Lista 6

Integração funções racionais de seno e cosseno

Exercícios. (Leithold pg. 569)

1. $\int \frac{3\,dx}{8+7\cos x}$

$z = \operatorname{tg}\frac{x}{2} \rightarrow \begin{cases} dx = \frac{2\,dz}{1+z^2} \\ \cos x = \frac{1-z^2}{1+z^2} \end{cases}$

$= \int \frac{3\,\frac{2\,dz}{1+z^2}}{8+7\,\frac{1-z^2}{1+z^2}}$

$= \int \frac{\frac{6\,dz}{1+z^2}}{\frac{8+8z^2+7-7z^2}{1+z^2}} = \int \frac{6\,dz}{z^2+15}$

$= 6\int \frac{dz}{z^2+\sqrt{15}^2}$

$= 6\,\frac{1}{\sqrt{15}}\,\operatorname{tg}^{-1}\frac{z}{\sqrt{15}} + C$

$= \frac{6}{\sqrt{15}}\,\operatorname{tg}^{-1}\left(\frac{1}{\sqrt{15}}\operatorname{tg}\frac{x}{2}\right) + C$ //

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2. $\int \frac{dx}{1+\sin x} =$

$z = \operatorname{tg}\frac{x}{2}, \begin{cases} dx = \frac{2\,dz}{1+z^2} \\ \sin x = \frac{2z}{1+z^2} \end{cases}$

$= \int \frac{\frac{2\,dz}{1+z^2}}{1+\frac{2z}{1+z^2}} = \int \frac{\frac{2\,dz}{1+z^2}}{\frac{1+z^2+2z}{1+z^2}} = \int \frac{2\,dz}{z^2+2z+1}$

$= \int \frac{2\,dz}{(z+1)^2} =$

$$= \int \frac{2\,dz}{(z+1)^2} = 2\,\frac{1}{-(z+1)} + C$$

$$= -\frac{2}{(z+1)} + C$$

$$\therefore \boxed{\int \frac{dx}{1+\sin x} = -\frac{2}{\tan\frac{x}{2}+1} + C}$$

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3. $\int \frac{3\,dx}{7+8\cos x}$ $\qquad z = \tan\frac{x}{2} \rightarrow dx = \frac{2\,dz}{1+z^2}$

$\cos x = \frac{1-z^2}{1+z^2}$

$$= \int \frac{3\,\frac{2\,dz}{1+z^2}}{7+8\,\frac{1-z^2}{1+z^2}}$$

$$= \int \frac{\frac{6\,dz}{1+z^2}}{\frac{7+7z^2+8-8z^2}{1+z^2}} = \int \frac{6\,dz}{15-z^2}$$

$$\int \frac{dz}{a^2-z^2} = \frac{1}{2a}\ln\left|\frac{z+a}{z-a}\right|$$

$$= 6\int \frac{dz}{\sqrt{15}^2 - z^2}$$

$$= 6\,\frac{1}{2\sqrt{15}}\ln\left|\frac{z+\sqrt{15}}{z-\sqrt{15}}\right| + C$$

$$\boxed{\int \frac{3\,dx}{7+8\cos x} = \frac{3}{\sqrt{15}}\ln\left|\frac{\tan\frac{x}{2}+\sqrt{15}}{\tan\frac{x}{2}-\sqrt{15}}\right| + C}$$

4. $\int \frac{dx}{\sin x - \cos x + 2}$ $\quad z = \tan\frac{x}{2}, \; dx = \frac{2\,dz}{1+z^2}$

$\sin x = \frac{2z}{1+z^2}$

$\cos x = \frac{1-z^2}{1+z^2}$

$= \int \frac{\frac{2\,dz}{1+z^2}}{\frac{2z}{1+z^2} - \frac{1-z^2}{1+z^2} + 2}$

$= \int \frac{\frac{2\,dz}{1+z^2}}{\frac{2z-1+z^2+2+2z^2}{1+z^2}}$

$= \int \frac{2\,dz}{3z^2+2z+1}$

$= \int \frac{2\,dz}{3\left(z^2+\frac{2}{3}z+\frac{1}{3}\right)} = \frac{2}{3}\int \frac{dz}{z^2+\frac{2}{3}z+\frac{1}{3}}$

$\frac{1}{9}+x=\frac{1}{3}$

$x = \frac{1}{3}-\frac{1}{9}$

$= \frac{2}{9}$

$= \frac{2}{3}\int \frac{dz}{\left(z+\frac{1}{3}\right)^2+\frac{2}{9}}$ $\quad z+\frac{1}{3} = u$

$= \frac{2}{3}\int \frac{du}{u^2+\frac{2}{9}} =$

$= \frac{2}{3}\int \frac{du}{u^2+\left(\frac{\sqrt{2}}{3}\right)^2}$

$= \frac{2}{3}\cdot\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\frac{u}{\frac{\sqrt{2}}{3}}$

$= \sqrt{2}\tan^{-1}\frac{3u}{\sqrt{2}}$

$= \sqrt{2}\tan^{-1}\frac{3z+1}{\sqrt{2}} + C \longrightarrow$

$\therefore = \sqrt{2}\, tg^{-1}\frac{3g+1}{2} + C$

$$\boxed{\int \frac{dx}{\sin x - \cos x + 2} = \sqrt{2}\, tg^{-1}\left(\frac{3tg\frac{x}{2}+1}{2}\right) + C}$$

5. $\int \frac{dx}{\sin x + tg x}$ $\quad z = tg\frac{x}{2} \rightarrow dx = \frac{2dz}{1+z^2}$

$\sin x = \frac{2z}{1+z^2}$

$\cos x = \frac{1-z^2}{1+z^2}$

$tg x = \frac{2z}{1-z^2}$

$$= \int \frac{\frac{2dz}{1+z^2}}{\frac{2z}{1+z^2} + \frac{2z}{1-z^2}}$$

$$= \int \frac{\frac{2dz}{1+z^2}}{\frac{2z(1-z^2)+2z(1+z^2)}{(1+z^2)(1-z^2)}}$$

$$= \int \frac{2(1-z^2)}{2z(1-z^2)+2z(1+z^2)}\, dz$$

$$= \int \frac{2(1-z^2)}{2z - 2z^3 + 2z + 2z^3}\, dz$$

$$= \int \frac{2(1-z^2)}{4z}\, dz = \frac{1}{2}\int \frac{(1-z^2)}{z}\, dz$$

$$= \frac{1}{2}\int \frac{dz}{z} - \frac{1}{2}\int z\, dz$$

$$= \frac{1}{2}\ln|z| - \frac{1}{2}\frac{z^2}{2} + C \rightarrow$$

$$= \frac{1}{2}\ln|z| - \frac{z^2}{4} + C$$

$\therefore$

$$\boxed{\int \frac{dx}{\sin x + \tan x} = \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| - \frac{1}{4}\tan^2\frac{x}{2} + C}$$

6. $\int \frac{dx}{5+4\cos x}$ $\qquad z = \tan\frac{x}{2}$

$$= \int \frac{\frac{2\,dz}{1+z^2}}{5+4\,\frac{1-z^2}{1+z^2}} = \int \frac{\frac{2\,dz}{1+z^2}}{\frac{5+5z^2+4-4z^2}{1+z^2}}$$

$$= \int \frac{2\,dz}{z^2+9}$$

$$= 2\int \frac{dz}{z^2+3^2}$$

$$= 2\cdot\frac{1}{3}\tan^{-1}\frac{z}{3} + C$$

$\therefore$

$$\boxed{\int \frac{dx}{5+4\cos x} = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right) + C}$$

7. $\int \frac{dx}{3-5\sin x}$

$= \int \frac{\frac{2\,dz}{1+z^2}}{3-5\frac{2z}{1+z^2}} = \int \frac{\frac{2\,dz}{1+z^2}}{\frac{3+3z^2-10z}{1+z^2}}$

$= \int \frac{2\,dz}{3z^2-10z+3}$

$= \int \frac{2\,dz}{3(z^2-\frac{10}{3}z+1)}$

$= \frac{2}{3}\int \frac{dz}{z^2-\frac{10}{3}z+1}$

$= \frac{2}{3}\int \frac{dz}{(z-\frac{5}{3})^2-\frac{16}{9}}$ $\quad z-\frac{5}{3}=u$

$= \frac{2}{3}\int \frac{du}{u^2-(\frac{4}{3})^2}$

$= \frac{2}{3}\left(\frac{1}{2\cdot\frac{4}{3}}\ln\left|\frac{u-\frac{4}{3}}{u+\frac{4}{3}}\right|\right)$

$= \frac{2}{3}\left(\frac{3}{8}\ln\left|\frac{z-\frac{5}{3}-\frac{4}{3}}{z-\frac{5}{3}+\frac{4}{3}}\right|\right)+C$

$= \frac{1}{4}\ln\left|\frac{z-3}{z-\frac{1}{3}}\right|+C$ $\Longrightarrow$

$\frac{25}{9}+c=1$

$c=1-\frac{25}{9}$

$=\frac{9-25}{9}=-\frac{16}{9}$

$\int \frac{du}{u^2-a^2} =$

$= \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|$

$$\int \frac{dx}{3-5\sin x} = \frac{1}{4} \ln \left| \frac{\tan \frac{x}{2} - 3}{\tan \frac{x}{2} - \frac{1}{3}} \right|$$

8. $\int \frac{dx}{4\sin x - 3\cos x} =$

$$= \int \frac{\frac{2dz}{1+z^2}}{4\frac{2z}{1+z^2} - 3\frac{1-z^2}{1+z^2}}$$

$$= \int \frac{\frac{2dz}{1+z^2}}{\frac{8z}{1+z^2} + \frac{-3+3z^2}{1+z^2}}$$

$$= \int \frac{2dz}{8z - 3 + 3z^2}$$

$$= \int \frac{2dz}{3z^2 + 8z - 3}$$

$$= \frac{2}{3} \int \frac{dz}{z^2 + \frac{8}{3}z - 1}$$

$\frac{16}{9} + x = -1$

$x = -1 - \frac{16}{9}$

$= -\frac{25}{9}$

$$= \frac{2}{3} \int \frac{dz}{\left(z + \frac{4}{3}\right)^2 - \frac{25}{9}} \longrightarrow$$

$$= \frac{2}{3} \int \frac{dz}{(z+\frac{4}{3})^2 - \frac{25}{9}} \qquad u = z + \frac{4}{3}$$

$$= \frac{2}{3} \int \frac{du}{u^2 - (\frac{5}{3})^2} \qquad \left\{ \int \frac{du}{u^2 - a^2} = \right.$$

$$= \frac{2}{3} \cdot \frac{1}{2 \cdot \frac{5}{3}} \ln \left| \frac{u - \frac{5}{3}}{u + \frac{5}{3}} \right| \qquad \left. = \frac{1}{2a} \ln \left| \frac{u-a}{u+a} \right| \right.$$

$$= \frac{1}{5} \ln \left| \frac{z + \frac{4}{3} - \frac{5}{3}}{z + \frac{4}{3} + \frac{5}{3}} \right|$$

$$= \frac{1}{5} \ln \left| \frac{z - \frac{1}{3}}{z + 3} \right|$$

$$= \frac{1}{5} \ln \left| \frac{tg\frac{x}{2} - \frac{1}{3}}{tg\frac{x}{2} + 3} \right| + C$$