIGURA 1

Considerando agora a outra integral no segundo membro de (7), teremos

$$\int \frac{dx}{(x^2-4x+4)+1} = \int \frac{dx}{(x-2)^2+1}$$

$$\int \frac{dx}{(x^2-4x+4)+1} = \operatorname{tg}^{-1}(x-2) + C_2$$

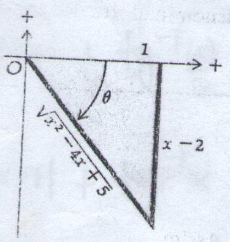


FIGURA 2

Substituindo essa relação e (8) em (7), teremos

$$\int \frac{(x-2)\,dx}{x(x^2-4x+5)^2}$$

$$= -\frac{2}{25}\ln|x| - \frac{1}{5(x^2-4x+5)} + \frac{1}{10}\operatorname{tg}^{-1}(x-2) + \frac{x-2}{10(x^2-4x+5)}$$

$$+ \frac{1}{25}\ln|x^2-4x+5| - \frac{4}{25}\operatorname{tg}^{-1}(x-2) + C$$

$$= \frac{1}{25}\ln\left|\frac{x^2-4x+5}{x^2}\right| - \frac{3}{50}\operatorname{tg}^{-1}(x-2) + \frac{x-4}{10(x^2-4x+5)} + C$$

Lista 5: #1 à 19

EXERCÍCIOS 9.6

Nos Exercícios de 1 a 20, calcule a integral indefinida.

1. $\int \frac{dx}{2x^3+x}$
2. $\int \frac{(x+4)\,dx}{x(x^2+4)}$
3. $\int \frac{dx}{16x^4-1}$
4. $\int \frac{(x^2-4x-4)\,dx}{x^3-2x^2+4x-8}$
5. $\int \frac{(t^2+t+1)\,dt}{(2t+1)(t^2+1)}$
6. $\int \frac{3w^3+13w+4}{w^3+4w}\,dw$
7. $\int \frac{(x^2+x)\,dx}{x^3-x^2+x-1}$
8. $\int \frac{dx}{9x^4+x^2}$
9. $\int \frac{dx}{x^3+x^2+x}$
10. $\int \frac{(x+3)\,dx}{4x^4+4x^3+x^2}$
11. $\int \frac{(2x^2-x+2)\,dx}{x^5+2x^3+x}$
12. $\int \frac{(2x^3+9x)\,dx}{(x^2+3)(x^2-2x+3)}$
13. $\int \frac{(5z^3-z^2+15z-10)\,dz}{(z^2-2z+5)^2}$
14. $\int \frac{dt}{(t^2+1)^3}$
15. $\int \frac{(x^2+2x-1)\,dx}{27x^3-1}$
16. $\int \frac{e^{5x}\,dx}{(e^{2x}+1)^2}$
17. $\int \frac{18\,dx}{(4x^2+9)^2}$
18. $\int \frac{(2x^2+3x+2)\,dx}{x^3+4x^2+6x+4}$
19. $\int \frac{(\sec^2 x+1)\sec^2 x\,dx}{1+\operatorname{tg}^3 x}$
20. $\int \frac{(6w^4+4w^3+9w^2+24w+32)\,dw}{(w^3+8)(w^2+3)}$

Nos Exercícios de 21 a 29, calcule a integral definida.

21. $\int_1^4 \frac{(4+5x^2)\,dx}{x^3+4x}$
22. $\int_0^1 \frac{x\,dx}{x^3+2x^2+x+2}$

ta 5- Respostas

1. $\ln|x| - \frac{1}{2}\ln|2x^2+1| + C$

2. $\ln|x| - \frac{1}{2}\ln|x^2+4| + \frac{1}{2}tg^{-1}\frac{x}{2} + C$

3. $\frac{1}{8}\ln\left|\frac{2x-1}{2x+1}\right| - \frac{1}{4}tg^{-1}2x + C$

4. $\ln\left|\frac{x^2+4}{x-2}\right| + C$

5. $\frac{3}{10}\ln|2t+1| + \frac{1}{10}\ln|t^2+1| + \frac{2}{5}tg^{-1}t + C$

6. $3w + \frac{1}{2}tg^{-1}\frac{w}{2} + \ln\left|\frac{w}{\sqrt{w^2+4}}\right| + C$

7. $\ln|x-1| + tg^{-1}x + C$

8. $-\frac{1}{x} - 3\,tg^{-1}3x + C$

9. $\ln|x| - \frac{1}{2}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}tg^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

10. $-11\ln|x| + \frac{11}{2}\ln|4x^2+4x+1| - \frac{3}{x} - \frac{5}{2x+1} + C$

(também : $11\ln\left|\frac{2x+1}{x}\right| - \frac{3}{x} - \frac{5}{2x+1}$)

11. $2\ln|x| - \ln|x^2+1| - \frac{1}{2}tg^{-1}x - \frac{1}{2}\frac{x}{1+x^2} + C$

12. $\ln|x^2-2x+3| - \frac{\sqrt{3}}{2}tg^{-1}\frac{x}{\sqrt{3}} + \frac{7}{4}\sqrt{2}\,tg^{-1}\frac{x-1}{\sqrt{2}} + C$

13. $\frac{5}{2}\ln|z^2-2z+5| + \frac{-47z+15}{8(z^2-2z+5)} + \frac{65}{16}tg^{-1}\frac{z-1}{2} + C$

14. $\frac{3}{8}tg^{-1}t + \frac{1}{2}\frac{t}{t^2+1} + \frac{1}{8}\frac{t-t^3}{(t^2+1)^2} + C$

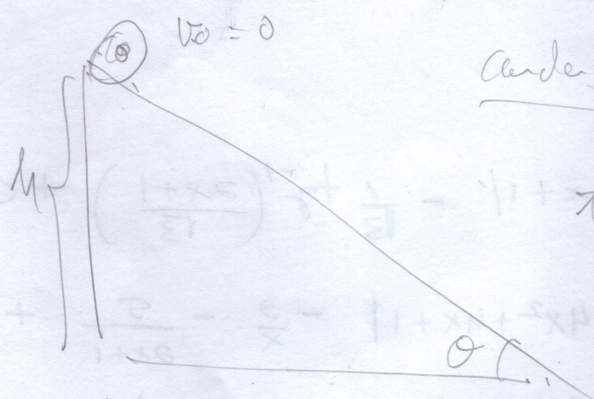
15. $-\frac{2}{81}\ln|3x-1| + \frac{5}{162}\ln|9x^2+3x+1| + \frac{5}{9\sqrt{3}}tg^{-1}\left(\frac{6x+1}{\sqrt{3}}\right) + C$

16. $e^x - \frac{3}{2}tg^{-1}e^x + \frac{e^x}{2(1+e^{2x})} + C$

17. $\frac{1}{6}tg^{-1}\frac{2x}{3} + \frac{x}{(4x^2+9)} + C$

18. $2\ln|x+2| - tg^{-1}|x+1| + C$

19. $\ln|tg\,x+1| + \frac{2}{\sqrt{3}}tg^{-1}\left(\frac{2tg\,x-1}{\sqrt{3}}\right) + C$



corda comprimento L

tempo para arrudar totalmente

Lista 5

Integração por frações parciais:
→ Denominador contém fatores quadráticos

Ex. 9.6

1. $\displaystyle\int \frac{dx}{2x^3+x}$

$$\frac{1}{2x^3+x} = \frac{1}{x(2x^2+1)} = \frac{A}{x} + \frac{B(4x)+C}{2x^2+1}$$

$$\Longleftrightarrow \qquad \text{mmc}: x(2x^2+1)$$

$$1 = A(2x^2+1) + (4Bx+C)x$$

$$1 = 2Ax^2 + A + 4Bx^2 + Cx$$

$$1 = (2A+4B)x^2 + Cx + A$$

$$\Rightarrow \begin{cases} 2A+4B=0 \\ C=0 \\ A=1 \end{cases} \Rightarrow \quad 2+4B=0, \quad B=-\frac{1}{2}$$

$\therefore$

$$\frac{1}{2x^3+x} = \frac{1}{x} + \frac{-\frac{1}{2}(4x)+0}{2x^2+1} = \frac{1}{x} - \frac{2x}{2x^2+1}$$

$$\int \frac{1}{2x^3+x}\,dx = \int \frac{1}{x}\,dx - 2\int \frac{x}{2x^2+1}\,dx$$

$$= \ln|x| - \cancel{2}\,\frac{1}{\cancel{4}\,2}\ln|2x^2+1| + C$$

$$\boxed{\int \frac{dx}{2x^3+x} = \ln\left|\frac{x}{\sqrt{2x^2+1}}\right| + C} = \frac{1}{2}\ln\left|\frac{x^2}{2x^2+1}\right|$$

2. $\displaystyle\int \frac{(x+4)}{x(x^2+4)}\,dx$

$$\frac{x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\Leftrightarrow$$

$$x+4 = A(x^2+4) + (Bx+C)x$$

$$= Ax^2+4A + Bx^2+Cx$$

$$x+4 = (A+B)x^2 + Cx + 4A$$

$$\Rightarrow \begin{cases} A+B=0 \Rightarrow B=-1 \\ \boxed{C=1} \\ 4A=4 \Rightarrow \boxed{A=1} \end{cases}$$

$\therefore$

$$\frac{x+4}{x(x^2+4)} = \frac{1}{x} + \frac{-x+1}{x^2+4}$$

$\therefore$

$$\int \frac{x+4}{x(x^2+4)}\,dx = \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+4} + \int \frac{dx}{x^2+4}$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+4| + \underbrace{\int \frac{dx}{x^2+4}}_{(*)}$$

$$(*) = \int \frac{dx}{x^2+4} = \quad \begin{cases} x = 2\,tg\,\theta \\ dx = 2\sec^2\theta\,d\theta \end{cases} \rightarrow \theta = tg^{-1}\frac{x}{2}$$

$$= \int \frac{2\sec^2\theta\,d\theta}{4(1+tg^2\theta)} = \frac{1}{2}\int \frac{\cancel{\sec^2\theta}\,d\theta}{\cancel{\sec^2\theta}} = \frac{1}{2}\theta = \frac{1}{2}tg^{-1}\frac{x}{2}$$

$$\therefore \boxed{\int \frac{x+4}{x(x^2+4)}\,dx = \ln|x| - \frac{1}{2}\ln|x^2+4| + \frac{1}{2}\operatorname{tg}^{-1}\frac{x}{2} + C}$$

3. $\displaystyle\int \frac{dx}{16x^4-1}$

$$\frac{1}{16x^4-1} = \frac{1}{\underbrace{(4x^2-1)}(4x^2+1)} =$$

$$= \frac{1}{\underbrace{(2x-1)(2x+1)}(4x^2+1)}$$

$$\frac{1}{(2x-1)(2x+1)(4x^2+1)} = \frac{A}{2x-1} + \frac{B}{2x+1} + \frac{Cx+D}{4x^2+1}$$

mmc : $(2x-1)(2x+1)(4x^2+1)$

$\Leftrightarrow$

$$1 = A(2x+1)(4x^2+1) + B(2x-1)(4x^2+1) + (Cx+D)(2x-1)(2x+1)$$

$$1 = A(8x^3+4x^2+2x+1) + B(8x^3-4x^2+2x-1) +$$

$$+ (Cx+D)(4x^2-1)$$

$$1 = 8Ax^3 + 4Ax^2 + 2Ax + A + 8Bx^3 - 4Bx^2 + 2Bx - B$$

$$+ 4Cx^3 - Cx + 4Dx^2 - D$$

$$1 = (8A+8B+4C)x^3 + (4A-4B+4D)x^2 + (2A+2B-C)x$$

$$+ (A-B-D)$$

$\Longrightarrow$

$$\begin{cases} 8A + 8B + 4C = 0 & (*) \\ 4A - 4B + 4D = 0 & (\star) \\ 2A + 2B - C = 0 & (**) \\ A - B - D = 1 & (\star\star) \end{cases}$$

$(*)$: $8A + 8B + 4C = 0 \quad \Leftrightarrow \quad A + B = -\frac{1}{2}C$

$(**)$: $2A + 2B - C = 0 \qquad A + B = +\frac{1}{2}C$

$$\Rightarrow \| A = -B \|$$

$$\boxed{C = 0}$$

$(\star)$: $4A - 4B + 4D = 0 \quad \Leftrightarrow \quad A - B = -D$

$(\star\star)$: $A - B - D = 1 \quad \Leftrightarrow \quad A - B = 1 + D$

$1 + D = -D \quad \Rightarrow \quad \boxed{D = -\frac{1}{2}}$

$(\star\star)$: $A - B - D = 1$

$A - (-A) - (-\frac{1}{2}) = 1$

$2A + \frac{1}{2} = 1$

$2A = \frac{1}{2} \quad \Rightarrow \quad \boxed{A = \frac{1}{4}}$

$\therefore \quad \boxed{A = \frac{1}{4}, \quad B = -\frac{1}{4}, \quad C = 0, \quad D = -\frac{1}{2}}$

$$\therefore \quad \frac{1}{(2x-1)(2x+1)(4x^2+1)} = \frac{1}{4(2x-1)} - \frac{1}{4(2x+1)} - \frac{1}{2(4x^2+1)}$$

$$\therefore \quad \int \frac{dx}{(2x-1)(2x+1)(4x^2+1)} = \int \frac{dx}{4(2x-1)} - \frac{1}{4}\int \frac{dx}{2x+1} - \frac{1}{2}\int \frac{dx}{4x^2+1}$$

$$= \frac{1}{4}\cdot\frac{1}{2}\ln|2x-1| - \frac{1}{4}\,\frac{\ln|2x+1|}{2} - \frac{1}{2}\int \frac{dx}{4x^2+1}$$

$$= \frac{1}{8}\ln|2x-1| - \frac{1}{8}\ln|2x+1| - \frac{1}{2}\int \frac{dx}{4x^2+1} \quad \circledast$$

Now

$$// \circledast = -\frac{1}{2}\int \frac{dx}{4x^2+1} \qquad \left\{ \begin{array}{l} x = \frac{1}{2}\operatorname{tg}\theta \Leftrightarrow \theta = \operatorname{tg}^{-1} 2x \\ dx = \frac{1}{2}\sec^2\theta\, d\theta \end{array} \right.$$

$$= -\frac{1}{2}\int \frac{\frac{1}{2}\sec^2\theta\, d\theta}{\operatorname{tg}^2\theta + 1}$$

$$= -\frac{1}{4}\int \frac{\cancel{\sec^2\theta}\, d\theta}{\cancel{\sec^2\theta}}$$

$$= -\frac{1}{4}\theta$$

$$= -\frac{1}{4}\operatorname{tg}^{-1} 2x \, //$$

$$\therefore \quad \int \frac{dx}{16x^4-1} = \frac{1}{8}\ln|2x-1| - \frac{1}{8}\ln|2x+1| - \frac{1}{4}\operatorname{tg}^{-1} 2x + C$$

$$\boxed{\int \frac{dx}{16x^4-1} = \frac{1}{8}\ln\left|\frac{2x-1}{2x+1}\right| - \frac{1}{4}\operatorname{tg}^{-1} 2x + C}$$

$$\therefore \begin{cases} A = 1 - B \\ C - 2B = -4 \\ C + 2B = 4 \end{cases} \quad 2C = 0 \Rightarrow \boxed{C = 0}$$

$$C + 2B = +4$$

$$2B = 4 \Rightarrow \boxed{B = 2}$$

$$A = 1 - B = 1 - 2 = -1 \quad \therefore \boxed{A = -1}$$

$$\therefore \frac{x^2 - 4x - 4}{(x-2)(x^2+4)} = \frac{-1}{x-2} + \frac{2x}{x^2+4}$$

$$\therefore \int \frac{x^2 - 4x - 4}{(x-2)(x^2+4)}\, dx = -\int \frac{dx}{x-2} + 2\int \frac{x\, dx}{x^2+4}$$

$$= -\ln|x-2| + \cancel{2}\, \frac{\ln|x^2+4|}{\cancel{2}} + C$$

$$\therefore \boxed{\int \frac{x^2 - 4x - 4}{(x-2)(x^2+4)}\, dx = \ln\left|\frac{x^2+4}{x-2}\right| + C}$$

5. $\displaystyle\int \frac{(x^2+x+1)\,dx}{(2x+1)(x^2+1)}$

$$\frac{x^2+x+1}{(2x+1)(x^2+1)} = \frac{A}{2x+1} + \frac{Bx+C}{x^2+1}$$

$$\Leftrightarrow$$

$$x^2+x+1 = A(x^2+1) + (Bx+C)(2x+1)$$

$$= Ax^2 + A + 2Bx^2 + Bx + 2Cx + C$$

$$x^2+x+1 = (A+2B)x^2 + (B+2C)x + (A+C)$$

$$\Rightarrow \begin{cases} A+2B=1 \\ B+2C=1 \\ A+C=1 \end{cases} \Rightarrow A = 1-C$$

$$A+2B=1 \Leftrightarrow \cancel{1}-C+2B=\cancel{1}$$

$$\boxed{2B=C}$$

$$B+2C=1 \Rightarrow B+2(2B)=1$$

$$5B=1 \Rightarrow \boxed{B=\frac{1}{5}}$$

$$2B=C \Rightarrow \boxed{C=\frac{2}{5}}$$

$$A = 1-C = 1-\frac{2}{5}$$

$$\therefore \boxed{A=\frac{3}{5}}$$

$$\therefore \quad \frac{x^2+x+1}{(2x+1)(x^2+1)} = \frac{3}{5(2x+1)} + \frac{\frac{1}{5}x+\frac{2}{5}}{x^2+1}$$

$$= \frac{3}{5(2x+1)} + \frac{x+2}{5(x^2+1)}$$

$$\therefore \quad \int \frac{x^2+x+1}{(2x+1)(x^2+1)}\,dx = \frac{3}{5}\int \frac{dx}{2x+1} + \frac{1}{5}\int \frac{(x+2)\,dx}{x^2+1}$$

$$= \frac{3}{5}\,\frac{\ln|2x+1|}{2} + \frac{1}{5}\int \frac{x}{x^2+1}\,dx + \underbrace{\frac{2}{5}\int \frac{dx}{x^2+1}}$$

$$= \frac{3}{10}\ln|2x+1| + \frac{1}{5}\,\frac{\ln|x^2+1|}{2} + \circledast$$

$$// \quad \circledast = \frac{2}{5}\int \frac{dx}{x^2+1} \qquad x = tg\,\theta, \quad dx = \sec^2\theta\,d\theta$$

$$= \frac{2}{5}\int \frac{\sec^2\theta\,d\theta}{tg^2\theta+1}$$

$$= \frac{2}{5}\int \frac{\cancel{\sec^2\theta}\,d\theta}{\cancel{\sec^2\theta}}$$

$$= \frac{2}{5}\theta = \frac{2}{5}\,tg^{-1}x \quad //$$

$$\therefore \quad \int \frac{x^2+x+1}{(2x+1)(x^2+1)}\,dx = \frac{3}{10}\ln|2x+1| + \frac{1}{10}\ln|x^2+1| + \frac{2}{5}tg^{-1}x + C$$

$$\therefore \quad \boxed{\int \frac{x^2+x+1}{(2x+1)(x^2+1)}\,dx = \frac{1}{10}\ln\left|(2x+1)^3(x^2+1)\right| + \frac{2}{5}tg^{-1}x + C}$$

6. $\int \frac{3w^3+13w+4}{w^3+4w}\,dw$

$$\begin{array}{r|l} 3w^3+13w+4 & w^3+4w \\ \cline{2-2} -3w^3-12w & 3 \\ \hline w+4 & \end{array}$$

$\therefore$ $\frac{3w^3+13w+4}{w^3+4w} = 3 + \frac{w+4}{w^3+4w}$

$= 3 + \frac{w+4}{w(w^2+4)}$

Mas:

$$\frac{w+4}{w(w^2+4)} = \frac{A}{w} + \frac{Bw+C}{w^2+4}$$

$\Leftrightarrow$

$$w+4 = A(w^2+4) + (Bw+C)w$$

$$w+4 = Aw^2+4A + Bw^2+Cw$$

$$w+4 = (A+B)w^2 + Cw + 4A$$

$$\Rightarrow \begin{cases} A+B=0 & \Rightarrow \quad B=-A=-1 \\ C=1 & \\ 4A=4 & \Rightarrow \quad A=1 \end{cases}$$

$\therefore \quad \dfrac{\omega+4}{\omega(\omega^2+4)} = \dfrac{1}{\omega} + \dfrac{-\omega+1}{\omega^2+4}$

$\therefore$

$$\frac{3\omega^3+13\omega+4}{\omega^3+4\omega} = 3 + \frac{1}{\omega} + \frac{1-\omega}{\omega^2+4}$$

$\therefore$

$$\int \frac{3\omega^3+13\omega+4}{\omega^3+4\omega}\,d\omega = \int 3\,d\omega + \int \frac{d\omega}{\omega} + \int \frac{d\omega}{\omega^2+4} - \int \frac{\omega}{\omega^2+4}\,d\omega$$

$$= 3\omega + \ln|\omega| + \int \frac{d\omega}{\omega^2+4} - \frac{\ln|\omega^2+4|}{2} + C$$

$\circledast$

// $\circledast = \displaystyle\int \frac{d\omega}{\omega^2+4} =$; $\omega = 2\,tg\,\theta$, $d\omega = 2\sec^2\theta\,d\theta$

$$= \int \frac{2\sec^2\theta\,d\theta}{4(tg^2\theta+1)}$$

$$= \frac{1}{2}\int \frac{\cancel{\sec^2\theta}}{\cancel{\sec^2\theta}}\,d\theta$$

$$= \frac{1}{2}\int d\theta = \frac{1}{2}\theta = \frac{1}{2}\,tg^{-1}\frac{\omega}{2} \ //$$

$\therefore$

$$\int \frac{3\omega^3+13\omega+4}{\omega^3+4\omega}\,d\omega = 3\omega + \ln|\omega| + \frac{1}{2}\,tg^{-1}\frac{\omega}{2} - \frac{1}{2}\ln|\omega^2+4| + C$$

$$\boxed{\int \frac{3\omega^3+13\omega+4}{\omega^3+4\omega}\,d\omega = 3\omega + \frac{1}{2}\,tg^{-1}\frac{\omega}{2} + \ln\left|\frac{\omega}{\sqrt{\omega^2+4}}\right| + C}$$

7. $\displaystyle\int \frac{(x^2+x)}{x^3-x^2+x-1}\,dx$

$x^3-x^2+x-1 = (x-1)\,H(x)$

$$\begin{array}{l|l} x^3-x^2+x-1 & x-1 \\ \hline -x^3+x^2 & x^2+1 \\ \hline 0+0+x-1 & \\ -x+1 & \\ \hline 0 & \end{array}$$

$\therefore \quad x^3-x^2+x-1 = (x-1)(x^2+1)$

$$\frac{x^2+x}{x^3-x^2+x-1} = \frac{x^2+x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$\Leftrightarrow$

$(x^2+x) = A(x^2+1) + (Bx+C)(x-1)$

$= Ax^2+A + Bx^2-Bx+Cx-C$

$x^2+x = (A+B)x^2 + (C-B)x + A-C$

$$\Longrightarrow \begin{cases} A+B=1 \\ C-B=1 \\ A-C=0 \Rightarrow \boxed{A=C} \end{cases}$$

$\begin{array}{l} A+B=1 \\ C-B=1 \end{array} \rightarrow \begin{array}{l} C+B=1 \\ C-B=1 \\ \hline 2C=2 \Longrightarrow \boxed{C=1} \end{array} \Longrightarrow 1+B=1 \Longrightarrow \boxed{B=0}$

esto es:

$$\boxed{A = 1 \,,\; B = 0 \,,\; C = 1}$$

Así,

$$\frac{x^2+x}{x^3-x^2+x-1} = \frac{1}{x-1} + \frac{1}{x^2+1}$$

$$\therefore \int \frac{x^2+x}{x^3-x^2+x-1}\,dx = \int \frac{dx}{x-1} + \underbrace{\int \frac{dx}{x^2+1}}$$

$$= \ln|x-1| + tg^{-1}x + C$$

$$\therefore \boxed{\int \frac{x^2+x}{x^3-x^2+x-1} = \ln|x-1| + tg^{-1}x + C}$$

8. $\displaystyle\int \frac{dx}{9x^4+x^2} =$

$$\frac{1}{9x^4+x^2} = \frac{1}{x^2(9x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{9x^2+1}$$

$$\Leftrightarrow \qquad \text{mmc}: x^2(9x^2+1)$$

$$1 = Ax(9x^2+1) + B(9x^2+1) + (Cx+D)x^2$$

$$= 9Ax^3 + Ax + 9Bx^2 + B + Cx^3 + Dx^2$$

$$1 = (9A+C)x^3 + (9B+D)x^2 + Ax + B$$

$\rightarrow$

$$\begin{cases} 9A + C = 0 \Rightarrow C = 0 \\ 9B + D = 0 \Rightarrow 9 + D = 0 \Rightarrow D = -9 \\ A = 0 \\ B = 1 \end{cases}$$

$\therefore$

$$\frac{1}{9x^4+x^2} = \frac{1}{x^2} + \frac{-9}{9x^2+1}$$

$\therefore$

$$\int \frac{dx}{9x^4+x^2} = \int \frac{dx}{x^2} - 9\underbrace{\int \frac{1}{9x^2+1}\,dx}$$

$$= \frac{1}{-x} - 9(*) + C$$

$$//\ (*) = \int \frac{1}{9x^2+1}\,dx \qquad \begin{cases} x = \frac{1}{3}\operatorname{tg}\theta \\ dx = \frac{1}{3}\sec^2\theta\, d\theta \end{cases}$$

$$= \int \frac{\frac{1}{3}\sec^2\theta\, d\theta}{\operatorname{tg}^2\theta + 1} =$$

$$= \frac{1}{3}\int d\theta$$

$$= \frac{1}{3}\theta = \frac{1}{3}\operatorname{tg}^{-1} 3x \ //$$

$$\Rightarrow \ // -9(*) = -3\operatorname{tg}^{-1} 3x \ //$$

$\therefore$

$$\boxed{\int \frac{dx}{9x^4+x^2} = -\frac{1}{x} - 3\operatorname{tg}^{-1} 3x + C}$$

9. $\displaystyle\int \frac{dx}{x^3+x^2+x}$

$$\frac{1}{x^3+x^2+x} = \frac{1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$

$$\Leftrightarrow$$

mmc: $x(x^2+x+1)$

$$1 = A(x^2+x+1) + (Bx+C)x$$

$$1 = \underbrace{Ax^2}+Ax+A+\underbrace{Bx^2}+Cx$$

$$1 = (A+B)x^2+(A+C)x+A$$

$$\Rightarrow \begin{cases} A+B=0 \Rightarrow B=-1 \\ A+C=0 \Rightarrow C=-1 \\ A=1 \end{cases}$$

$\therefore$

$$\frac{1}{x^3+x^2+x} = \frac{1}{x} + \frac{-x-1}{x^2+x+1}$$

$\therefore$

(**) $$\begin{cases} \displaystyle\int \frac{dx}{x^3+x^2+x} = \int \frac{dx}{x} - \int \frac{x+1}{x^2+x+1}\,dx \\ \displaystyle = \ln|x| - \underbrace{\int \frac{x}{x^2+x+1}\,dx}_{(*)} - \int \frac{dx}{x^2+x+1} \end{cases}$$

$$\circledast = \int \frac{x+1}{x^2+x+1}\,dx$$

$$= \int \frac{x+\frac{1}{2}-\frac{1}{2}}{x^2+x+1}\,dx$$

$$= \int \frac{x+\frac{1}{2}}{x^2+x+1}\,dx - \frac{1}{2}\int \frac{dx}{x^2+x+1}$$

$$= \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{1}{2}\int \frac{dx}{x^2+x+1}$$

$$= \frac{1}{2}\ln|x^2+x+1| - \frac{1}{2}\int \frac{dx}{x^2+x+1}$$

$\circledast \longrightarrow \circledast\circledast$:

$$\int \frac{dx}{x^3+x^2+x} = \ln|x| - \left(\frac{1}{2}\ln|x^2+x+1| - \frac{1}{2}\int\frac{dx}{x^2+x+1}\right) - \int\frac{dx}{x^2+x+1}$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+x+1| + \underbrace{\frac{1}{2}\int\frac{dx}{x^2+x+1} - \int\frac{dx}{x^2+x+1}}$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+x+1| - \frac{1}{2}\underbrace{\int\frac{dx}{x^2+x+1}}_{(\star)}$$

$$\star = \int\frac{dx}{x^2+x+1} = \int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} \overset{u=x+\frac{1}{2}}{=} \int\frac{du}{u^2+\frac{3}{4}}$$

$\frac{1}{4}+x=1$

$x=1-\frac{1}{4}$

$$\text{☆} = \int \frac{du}{u^2+\frac{3}{4}} \qquad u = \frac{\sqrt{3}}{2}\, tg\,\theta \rightarrow du = \frac{\sqrt{3}}{2} \sec^2\theta\, d\theta$$

$$= \int \frac{\frac{\sqrt{3}}{2}\sec^2\theta\, d\theta}{\frac{3}{4}(tg^2\theta+1)}$$

$$= \frac{2}{\sqrt{3}}\int d\theta$$

$$= \frac{2}{\sqrt{3}}\,\theta$$

$$= \frac{2}{\sqrt{3}}\, tg^{-1}\frac{2u}{\sqrt{3}} \qquad u = x+\frac{1}{2}$$

$$= \frac{2}{\sqrt{3}}\, tg^{-1}\frac{2(x+\frac{1}{2})}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}}\, tg^{-1}\frac{(2x+1)}{\sqrt{3}}$$

$\therefore$

$$\int\frac{dx}{x^3+x^2+x} = \ln|x| - \frac{1}{2}\ln|x^2+x+1| - \frac{1}{2}\overbrace{\left(\frac{2}{\sqrt{3}}\, tg^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right)}^{\text{☆}} + c$$

$$\boxed{\int\frac{dx}{x^3+x^2+x} = \frac{1}{2}\ln\left|\frac{x^2}{x^2+x+1}\right| - \frac{1}{\sqrt{3}}\, tg^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C}$$

Obs.: Uma maneira simplificada

Escrevemos

$$\frac{1}{x(x^2+x+1)} = \frac{A}{x} + \frac{B(2x+1)+C}{x^2+x+1}$$

$$\therefore \quad \frac{1}{x(x^2+x+1)} = \frac{A}{x} + \frac{B(2x+1)+C}{x^2+x+1}$$

$$\Leftrightarrow$$

$$1 = A(x^2+x+1) + B(2x+1)x + Cx$$

$$1 = Ax^2 + Ax + A + 2Bx^2 + Bx + Cx$$

$$1 = (A+2B)x^2 + (A+B+C)x + A$$

$$\Rightarrow \begin{cases} A+2B=0 \Rightarrow 2B=-1 \Rightarrow B=-\frac{1}{2} \\ A+B+C=0 \Rightarrow 1 + -\frac{1}{2} + C = 0 \\ A=1 \qquad\qquad C=-\frac{1}{2} \end{cases}$$

$$\frac{1}{x(x^2+x+1)} = \frac{1}{x} + \frac{-\frac{1}{2}(2x+1) - \frac{1}{2}}{x^2+x+1}$$

$$\int \frac{1}{x(x^2+x+1)}\,dx = \int \frac{dx}{x} - \frac{1}{2}\int \frac{(2x+1)\,dx}{x^2+x+1} - \frac{1}{2}\int \frac{dx}{x^2+x+1}$$

(A)

$$= \ln|x| - \frac{1}{2}\ln|x^2+x+1| - \frac{1}{2}\left(\frac{2}{\sqrt{3}}\, tg^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right)$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\, tg^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

10. $\displaystyle\int \frac{(x+3)\,dx}{4x^4+4x^3+x^2}$

$$\frac{x+3}{4x^4+4x^3+x^2} = \frac{(x+3)}{x^2(4x^2+4x+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C(8x+4)+D}{4x^2+4x+1}$$

$\Leftrightarrow$ mmc $= x^2(4x^2+4x+1)$

$$x+3 = Ax(4x^2+4x+1)+B(4x^2+4x+1)+C(8x+4)x^2+Dx^2$$

$$= 4Ax^3+4Ax^2+Ax+4Bx^2+4Bx+B+8Cx^3+4Cx^2+Dx^2$$

$$x+3 = (4A+8C)x^3+(4A+4B+4C+D)x^2+(A+4B)x+B$$

$\Rightarrow \begin{cases} 4A+8C=0 & \Rightarrow -44+8C=0 \quad C=\frac{11}{2} \\ 4A+4B+4C+D=0 \\ A+4B=1 & \Rightarrow A=-11 \\ B=3 \end{cases}$

$$D = -4(A+B+C)$$

$$= -4\left(-11+3+\frac{11}{2}\right)$$

$$= -4\left(-\frac{11}{2}+3\right)$$

$$= -4\left(-\frac{5}{2}\right)$$

$$D = +10$$

$$\frac{x+3}{4x^4+4x^3+x^2} = \frac{-11}{x} + \frac{3}{x^2} + \frac{\frac{11}{2}(8x+4) + 10}{4x^2+4x+1}$$

$\therefore$

$$\int \frac{x+3}{4x^4+4x^3+x^2}\,dx = -11\int\frac{dx}{x} + 3\int\frac{dx}{x^2} + \frac{11}{2}\int\frac{(8x+4)\,dx}{4x^2+4x+1}$$

$$+ 10\int\frac{dx}{4x^2+4x+1}$$

$$= \underbrace{-11\ln|x| + 3\frac{1}{-x} + \frac{11}{2}\ln|4x^2+4x+1|}$$

$$+ 10\int\frac{dx}{4x^2+4x+1}$$

$$= \frac{11}{2}\ln\left|\frac{4x^2+4x+1}{x^2}\right| - \frac{3}{x} + 10\underbrace{\int\frac{dx}{4x^2+4x+1}}_{(\star)}$$

$$(\star) = \int\frac{dx}{4x^2+4x+1} = \int\frac{dx}{(2x+1)^2}, \quad u = 2x+1, \quad du = 2\,dx$$

$$= \int\frac{\frac{1}{2}du}{u^2}$$

$$= \frac{1}{2}\frac{1}{-u} = -\frac{1}{2}\frac{1}{u}$$

$$= -\frac{1}{2}\frac{1}{(2x+1)} \quad //$$

$\therefore$

$$\int\frac{x+3}{4x^4+4x^3+x^2}\,dx = \frac{11}{2}\ln\left|\frac{4x^2+4x+1}{x^2}\right| - \frac{3}{x} + 10\left(-\frac{1}{2}\frac{1}{(2x+1)}\right) + C$$

$$\boxed{\int\frac{x+3}{4x^4+4x^3+x^2}\,dx = \frac{11}{2}\ln\left|\frac{4x^2+4x+1}{x^2}\right| - \frac{3}{x} - \frac{5}{2x+1} + C}$$

Outra solução

10. $\displaystyle\int \frac{(x+3)}{4x^4+4x^3+x^2}\,dx$

$$\frac{x+3}{4x^4+4x^3+x^2} = \frac{x+3}{x^2(4x^2+4x+1)}$$

$$= \frac{x+3}{x^2(2x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{2x+1} + \frac{D}{(2x+1)^2}$$

$$x+3 = Ax(2x+1)^2 + B(2x+1)^2 + Cx^2(2x+1) + Dx^2$$

$$= Ax(4x^2+4x+1) + B(4x^2+4x+1) + Cx^2(2x+1) + Dx^2$$

$$= 4Ax^3 + 4Ax^2 + Ax + 4Bx^2 + 4Bx + B + 2Cx^3 + Cx^2 + Dx^2$$

$$x+3 = (4A+2C)x^3 + (4A+4B+C+D)x^2 + (A+4B)x + B$$

$$\Rightarrow \begin{cases} 4A+2C=0 \Rightarrow C=-2A \Rightarrow C=22 \\ 4A+4B+C+D=0 \Rightarrow -44+12+22+D=0 \quad D=10 \\ A+4B=1 \Rightarrow A=-11 \\ B=3 \end{cases}$$

$\therefore$

$$\frac{x+3}{x^2(2x+1)^2} = \frac{-11}{x} + \frac{3}{x^2} + \frac{22}{2x+1} + \frac{10}{(2x+1)^2}$$

$\therefore$

$$\int \frac{x+3}{4x^4+4x^3+x^2}\,dx = -11\int\frac{dx}{x} + 3\int\frac{dx}{x^2} + 22\int\frac{dx}{2x+1}$$

$$+ 10\int\frac{dx}{(2x+1)^2}$$

$$= -11\ln|x| + 3\frac{1}{-x} + 22\frac{\ln|2x+1|}{2}$$

$$+ 10\left(-\frac{1}{2}\frac{1}{(2x+1)}\right)$$

$$= -11\ln|x| - \frac{3}{x} + 11\ln|2x+1| - \frac{5}{2x+1}$$

$$\boxed{\int \frac{x+3}{4x^4+4x^3+x^2}\,dx = 11\ln\left|\frac{2x+1}{x}\right| - \frac{3}{x} - \frac{5}{2x+1}}$$

Note: a resposta anterior

$$\frac{11}{2}\ln\left|\frac{4x^2+4x+1}{x^2}\right| - \frac{3}{x} - \frac{5}{2x+1}$$

$$= \frac{11}{2}\ln\left|\left(\frac{2x+1}{x}\right)^2\right| - \frac{3}{x} - \frac{5}{2x+1}$$

$$= 11\ln\left|\frac{2x+1}{x}\right| - \frac{3}{x} - \frac{5}{2x+1} \quad //$$

OK!

11. $\displaystyle\int \frac{2x^2 - x + 2}{x^5 + 2x^3 + x}\, dx$

$$\frac{2x^2 - x + 2}{x^5 + 2x^3 + x} = \frac{2x^2 - x + 2}{x(x^4 + 2x^2 + 1)} = \frac{2x^2 - x + 2}{x(x^2+1)^2}$$

$\therefore$

$$\frac{2x^2 - x + 2}{x(x^2+1)^2} = \frac{A}{x} + \frac{B(2x) + C}{x^2+1} + \frac{D(2x) + E}{(x^2+1)^2}$$

$\Leftrightarrow$

mmc : $(x^2+1)^2 x$

$$2x^2 - x + 2 = A(x^2+1)^2 + (B(2x) + C)\,x(x^2+1) + (D2x + E)\,x$$

$$= A(x^4 + 2x^2 + 1) + (2Bx + C)(x^3 + x) + 2x^2D + Ex$$

$$= Ax^4 + 2Ax^2 + A + 2Bx^4 + 2Bx^2 + Cx^3 + Cx + 2x^2D + Ex$$

$$2x^2 - x + 2 = (A + 2B)x^4 + Cx^3 + (2A + 2B + 2D)x^2 + (C+E)x + A$$

$$\Rightarrow \begin{cases} A + 2B = 0 \Rightarrow 2 + 2B = 0 \Rightarrow B = -1 \\ C = 0 \\ 2A + 2B + 2D = 2 \Rightarrow \cancel{4} + \cancel{(-2)} + 2D = \cancel{2} \quad D = 0 \\ C + E = -1 \Rightarrow E = -1 \\ A = 2 \end{cases}$$

$$\therefore \left\{ \frac{2x^2 - x + 2}{x(x^2+1)^2} = \frac{2}{x} + \frac{-2x}{x^2+1} + \frac{-1}{(x^2+1)^2} \right. \quad \Longrightarrow$$

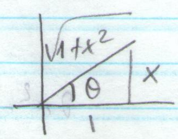
$$\int \frac{2x^2-x+2}{x^5+2x^3+x}\,dx =$$

$$= \int \frac{2}{x}\,dx - 2\int \frac{x\,dx}{x^2+1} - \underbrace{\int \frac{dx}{(x^2+1)^2}}_{(*)}$$

$$= 2\ln|x| - \ln|x^2+1| - (*)$$

$$// \; (*) = \int \frac{dx}{(x^2+1)^2} \qquad \begin{cases} x = tg\,\theta \\ dx = \sec^2\theta\,d\theta \end{cases} \Rightarrow \theta = tg^{-1}x$$

$$= \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta\,d\theta}$$



$$= \int \cos^2\theta\,d\theta$$

$$= \int \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{1}{2}\theta + \frac{\sin 2\theta}{4}$$

$$= \frac{1}{2}\theta + \frac{2\sin\theta\cos\theta}{4}$$

$$= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta$$

$$= \frac{1}{2}tg^{-1}x + \frac{1}{2}\frac{x}{\sqrt{1+x^2}}\,\frac{1}{\sqrt{1+x^2}}$$

$$= \frac{1}{2}tg^{-1}x + \frac{1}{2}\frac{x}{(1+x^2)} \; //$$

$$\int \frac{2x^2 - x + 2}{x^5 + 2x^3 + x}\,dx = 2\ln|x| - \ln|x^2+1| - \left(\frac{1}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{1+x^2}\right)$$

$$\boxed{\int \frac{2x^2 - x + 2}{x^5 + 2x^3 + x}\,dx = \ln\left|\frac{x^2}{x^2+1}\right| - \frac{1}{2}\tan^{-1}x - \frac{1}{2}\frac{x}{(1+x^2)} + C}$$

---

12. $\int \frac{(2x^3+9x)}{(x^2+3)(x^2-2x+3)}\,dx$

$$\frac{2x^3+9x}{(x^2+3)(x^2-2x+3)} = \frac{A2x+B}{x^2+3} + \frac{C(2x-2)+D}{x^2-2x+3}$$

$\Leftrightarrow$

$$2x^3+9x = (2Ax+B)(x^2-2x+3) + (C(2x-2)+D)(x^2+3)$$

$$= 2Ax^3 - 4Ax^2 + 6Ax + Bx^2 - 2Bx + 3B +$$

$$+ (2Cx - 2C + D)(x^2+3)$$

$$= 2Ax^3 - 4Ax^2 + 6Ax + Bx^2 - 2Bx + 3B + 2Cx^3$$

$$+ 6Cx - 2Cx^2 - 6C + Dx^2 + 3D$$

$$\begin{cases} 2x^3 + 9x = (2A+2C)x^3 + (-4A+B-2C+D)x^2 + (6A-2B+6C)x \\ \qquad + 3B - 6C + 3D \end{cases}$$

$\Longrightarrow$

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$$\begin{cases} 2A + 2C = 2 & \Rightarrow \ \| C = 1 - A \| \\ -4A + B - 2C + D = 0 & \Rightarrow \\ 6A - 2B + 6C = 9 & \\ 3B - 6C + 3D = 0 & \end{cases}$$

$$-4A + B - 2C + D = 0 \Leftrightarrow -4A + B - 2(1-A) + D = 0$$

$$-4A + B - 2 + 2A + D = 0$$

$$\| -2A + B + D = 2 \|$$

$$6A - 2B + 6C = 9 \Leftrightarrow 6A - 2B + 6(1-A) = 9$$

$$\cancel{6A} - 2B + 6 - \cancel{6A} = 9$$

$$-2B = 3 \Rightarrow B = -\frac{3}{2}$$

$$3B - 6C + 3D = 0 \Leftrightarrow -\frac{9}{2} - 6(1-A) + 3D = 0$$

$$-\frac{9}{2} - 6 + 6A + 3D = 0$$

$$6A + 3D = +\frac{21}{2}$$

$$\| 2A + D = \frac{7}{2} \|$$

tenemos entonces:

$$\begin{cases} C = 1 - A \\ -2A + B + D = 2 \Rightarrow -2A - \frac{3}{2} + D = 2 \\ B = -\frac{3}{2} \qquad \| -2A + D = \frac{7}{2} \| \\ 2A + D = \frac{7}{2} \end{cases}$$

$$\begin{cases} -2A + D = \frac{7}{2} \\ 2A + D = \frac{7}{2} \end{cases} \Rightarrow 2D = 7$$

$D = 7/2$

$$2A + D = \frac{7}{2}$$

$$2A + \frac{7}{2} = \frac{7}{2}$$

$\frac{7}{2} - 4 = \frac{1}{2}$

$$2A = 0 \Rightarrow A = 0$$

$$C = 1 - A = 1 \Rightarrow C = 1$$

i.e. :

$$\boxed{A = 0,\ B = -\frac{3}{2},\ C = 1,\ D = \frac{7}{2}}$$

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$$\frac{2x^3+9x}{(x^2+3)(x^2-2x+3)} = \frac{-3/2}{x^2+3} + \frac{2x-2+\frac{7}{2}}{x^2-2x+3}$$

$$= -\frac{3}{2}\frac{1}{x^2+3} + \frac{2x-2}{x^2-2x+3} + \frac{7}{2}\frac{1}{x^2-2x+3}$$

$$\int \frac{2x^3+9x}{(x^2+3)(x^2-2x+3)}\,dx = -\frac{3}{2}\int\frac{dx}{x^2+3} + \underbrace{\int \frac{2x-2}{x^2-2x+3}\,dx} + \frac{7}{2}\int\frac{dx}{x^2-2x+3}$$

$$= \ln|x^2-2x+3| - \underbrace{\frac{3}{2}\int\frac{dx}{x^2+3}}_{*} + \underbrace{\frac{7}{2}\int\frac{dx}{x^2-2x+3}}_{**}$$

// $(*) = -\frac{3}{2}\int\frac{dx}{x^2+3}$ $\qquad x = \sqrt{3}\,tg\,\theta$, $dx = \sqrt{3}\sec^2\theta\,d\theta$

$$= -\frac{3}{2}\int\frac{\sqrt{3}\sec^2\theta\,d\theta}{3(tg^2\theta+1)} = -\frac{\sqrt{3}}{2}\int d\theta$$

$$= -\frac{\sqrt{3}}{2}\theta$$

$$= -\frac{\sqrt{3}}{2}\,tg^{-1}\frac{x}{\sqrt{3}} \quad //$$

// $(**) = \frac{7}{2}\int\frac{dx}{x^2-2x+3} = \frac{7}{2}\int\frac{dx}{(x-1)^2+2}$ $\qquad u = x-1$, $du = dx$

$$= \frac{7}{2}\int\frac{du}{u^2+2} \qquad u = \sqrt{2}\,tg\,\theta,\ du = \sqrt{2}\sec^2\theta\,d\theta$$

$$= \frac{7}{2}\int\frac{\sqrt{2}\sec^2\theta\,d\theta}{2(tg^2\theta+1)}$$

$$= \frac{7\sqrt{2}}{4}\,tg^{-1}\frac{u}{\sqrt{2}} = \frac{7\sqrt{2}}{4}\,tg^{-1}\frac{x-1}{\sqrt{2}} \quad //$$

$\therefore$

$$\boxed{\int \frac{2x^3+9x}{(x^2+3)(x^2-2x+3)}\,dx = \ln|x^2-2x+3| - \frac{\sqrt{3}}{2}\,tg^{-1}\frac{x}{\sqrt{3}} + \frac{7\sqrt{2}}{4}\,tg^{-1}\frac{x-1}{\sqrt{2}} + C}$$

13. $\displaystyle\int \frac{5z^3-z^2+15z-10}{(z^2-2z+5)^2}\,dz$

$$\frac{5z^3-z^2+15z-10}{(z^2-2z+5)^2} = \frac{A(2z-2)+B}{(z^2-2z+5)} + \frac{C(2z-2)+D}{(z^2-2z+5)^2}$$

$$\Leftrightarrow$$

$$5z^3-z^2+15z-10 = \left[A(2z-2)+B\right](z^2-2z+5) + C(2z-2)+D$$

$$= (2Az-2A+B)(z^2-2z+5) + 2Cz-2C+D$$

$$= 2Az^3 - 2Az^2 + Bz^2 - 4Az^2 + 4Az - 2Bz + 10Az - 10A + 5B + 2Cz - 2C + D$$

$$5z^3-z^2+15z-10 = 2Az^3 + (-6A+B)z^2 + (+14A-2B+2C)z - 10A + 5B - 2C + D$$

$\Rightarrow$

$$\begin{cases} 2A = 5 & \Longrightarrow \quad A = 5/2 \\ -6A + B = -1 & \Longrightarrow \quad -6\times\frac{5}{2} + B = -1 \Rightarrow B = 14 \\ 14A - 2B + 2C = 15 & \Longrightarrow \quad \underbrace{14\times\frac{5}{2}} - 28 + 2C = 15 \\ \underbrace{-10A + 5B - 2C + D = -10} & \end{cases}$$

$$7 + 2C = 15$$

$$2C = 8$$

$$C = 4$$

$$-10\times\frac{5}{2} + 5\times 14 - 2\times 4 + D = -10$$

$$-25 + 70 - 8 + D = -10$$

$$-33 + 70 + D = -10$$

$$37 + D = -10 \quad \Longrightarrow \quad D = -47$$

$$\Longrightarrow$$

$$\frac{5z^3 - z^2 + 15z - 10}{(z^2 - 2z + 5)^2} = \frac{\frac{5}{2}(2z-2) + 14}{(z^2 - 2z + 5)} + \frac{4(2z-2) - 47}{(z^2 - 2z + 5)^2}$$

$$\int \frac{5z^3 - z^2 + 15z - 10}{(z^2 - 2z + 5)^2}\, dz = \underbrace{\frac{5}{2}\int \frac{2z-2}{z^2 - 2z + 5}\, dz} + $$

$$+ 14 \int \frac{dz}{(z^2 - 2z + 5)}$$

$$+ 4 \underbrace{\int \frac{(2z-2)\, dz}{(z^2 - 2z + 5)^2}}$$

$$- 47 \int \frac{dz}{(z^2 - 2z + 5)^2}$$

$$= \underbrace{\frac{5}{2}\ln|z^2-2z+5|}+4\,\underbrace{\frac{-1}{z^2-2z+5}}$$

$$+14\underbrace{\int\frac{dz}{z^2-2z+5}}_{(\star)} - 47\underbrace{\int\frac{dz}{(z^2-2z+5)^2}}_{(\star\star)}$$

$$= \frac{5}{2}\ln|z^2-2z+5| - \frac{4}{z^2-2z+5} + \star + \star\star$$

May

$$/\!/ \star = 14\int\frac{dz}{z^2-2z+5} = 14\int\frac{dz}{(z-1)^2+4}$$

$u = z-1$
$du = dz$

$$= 14\int\frac{du}{u^2+4}$$

$u = 2\,tg\,\theta$
$du = 2\sec^2\theta\, d\theta$

$$= 14\int\frac{\cancel{2\sec^2\theta}\, d\theta}{_2\cancel{4}\,\cancel{\sec^2\theta}}$$

$$= 7\int d\theta = 7\theta$$

$$= 7\,tg^{-1}\frac{u}{2}$$

$$= 7\,tg^{-1}\left(\frac{z-1}{2}\right) /\!/$$

$$\star\star = -47\int\frac{dz}{(z^2-2z+5)^2} = -47\int\frac{dz}{((z-1)^2+4)^2}$$

$z-1 = u$
$dz = du$

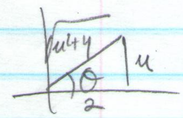
$$= -47\int\frac{du}{(u^2+4)^2}$$

$u = 2\,tg\,\theta$
$du = 2\sec^2\theta\, d\theta$

$$= -47\int\frac{2\sec^2\theta\, d\theta}{(4\sec^2\theta)^2}$$

$$= -47\int \frac{2\sec^2\theta}{(4\sec^2\theta)^2}\,d\theta$$

$\operatorname{tg}\theta = \frac{u}{2}$



$$= -47\int \frac{\cancel{2}\sec^2\theta}{\cancel{16}\,\sec^4\theta}\,d\theta$$

$$= -\frac{47}{8}\int \cos^2\theta\,d\theta$$

$\sin\theta = \frac{u}{\sqrt{u^2+4}}$

$\cos\theta = \frac{2}{\sqrt{u^2+4}}$

$$= -\frac{47}{8}\int \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= -\frac{47}{16}\int (1+\cos 2\theta)\,d\theta$$

$$= \frac{-47}{16}\left(\theta + \frac{\sin 2\theta}{2}\right)$$

$$= -\frac{47}{16}\left(\theta + \sin\theta\cos\theta\right)$$

$$= \frac{-47}{16}\left(\operatorname{tg}^{-1}\frac{u}{2} + \frac{u}{\sqrt{u^2+4}}\,\frac{2}{\sqrt{u^2+4}}\right)$$

$$= \frac{-47}{16}\left(\operatorname{tg}^{-1}\frac{u}{2} + \frac{2u}{u^2+4}\right) \quad \Big\downarrow\ u = z-1$$

$$= \frac{-47}{16}\left(\operatorname{tg}^{-1}\frac{z-1}{2} + \frac{2(z-1)}{(z-1)^2+4}\right)$$

$$//\ \bigstar\bigstar = \frac{-47}{16}\operatorname{tg}^{-1}\frac{z-1}{2} - \frac{47}{8}\,\frac{z-1}{(z^2-2z+5)}\ //$$

$$\Rightarrow \int \frac{5z^3 - z^2 + 15z - 10}{(z^2-2z+5)^2}\,dz = \frac{5}{2}\ln|z^2-2z+5| - \frac{4}{z^2-2z+5}$$

$$+ 7\,tg^{-1}\left(\frac{z-1}{2}\right) - \frac{47}{16}\,tg^{-1}\frac{z-1}{2}$$

$$- \frac{47}{8}\,\frac{z-1}{z^2-2z+5}$$

Mas:

$$\frac{-4}{z^2-2z+5} - \frac{47}{8}\frac{z-1}{z^2-2z+5} = \frac{-32-47z+47}{8(z^2-2z+5)}$$

$$= \frac{-47z+15}{8(z^2-2z+5)}$$

$$7\,tg^{-1}\frac{z-1}{2} - \frac{47}{16}\,tg^{-1}\frac{z-1}{2} = \left(7 - \frac{47}{16}\right) tg^{-1}\frac{z-1}{2}$$

$$= \frac{112-47}{16}\,tg^{-1}\frac{z-1}{2}$$

$$= \frac{65}{16}\,tg^{-1}\frac{z-1}{2}$$

$$\therefore \boxed{\int \frac{5z^3 - z^2 + 15z - 10}{(z^2-2z+5)^2}\,dz = \frac{5}{2}\ln|z^2-2z+5| + \frac{-47z+15}{8(z^2-2z+5)} + \frac{65}{16}\,tg^{-1}\frac{z-1}{2} + C}$$

14. $\displaystyle\int \frac{dx}{(x^2+1)^3}$

$$\frac{1}{(x^2+1)^3} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{Ex+F}{(x^2+1)^3}$$

$\Leftrightarrow$

$$1 = (Ax+B)(x^2+1)^2 + (Cx+D)(x^2+1) + Ex+F$$

$$= (Ax+B)(x^4+2x^2+1) + Cx^3+Dx^2+Cx+D+Ex+F$$

$$= Ax^5 + Bx^4 + 2Ax^3 + 2Bx^2 + Ax + B + Cx^3 + Dx^2 + Cx + D + Ex + F$$

$$= Ax^5 + Bx^4 + (2A+C)x^3 + (2B+D)x^2 + (A+C+E)x + B+D+F$$

$\Rightarrow$

$$\begin{cases} A=0 \\ B=0 \\ 2A+C=0 \Rightarrow C=0 \\ 2B+D=0 \Rightarrow D=0 \\ A+C+E=0 \Rightarrow E=0 \\ B+D+F=1 \Rightarrow F=1 \end{cases}$$

$\therefore \quad \dfrac{1}{(x^2+1)^3} = \dfrac{1}{(x^2+1)^3}$ não há decomposição em frações parciais

$$\int \frac{dt}{(t^2+1)^3} = \int \frac{\sec^2\theta \, d\theta}{(\sec^2\theta)^3} \qquad \begin{cases} t = \tan\theta \\ dt = \sec^2\theta \, d\theta \end{cases}$$

$$= \int \frac{\cancel{\sec^2\theta} \, d\theta}{\sec^6\theta^4}$$

$$= \int \cos^4\theta \, d\theta$$

$$= \int (\cos^2\theta)^2 d\theta$$

$$= \int \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{1}{4}\int (1 + 2\cos 2\theta + \cos^2 2\theta) \, d\theta$$

$$= \frac{1}{4}\left(\theta + \sin 2\theta\right) + \frac{1}{4}\int \cos^2 2\theta \, d\theta$$

$$= \frac{1}{4}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{4}\int \left(\frac{1+\cos 4\theta}{2}\right) d\theta$$

$$= \frac{1}{4}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\theta + \frac{1}{8}\int \cos 4\theta \, d\theta$$

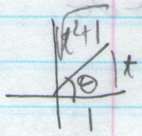
$$= \frac{3\theta}{8} + \frac{1}{4}\sin 2\theta + \frac{1}{8}\,\frac{\sin 4\theta}{4} + C$$

$$= \frac{3\theta}{8} + \frac{1}{2}\sin\theta\cos\theta + \frac{1}{8}\,\frac{\cancel{2}\sin 2\theta\cos 2\theta}{\cancel{4}\,2} + C$$

$$= \frac{3\theta}{8} + \frac{1}{2}\sin\theta\cos\theta + \frac{1}{16}\sin 2\theta\cos 2\theta + C$$

$$= \frac{3\theta}{8} + \frac{1}{2}\sin\theta\cos\theta + \frac{1}{16}\,2\sin\theta\cos\theta\,(\cos^2\theta - \sin^2\theta) + C$$

$\operatorname{tg}\theta = x$



$\sin\theta = \dfrac{x}{\sqrt{x^2+1}}$

$\cos\theta = \dfrac{1}{\sqrt{x^2+1}}$

$$= \frac{3}{8}\theta + \frac{1}{2}\sin\theta\cos\theta + \frac{1}{8}\sin\theta\cos^3\theta - \frac{1}{8}\sin^3\theta\cos\theta + C$$

$$= \frac{3}{8}\operatorname{tg}^{-1}x + \frac{1}{2}\frac{x}{\sqrt{x^2+1}}\frac{1}{\sqrt{x^2+1}} + \frac{1}{8}\frac{x}{\sqrt{x^2+1}}\left(\frac{1}{\sqrt{x^2+1}}\right)^3$$

$$- \frac{1}{8}\frac{x^3}{(\sqrt{x^2+1})^3}\frac{1}{\sqrt{x^2+1}}$$

$$= \frac{3}{8}\operatorname{tg}^{-1}x + \frac{1}{2}\frac{x}{(x^2+1)} + \frac{1}{8}\frac{x}{(x^2+1)^2} - \frac{1}{8}\frac{x^3}{(x^2+1)^2}$$

$$\boxed{\int\frac{dx}{(x^2+1)^3} = \frac{3}{8}\operatorname{tg}^{-1}x + \frac{1}{2}\frac{x}{x^2+1} + \frac{1}{8}\frac{x-x^3}{(x^2+1)^2}}$$

15. $\displaystyle\int \frac{x^2+2x-1}{27x^3-1}\,dx$

$$27x^3-1 = \left(x-\tfrac{1}{3}\right) H(x)$$

$$\begin{array}{l|l}
27x^3-1 & x-\frac{1}{3} \\ \cline{2-2}
-\cancel{27x^3}+9x^2 & 27x^2+9x+3 \\ \cline{1-1}
\quad 9x^2-1 & \\
\quad -9x^2+3x & \\ \cline{1-1}
\quad\quad 3x-1 & \\
\quad\quad -3x+1 & \\ \cline{1-1}
\quad\quad\quad 0 &
\end{array}$$

$\therefore$

$$27x^3-1 = \left(x-\tfrac{1}{3}\right)(27x^2+9x+3)$$

$$\frac{x^2+2x-1}{27x^3-1} = \frac{x^2+2x-1}{\left(x-\frac{1}{3}\right)(27x^2+9x+3)}$$

$$= \frac{A}{x-\frac{1}{3}} + \frac{B(54x+9)+C}{27x^2+9x+3}$$

$$\Leftrightarrow$$

$$x^2+2x-1 = A(27x^2+9x+3) + [54Bx+9B+C]\left(x-\tfrac{1}{3}\right)$$

$$= 27Ax^2 + 9Ax + 3A + 54Bx^2 - 18Bx + 9Bx - 3B + Cx - \frac{C}{3}$$

$$x^2+2x-1 = (27A+54B)x^2 + (9A\underbrace{-18B+9B}_{-9B}+C)x + 3A - 3B - \frac{C}{3}$$

$\Longrightarrow$

$$27A + 54B = 1 \Rightarrow A = \frac{1-54B}{27}$$

$$9A - 9B + C = 2 \qquad \| A = \frac{1}{27} - 2B \|$$

$$3A - 3B - \frac{C}{3} = -1$$

$$9\left(\frac{1}{27} - 2B\right) - 9B + C = 2$$

$$\frac{1}{3} - 18B - 9B + C = 2$$

$$-27B + C = 2 - \frac{1}{3}$$

$$\| -27B + C = \frac{5}{3} \|$$

$$3\left(\frac{1}{27} - 2B\right) - 3B - \frac{C}{3} = -1$$

$$\frac{1}{9} - 6B - 3B - \frac{C}{3} = -1$$

$$-9B - \frac{C}{3} = -1 - \frac{1}{9}$$

$$-9B - \frac{C}{3} = \frac{-10}{9}$$

$$\| 9B + \frac{C}{3} = \frac{10}{9} \|$$

$$\therefore \begin{cases} -27B + C = \dfrac{5}{3} \\ 9B + \dfrac{C}{3} = \dfrac{10}{9} \quad (\times 3) \\ 27B + C = \dfrac{10}{3} \end{cases}$$

$$\therefore \begin{cases} -27B + C = \dfrac{5}{3} \\ 27B + C = \dfrac{10}{3} \end{cases}$$

$$2C = \frac{15}{3} = 5 \implies C = 5/2$$

$$27B + C = \frac{10}{3}$$

$$27B + \frac{5}{2} = \frac{10}{3}$$

$$27B = \frac{10}{3} - \frac{5}{2} = \frac{20-15}{6} = \frac{5}{6}$$

$$B = \frac{5}{27 \times 6} = \frac{5}{162} \implies B = \frac{5}{162}$$

$$\therefore A = \frac{1}{27} - 2B = \frac{1}{27} - \frac{10}{162} = \frac{6-10}{162} = \frac{-4}{162}$$

$$\boxed{A = -\frac{2}{81}}$$

$$\frac{x^2+2x-1}{27x^3-1} = \frac{-2}{81}\,\frac{1}{x-\frac{1}{3}} + \frac{\frac{5}{162}(54x+9)}{27x^2+9x+3} + \frac{5/2}{27x^2+9x+3}$$

$\Rightarrow$

$$\int \frac{x^2+2x-1}{27x^3-1}\,dx = -\frac{2}{81}\int \frac{dx}{x-\frac{1}{3}} + \frac{5}{162}\int \frac{54x+9}{27x^2+9x+3}\,dx$$

$$+\frac{5}{2}\int \frac{dx}{27x^2+9x+3}$$

$$= -\frac{2}{81}\ln\left|x-\tfrac{1}{3}\right| + \frac{5}{162}\ln\left|27x^2+9x+3\right|$$

$$+\frac{5}{2}\int \frac{dx}{27x^2+9x+3}$$

$\star$

$$\star = \frac{5}{2}\int \frac{dx}{27x^2+9x+3}$$

$$= \frac{5}{2}\int \frac{dx}{27\left(x^2+\frac{1}{3}x+\frac{1}{9}\right)}$$

$$= \frac{5}{54}\int \frac{dx}{x^2+\frac{1}{3}x+\frac{1}{9}}$$

$u = x+\frac{1}{6}$

$du = dx$

$$= \frac{5}{54}\int \frac{dx}{\left(x+\frac{1}{6}\right)^2+\frac{3}{36}}$$

$\frac{1}{36}+x=\frac{1}{9}$

$x=\frac{1}{9}-\frac{1}{36}$

$=\frac{4-1}{36}$

$=\frac{3}{36}$

$$= \frac{5}{54}\int \frac{du}{u^2+\frac{3}{36}}$$

$$\begin{cases} u = \sqrt{\frac{3}{36}}\,\operatorname{tg}\theta \\ du = \frac{\sqrt{3}}{6}\sec^2\theta\,d\theta \end{cases}$$

$$11\star \quad = \frac{5}{54}\int \frac{\frac{\sqrt{3}}{6}\sec^2\theta\, d\theta}{\frac{3}{36}(1+tg^2\theta)}$$

$$= \frac{5}{54}\ \frac{\sqrt{3}}{6}\ \frac{36}{3}\int d\theta \qquad u = \frac{\sqrt{3}}{6}\,tg\,\theta$$

$$= \frac{5}{\cancel{54}}\ \frac{\cancel{6}}{\sqrt{3}}\ \theta \qquad \theta = tg^{-1}\frac{6u}{\sqrt{3}}$$

$$= \frac{5}{9\sqrt{3}}\ tg^{-1}\frac{6u}{\sqrt{3}}$$

$$= \frac{5}{9\sqrt{3}}\ tg^{-1}\frac{6(x+\frac{1}{6})}{\sqrt{3}}$$

$$= \frac{5}{9\sqrt{3}}\ tg^{-1}\frac{6x+1}{\sqrt{3}} \ //$$

$\therefore (\star)$

$$\int \frac{x^2+2x-1}{27x^3-1}\,dx = -\frac{2}{81}\ln\left|x-\frac{1}{3}\right| + \frac{5}{162}\ln|27x^2+9x+3| + \frac{5}{9\sqrt{3}}\,tg^{-1}\frac{6x+1}{\sqrt{3}}$$

Mas

$$-\frac{2}{81}\ln\left|x-\frac{1}{3}\right| = -\frac{2}{81}\ln\left|\frac{3x-1}{3}\right| = -\frac{2}{81}\ln|3x-1| + C$$

$$\frac{5}{162}\ln|27x^2+9x+3| = \frac{5}{162}\ln|3(9x^2+3x+1)| = \frac{5}{162}\ln|9x^2+3x+1| + C$$

$\therefore$

$$\boxed{\int \frac{x^2+2x-1}{27x^3-1}\,dx = -\frac{2}{81}\ln|3x-1| + \frac{5}{162}\ln|9x^2+3x+1| + \frac{5}{9\sqrt{3}}\,tg^{-1}\left(\frac{6x+1}{\sqrt{3}}\right) + C}$$

16. $$\int \frac{e^{5x}}{(e^{2x}+1)^2}\,dx = \qquad \begin{cases} e^x = u \\ du = e^x\,dx \end{cases}$$

$$= \int \frac{e^{4x}\,e^x\,dx}{(e^{2x}+1)^2}$$

$$= \int \frac{u^4\,du}{(u^2+1)^2}$$

$$= \int \frac{u^4\,du}{u^4+2u^2+1}$$

$$\begin{array}{l|l} u^4 & u^4+2u^2+1 \\ \underline{-u^4-2u^2-1} & 1 \\ -2u^2-1 & \end{array}$$

$$\therefore \frac{u^4}{u^4+2u^2+1} = 1 - \frac{2u^2+1}{u^4+2u^2+1} \quad (\bigstar)$$

$$\frac{2u^2+1}{u^4+2u^2+1} = \frac{2u^2+1}{(u^2+1)^2} = \frac{Au+B}{u^2+1} + \frac{Cu+D}{(u^2+1)^2}$$

$$\Leftrightarrow$$

$$2u^2+1 = (Au+B)(u^2+1) + Cu+D$$

$$= Au^3+Au+Bu^2+B+Cu+D$$

$$2u^2+1 = Au^3+Bu^2+(A+C)u+B+D$$

$$\Rightarrow$$

$$\begin{cases} A=0 \\ B=2 \\ A+C=0 \implies C=0 \\ B+D=1 \implies 2+D=1 \implies D=-1 \end{cases}$$

$\therefore$

$$\frac{2u^2+1}{u^4+2u^2+1} = \frac{2}{u^2+1} - \frac{1}{(u^2+1)^2} \quad (**)$$

$\therefore$

$(**) \to *$:

$$\frac{u^4}{u^4+2u^2+1} = 1 - \frac{2}{u^2+1} + \frac{1}{(u^2+1)^2}$$

$\therefore$

$$\int \frac{e^{5x}}{(e^{2x}+1)^2}\,dx = \int \frac{u^4}{u^4+2u^2+1}\,du =$$

$$= \int du - 2\underbrace{\int \frac{du}{u^2+1}} + \int \frac{du}{(u^2+1)^2}$$

$$= u - 2\,\underline{tg^{-1} u} + \circledast$$

$$\int \frac{du}{(u^2+1)^2}$$

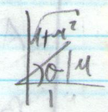
$$\circledast = \int \frac{du}{(u^2+1)^2} \quad \begin{cases} u = tg\theta \\ du = sec^2\theta\, d\theta \end{cases}$$

$$= \int \frac{sec^2\theta\, d\theta}{(1+tg^2\theta)^2} = \int \frac{sec^2\theta}{sec^4\theta}\,d\theta = \int cos^2\theta\, d\theta$$

$$= \int \cos^2\theta \, d\theta$$

$$= \int \frac{1+\cos 2\theta}{2} \, d\theta \qquad tg\theta = u$$

$$= \frac{1}{2}\theta + \frac{\sin 2\theta}{4}$$



$$= \frac{1}{2} tg^{-1} u + \frac{1}{2} \sin\theta \cos\theta$$

$$= \frac{1}{2} tg^{-1} u + \frac{1}{2} \frac{u}{\sqrt{1+u^2}} \frac{1}{\sqrt{1+u^2}}$$

$$= \frac{1}{2} tg^{-1} u + \frac{1}{2} \frac{u}{(1+u^2)} \qquad u = e^x$$

$$* \quad = \frac{1}{2} tg^{-1} e^x + \frac{1}{2} \frac{e^x}{(1+e^{2x})}$$

$$\int \frac{e^{5x}}{(e^{2x}+1)^2} dx = u - 2 tg^{-1} u + \frac{1}{2} tg^{-1} e^x + \frac{e^x}{2(1+e^{2x})}$$

$$= e^x \underbrace{- 2 tg^{-1} e^x + \frac{1}{2} tg^{-1} e^x}_{} + \frac{e^x}{2(1+e^{2x})}$$

$$\boxed{\int \frac{e^{5x}}{(e^{2x}+1)^2} dx = e^x - \frac{3}{2} tg^{-1} e^x + \frac{e^x}{2(1+e^{2x})} + C}$$

17. $\displaystyle\int \frac{18\,dx}{(4x^2+9)^2}$

$$\frac{1}{(4x^2+9)^2} = \frac{Ax+B}{4x^2+9} + \frac{Cx+D}{(4x^2+9)^2}$$

$$\Leftrightarrow$$

$$1 = (Ax+B)(4x^2+9) + Cx + D$$

$$= 4Ax^3 + 9Ax + 4Bx^2 + 9B + Cx + D$$

$$1 = 4Ax^3 + 4Bx^2 + (9A+C)x + 9B + D$$

$$\Rightarrow \begin{cases} 4A = 0 \Rightarrow A = 0 \\ 4B = 0 \Rightarrow B = 0 \\ 9A + C = 0 \Rightarrow C = 0 \\ 9B + D = 1 \Rightarrow D = 1 \end{cases}$$

$\dfrac{1}{(4x^2+9)^2}$ não se decompõe

$$\int \frac{18}{(4x^2+9)^2}\,dx = \int \frac{18\,dx}{\left(4\left(x^2+\frac{9}{4}\right)\right)^2} = \int \frac{18\,dx}{16\left(x^2+\frac{9}{4}\right)^2}$$

$$= \frac{9}{8}\int \frac{dx}{\left(x^2+\frac{9}{4}\right)^2} \qquad \begin{cases} x = \frac{3}{2}\,\mathrm{tg}\,\theta \\ dx = \frac{3}{2}\sec^2\theta\,d\theta \end{cases}$$

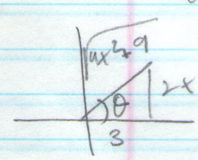
$$= \frac{9}{8}\int \frac{dx}{\left(x^2+\frac{9}{4}\right)^2} \qquad x = \frac{3}{2}\,tg\,\theta$$

$$dx = \frac{3}{2}\sec^2\theta\, d\theta$$

$$= \frac{9}{8}\int \frac{\frac{3}{2}\sec^2\theta\, d\theta}{\left(\frac{9}{4}\left(1+tg^2\theta\right)\right)^2}$$

$$= \frac{9}{8}\times\frac{3}{2}\,\frac{1}{\left(\frac{9}{4}\right)^2}\int \frac{\sec^2\theta\, d\theta}{\sec^4\theta}$$

$$tg\,\theta = \frac{2x}{3}$$



$$= \frac{\cancel{9}}{8}\times\frac{3}{\cancel{2}}\times\frac{16}{8\times 9}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{3}\int \frac{1+\cos 2\theta}{2}\, d\theta$$

$$\sin\theta = \frac{2x}{\sqrt{4x^2+9}}$$

$$= \frac{1}{6}\left(\theta + \frac{\sin 2\theta}{2}\right)$$

$$\cos\theta = \frac{3}{\sqrt{4x^2+9}}$$

$$= \frac{1}{6}\theta + \frac{1}{12}\sin 2\theta$$

$$= \frac{\theta}{6} + \frac{1}{6}\sin\theta\cos\theta$$

$$= \frac{1}{6}\,tg^{-1}\frac{2x}{3} + \frac{1}{6}\,\frac{2x}{\sqrt{4x^2+9}}\;\frac{3}{\sqrt{4x^2+9}}$$

$$\boxed{\int \frac{18\, dx}{(4x^2+9)^2} = \frac{1}{6}\,tg^{-1}\frac{2x}{3} + \frac{x}{(4x^2+9)} + C}$$

18. $$\int \frac{2x^2+3x+2}{x^3+4x^2+6x+4}\,dx$$

$$x^3+4x^2+6x+4 = (x+2)\,H(x)$$

$$\begin{array}{l|l} x^3+4x^2+6x+4 & x+2 \\ \cline{2-2} -x^3-2x^2 & x^2+2x+2 \\ \hline \quad 2x^2+6x+4 & \\ \quad -2x^2-4x & \\ \hline \qquad 2x+4 & \\ \qquad -2x-4 & \\ \hline \qquad\quad 0 & \end{array}$$

$\therefore$

$$x^3+4x^2+6x+4 = (x+2)(x^2+2x+2)$$

$\therefore$

$$\frac{2x^2+3x+2}{x^3+4x^2+6x+4} = \frac{2x^2+3x+2}{(x+2)(x^2+2x+2)}$$

$\therefore$

$$\frac{2x^2+3x+2}{(x+2)(x^2+2x+2)} = \frac{A}{x+2} + \frac{B(2x+2)+C}{x^2+2x+2}$$

$$\Leftrightarrow$$

$$2x^2+3x+2 = A(x^2+2x+2) + [2Bx+2B+C](x+2)$$

$$\Longleftarrow\ = Ax^2+2Ax+2A+2Bx^2+4Bx+2Bx+4B+Cx+2C$$

$$2x^2+3x+2 = (A+2B)x^2 + (2A+6B+C)x + 2A+4B+2C$$

$$\Longrightarrow$$

$$A+2B = 2 \Rightarrow \| A = 2(1-B) \|$$

$$2A+6B+C = 3$$

$$\cancel{2}A + \underset{2}{\cancel{4}}B + \cancel{2}C = \cancel{2}\,1$$

$$A+2B+C = 1$$

$$4(1-B)+6B+C = 3$$

$$4-4B+6B+C = 3$$

$$\| 2B+C = -1 \|$$

$$2(1-B)+2B+C = 1$$

$$2-2\cancel{B}+2\cancel{B}+C = 1$$

$$\boxed{C=-1}$$

$\therefore$

$$2B+C = -1 \Rightarrow 2B-1 = -1 \Rightarrow B=0$$

$$A = 2(1-B) \Rightarrow A=2$$

$$\frac{2x^2+3x+2}{(x+2)(x^2+2x+2)} = \frac{2}{x+2} - \frac{1}{x^2+2x+2}$$

$5 - \frac{8}{3}$

$$\int \frac{2x^2+3x+2}{(x+2)(x^2+2x+2)}\,dx = 2\int \frac{dx}{x+2} - \underbrace{\int \frac{dx}{x^2+2x+2}}_{(*)}$$

$$= 2\ln|x+2| - (*)$$

$$(*) = \int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x+1)^2+1} \qquad \begin{aligned} x+1 &= u \\ dx &= du \end{aligned}$$

$$= \int \frac{du}{u^2+1}$$

$$= tg^{-1} u$$

$$= tg^{-1}(x+1)$$

$\therefore$

$$\boxed{\int \frac{2x^2+3x+2}{(x+2)(x^2+2x+2)}\,dx = 2\ln|x+2| - tg^{-1}(x+1) + C}$$

---

19. $$\int \frac{(\sec^2 x+1)\sec^2 x}{1+tg^3 x}\,dx \qquad \sec^2 x = 1+tg^2 x$$

$$= \int \frac{(2+tg^2 x)\sec^2 x\,dx}{1+tg^3 x}$$

$$y = tg\,x \longrightarrow dy = \sec^2 y\,dy$$

$$= \int \frac{(2+y^2)\,dy}{1+y^3}$$

$$\frac{y^2+2}{y^3+1} = \frac{y^2+2}{(y+1)(y^2-y+1)} = \frac{A}{y+1} + \frac{B(2y-1)+C}{y^2-y+1}$$

$$y^3+1 = (y+1)(y^2-y+1)$$

$$\Leftrightarrow$$

$$y^2+2 = A(y^2-y+1) + [2By - B + C](y+1)$$

$$y^2+2 = Ay^2 - Ay + A + 2By^2 + 2B - By - B + Cy + C$$

$$y^2+2 = (A+2B)y^2 + (-A-B+C)y + A+2B+C$$

$$\Rightarrow \begin{cases} A+2B = 1 \Rightarrow \| A = 1-2B \| \\ -A-B+C = 0 \\ A+2B+C = 2 \end{cases}$$

$$\begin{cases} -1+2B-B+C = 0 \\ -1+B+C = 0 \\ \| B+C = 1 \| \Rightarrow B = 0 \end{cases}$$

$$1-\cancel{2B}+\cancel{2B}+C = 2$$

$$C = 1$$

$$A = 1-2B = 0$$

$$A = 1$$

$$\Rightarrow$$

$$\frac{y^2+2}{y^3+1} = \frac{1}{y+1} + \frac{1}{y^2-y+1}$$

$$\int \frac{(\sec^2 x+1)\sec^2 x\,dx}{1+\tg^3 x} = \int \frac{y^2+2}{y^3+1}\,dy =$$

$$= \int \frac{dy}{y+1} + \underbrace{\int \frac{1}{y^2-y+1}\,dy}$$

$$= \ln|y+1| + (\star)$$

$$//(\star) = \int \frac{dy}{y^2-y+1} = \int \frac{dy}{\left(y-\frac{1}{2}\right)^2+\frac{3}{4}} \quad \left\{ \begin{array}{l} u = y-\frac{1}{2} \\ du = dy \end{array} \right.$$

$\frac{1}{4}+x=1$

$x = 1-\frac{1}{4}$

$= \frac{3}{4}$

$$= \int \frac{du}{u^2+\frac{3}{4}} \qquad u = \frac{\sqrt{3}}{2}\tg\theta$$

$$\theta = \tg^{-1}\frac{2u}{\sqrt{3}}$$

$$= \int \frac{\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta}{\frac{3}{4}(\tg^2\theta+1)} \qquad du = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$$

$$= \frac{\sqrt{3}}{2}\times\frac{4}{3}\int d\theta$$

$$= \frac{2}{\sqrt{3}}\,\theta = \frac{2}{\sqrt{3}}\tg^{-1}\frac{2u}{\sqrt{3}} \qquad \downarrow u = y-\frac{1}{2}$$

$$= \frac{2}{\sqrt{3}}\tg^{-1}\left(\frac{2y-1}{\sqrt{3}}\right) //$$

$$\int \frac{(\sec^2 x+1)\sec^2 x\,dx}{1+\tg^3 x} = \ln|y+1| + \frac{2}{\sqrt{3}}\tg^{-1}\left(\frac{2y-1}{\sqrt{3}}\right) + C$$

$$(y = \tg x)$$

$\Longrightarrow$

$$\boxed{\int \frac{(\sec^2 x+1)\sec^2 x}{1+tg^3 x}\,dx = \ln|tg\,x+1| + \frac{2}{\sqrt{3}}\,tg^{-1}\left(\frac{2tg\,x-1}{\sqrt{3}}\right) + C}$$

20. $$\int \frac{(6w^4+4w^3+9w^2+24w+32)\,dw}{(w^3+8)(w^2+3)}$$

$$w^3+8 = (w+2)(w^2-2w+4)$$

$$\begin{array}{l|l} w^3+8 & w+2 \\ -w^3-2w^2 & w^2-2w+4 \\ -2w^2+8 & \\ +2w^2+4w & \\ 4w+8 & \\ -4w-8 & \\ 0 & \end{array}$$

$$\therefore \frac{6w^4+4w^3+9w^2+24w+32}{(w^3+8)(w^2+3)} = \frac{6w^4+4w^3+9w^2+24w+32}{(w+2)(w^2-2w+4)(w^2+3)}$$

$$\frac{6w^4+4w^3+9w^2+24w+32}{(w+2)(w^2-2w+4)(w^2+3)} = \frac{A}{w+2} + \frac{B(2w-2)+C}{w^2-2w+4} + \frac{Dw+E}{w^2+3}$$

$$\Leftrightarrow$$

$$6w^4+4w^3+9w^2+24w+32 = A(w^2-2w+4)(w^2+3) +$$
$$+ (2Bw-2B+C)(w+2)(w^2+3)$$
$$+ (Dw+E)(w+2)(w^2-2w+4)$$
$$= Aw^4+3Aw^2-2Aw^3-6Aw+4Aw^2+12A$$
$$(2Bw-2B+C)(w^3+3w+2w^2+6)$$
$$+ (Dw+E)(w^3+\cancel{2w^2}-\cancel{2w^2}-\cancel{4w}+\cancel{4w}+8)$$

$$6w^4+4w^3+9w^2+24w+32 = Aw^4+3Aw^2-2Aw^3-6Aw+$$

$$+4Aw^2+12A+2Bw^4-2Bw^3+Cw^3$$

$$+6Bw^2-6Bw+3Cw$$

$$+4Bw^3-4Bw^2+2Cw^2$$

$$+12Bw-12B+6C$$

$$+Dw^4+Ew^3+8Dw+8E$$

$$=(A+2B+D)w^4+(-2A-2B+C+4B+E)w^3$$

$$+(3A+4A+6B-4B+2C)w^2$$

$$+(-6A-6B+3C+12B+8D)w$$

$$+12A-12B+6C+8E$$

$$6w^4+4w^3+9w^2+24w+32 = (A+2B+D)w^4+(-2A+2B+C+E)w^3$$

$$+(7A+2B+2C)w^2+$$

$$+(-6A+6B+3C+8D)w$$

$$+12A-12B+6C+8E$$

$\Longrightarrow$

$A + 2B + D = 6 \Rightarrow // A = 6 - 2B - D //$

① $-2A + 2B + C + E = 4$

② $7A + 2B + 2C = 9$

③ $-6A + 6B + 3C + 8D = 24$

④ $12A - 12B + 6C + 8E = 32$

$\therefore$

① :

$-2(6 - 2B - D) + 2B + C + E = 4$

$-12 + 4B + 2D + 2B + C + E = 4$

$-12 + 6B + 2D + C + E = 4$

$// 6B + 2D + C + E = 16 //$

②

$7(6 - 2B - D) + 2B + 2C = 9$

$42 - 14B - 7D + 2B + 2C = 9$

$42 - 12B - 7D + 2C = 9$

$// -12B + 2C - 7D = -33 //$

(3)

$$-6A + 6B + 3C + 8D = 24$$

$$-3A + 3B + C + 4D = 12$$

$$-3(6 - 2B - D) + 3B + C + 4D = 12$$

$$-18 + 6B + 3D + 3B + C + 4D = 12$$

$$-18 + 9B + 7D + C = 12$$

$$\| 9B + C + 7D = 30 \|$$

(4)

$$12A - 12B + 6C + 8E = 32$$

$$6A - 6B + 3C + 4E = 16$$

$$6(6 - 2B - D) - 6B + 3C + 4E = 16$$

$$36 - 12B - 6D - 6B + 3C + 4E = 16$$

$$36 - 18B - 6D + 3C + 4E = 16$$

$$\| -18B + 3C - 6D + 4E = -20 \|$$

$$\begin{cases} A = 6-2B-D \\ \begin{cases} 6B+2D+C+E = 16 \\ -12B+2C-7D=-33 \\ 9B+C+7D=30 \\ -18B+3C-6D+4E=-20 \end{cases} \end{cases}$$