Lista 8 : Ints. Impróprias II

Exercícios (Pg. 676)

1. $$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{t \to 1^-} \int_0^t \frac{dx}{\sqrt{1-x}}$$

$$= \lim_{t \to 1^-} -2\sqrt{1-x}\Big]_0^t$$

$$= \lim_{t \to 1^-} \left(\cancel{-2\sqrt{1-t}}^{\,0} + 2 \right)$$

$$= 2$$

2. $$\int_0^{16} \frac{dx}{x^{3/4}} = \lim_{t \to 0^+} \int_t^{16} \frac{dx}{x^{3/4}}$$

$$= \lim_{t \to 0^+} 4x^{1/4}\Big]_t^{16}$$

$$\int \frac{dx}{x^{3/4}} = \frac{x^{-3/4+1}}{-\frac{3}{4}+1} = \frac{x^{1/4}}{\frac{1}{4}} = 4x^{1/4}$$

$$= \lim_{t \to 0^+} \left(4\,16^{1/4} - 4t^{1/4} \right)$$

$$= \lim_{t \to 0^+} \left(4 \times 2 - 4t^{1/4} \right)$$

$$= 8$$

3. $\displaystyle\int_{-5}^{-3} \frac{x\,dx}{\sqrt{x^2-9}} =$ f(x) é descontínua em x=-3

$$= \lim_{t \to -3^-} \int_{-5}^{t} \frac{x\,dx}{\sqrt{x^2-9}}$$

$$= \lim_{t \to -3^-} \left(\sqrt{x^2-9}\right)\Big|_{-5}^{t}$$

$$= \lim_{t \to -3^-} \left(\cancel{\sqrt{t^2-9}}^{\,0} - 4\right)$$

$$= -4 \;//$$

4. $\displaystyle\int_{0}^{4} \frac{x\,dx}{\sqrt{16-x^2}} =$ f(x) é descontínua em $x = \pm 4$

$$= \lim_{t \to 4^-} \int_{0}^{t} \frac{x\,dx}{\sqrt{16-x^2}}$$

$$= \lim_{t \to 4^-} -\sqrt{16-x^2}\,\Big|_{0}^{t}$$

$$= \lim_{t \to 4^-} \left(-\cancel{\sqrt{16-t^2}}^{\,0} + 4\right)$$

$$= +4 \;//$$

5. $\int_2^4 \frac{dt}{\sqrt{16-t^2}}$ $\qquad f(x) = \frac{1}{\sqrt{16-x^2}}$ descontínua em $x=4$

$$\int_2^4 \frac{dt}{\sqrt{16-t^2}} = \lim_{t\to 4^-} \underbrace{\int_2^t \frac{dt}{\sqrt{16-t^2}}}$$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

$$= \lim_{t\to 4^-} \sin^{-1}\frac{t}{4}\Big]_2^t$$

$$= \lim_{t\to 4^-}\left(\sin^{-1}\frac{t}{4} - \sin^{-1}\frac{1}{2}\right)$$

$\sin^{-1}x$:

Dom $\sin^{-1} = (-\infty, +\infty)$

Im $\sin^{-1} = [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$= \lim_{t\to 4^-}\left(\underbrace{\sin^{-1}\frac{t}{4}} - \frac{\pi}{6}\right)$$

$$= \left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{2\pi}{6} = \frac{\pi}{3} \; //$$

6. $\int_{-4}^1 \frac{dz}{(z+3)^3} =$ $\qquad f(z) = \frac{1}{z+3}$ é descontínua em $z=-3$.

$$= \lim_{t\to -3^-} \underbrace{\int_{-4}^t \frac{dz}{(z+3)^3}} + \lim_{t\to -3^+} \int_t^1 \frac{dz}{(z+3)^3}$$

$$= \lim_{t\to -3^-} \frac{-1}{2(z+3)^2}\Big]_{-4}^t + \lim_{t\to -3^+} \frac{-1/2}{(z+3)^2}\Big]_t^1$$

$$= \lim_{t\to -3^-}\left(\frac{-1}{2(t+3)^2} + \frac{1}{2(-4+3)^2}\right) + \lim_{t\to -3^+}\left(\frac{-1}{2(4)^2} + \frac{1}{2(t+3)^2}\right)$$

$$= \lim_{t \to -3^-} \left(\underbrace{\frac{-1}{2(t+3)^2}}_{-\infty} + \frac{1}{2} \right) + \lim_{t \to -3^+} \left(\frac{-1}{32} + \frac{1}{2(t+3)^2} \right)$$

$$= -\infty + \infty$$

= Divergente

$$\therefore \boxed{\int_{-4}^{1} \frac{dz}{(z+3)^3} = \text{divergente}}$$

7. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sec\theta \, d\theta =$ $\qquad \sec\theta = \frac{1}{\cos\theta} \rightarrow$ descontínuo em $\theta = \frac{\pi}{2}$

$$= \lim_{t \to \frac{\pi}{2}^-} \int_{\frac{\pi}{4}}^{t} \sec\theta \, d\theta \qquad \sec\frac{\pi}{4} = \frac{1}{\cos\frac{\pi}{4}} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}}$$

$$= \lim_{t \to \frac{\pi}{2}^-} \ln|\sec\theta + tg\,\theta| \Big]_{\frac{\pi}{4}}^{t}$$

$$= \lim_{t \to \frac{\pi}{2}^-} \left\{ \ln|\sec t + tg\,t| - \ln\left|\sec\frac{\pi}{4} + tg\frac{\pi}{4}\right| \right\}$$

$$= \lim_{t \to \frac{\pi}{2}^-} \left\{ \ln|\sec t + tg\,t| - \ln\left|\frac{2}{\sqrt{2}} + 1\right| \right\}$$

Mas,

$$\ln|\sec t + tg\,t| = \ln\left|\frac{1}{\cos t} + \frac{\sin t}{\cos t}\right| =$$

$$= \ln \left| \frac{1+\sin t}{\cos t} \right|$$

$$\lim_{t \to \frac{\pi}{2}^-} \ln \left| \frac{1+\sin t}{\cos t} \right| = \ln \left| \frac{1+1}{0} \right| = +\infty$$

Uma vez que o limite anterior não existe temos que

$$\boxed{\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sec \theta \, d\theta \; = \text{diverge}}$$

8. $\displaystyle\int_{-2}^{0} \frac{dx}{\sqrt{4-x^2}} =$ $\qquad f(x) = \frac{1}{\sqrt{4-x^2}}$ diverge em $x = -2$

$$= \lim_{t \to -2^+} \int_t^0 \frac{dx}{\sqrt{4-x^2}} \qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$= \lim_{t \to -2^+} \sin^{-1}\frac{x}{2} \Big]_t^0$$

$$= \lim_{t \to -2^+} \left(\sin^{-1}\frac{0}{2} - \sin^{-1}\frac{t}{2} \right) \qquad \sin^{-1} 0 = 0$$

$$= \lim_{t \to -2^+} \left(0 - \sin^{-1}\frac{t}{2} \right)$$

$\sin^{-1} x \Rightarrow$ Dom $= (-1, 1)$

Img $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$= \lim_{t \to -2^+} -\sin^{-1}\frac{t}{2} = -\underbrace{\sin^{-1}(-1)}_{-\frac{\pi}{2}} = +\frac{\pi}{2} \,//$$

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9. $\displaystyle\int_0^{+\infty} \frac{dx}{x^3} =$

$$= \lim_{t\to 0^+} \int_t^1 \frac{dx}{x^3} + \lim_{b\to +\infty} \int_1^b \frac{dx}{x^3}$$

$$= \lim_{t\to 0^+} \left. \frac{-1}{2x^2} \right]_t^1 + \lim_{b\to +\infty} \left. \frac{-1}{2x^2} \right]_1^b$$

$$= \lim_{t\to 0^+} \underbrace{\left(\frac{-1}{2} + \frac{1}{2t^2} \right)} + \lim_{b\to +\infty} \left(\frac{-1}{2b^2} + \frac{1}{2} \right)$$

$$= +\infty \qquad + \qquad \frac{1}{2}$$

$$= +\infty$$

$$\boxed{\int_0^{+\infty} \frac{dx}{x^3} = +\infty \quad \text{diverge}}$$

10. $\int_0^{\frac{\pi}{2}} tg\theta \, d\theta$ $\qquad$ $tg\theta$ diverge in $\theta = \frac{\pi}{2}$

$$= \lim_{t \to \frac{\pi}{2}^-} \int_0^t tg\theta \, d\theta$$

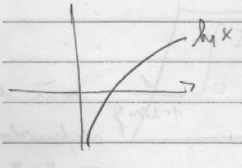
$$= \lim_{t \to \frac{\pi}{2}^-} \ln|\sec\theta| \Big|_0^t \qquad \Big\} \sec 0 = \frac{1}{\cos 0} = 1$$

$$= \lim_{t \to \frac{\pi}{2}^-} \left(\ln|\sec t| - \ln \sec 0 \right)$$

$$= \lim_{t \to \frac{\pi}{2}^-} \left(\ln|\sec t| - \underbrace{\ln 1}_{0} \right)$$

$$= \lim_{t \to \frac{\pi}{2}^-} \ln|\sec t| \qquad \lim_{t \to \frac{\pi}{2}^-} \sec t = \frac{1}{\cos\frac{\pi}{2}} = \frac{1}{0} = +\infty$$

$$= +\infty$$



$$\therefore \boxed{\int_0^{\frac{\pi}{2}} tg\theta \, d\theta \ \therefore \text{diverge}}$$

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11. $\int_0^{\frac{\pi}{2}} \frac{dy}{1-\sin y}$

$$\int \frac{dy}{1-\sin y} = \int \frac{(1+\sin y)\,dy}{(1-\sin y)(1+\sin y)}$$

$$= \int \frac{(1+\sin y)\,dy}{1-\sin^2 y}$$

$$= \int \frac{1+\sin y}{\cos^2 y}\,dy$$

$$= \int \frac{1}{\cos^2 y}\,dy + \int \sec y\, \operatorname{tg} y\, dy$$

$$= \int \sec^2 y\, dy + \int \sec y\, \operatorname{tg} y\, dy$$

$$= \operatorname{tg} y + \sec y$$

$\therefore$

$$\int_0^{\frac{\pi}{2}} \frac{dy}{1-\sin y} = \lim_{t \to \frac{\pi}{2}^-} \int_0^{t} \frac{dy}{1-\sin y}$$

$\frac{1}{1-\sin y}$ não está definida em $y = \frac{\pi}{2}$

$$= \lim_{t \to \frac{\pi}{2}^-} \left[\operatorname{tg} y + \sec y\right]_0^t$$

$\operatorname{tg} 0 = 0$

$\sec 0 = \frac{1}{\cos 0} = 1$

$$= \lim_{t \to \frac{\pi}{2}^-} \left(\operatorname{tg} t + \sec t - \operatorname{tg} 0 - \sec 0\right)$$

$$= \lim_{t \to \frac{\pi}{2}^-} \left(\overset{+\infty}{\cancel{\operatorname{tg}} t} + \overset{+\infty}{\sec t} - 0 - 1\right)$$

$\operatorname{tg} t \xrightarrow{t \to \frac{\pi}{2}^-} +\infty$

$\sec t \xrightarrow{t \to \frac{\pi}{2}^-} \frac{1}{0^+} = +\infty$

$= +\infty \quad \therefore \quad \boxed{\int_0^{\frac{\pi}{2}} \frac{dy}{1-\sin y} = \infty \quad \text{diverge}}$

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12. $\displaystyle\int_0^2 \frac{dx}{(x-1)^{2/3}}$ $\quad f(x) = \frac{1}{(x-1)^{2/3}}$ diverge em $x=1$

$$= \lim_{t\to 1^-} \int_0^t \frac{dx}{(x-1)^{2/3}} + \lim_{t\to 1^+} \int_t^2 \frac{dx}{(x-1)^{2/3}}$$

$-\frac{2}{3}+1 = \frac{1}{3}$

$$= \lim_{t\to 1^-} 3(x-1)^{1/3}\Big]_0^t + \lim_{t\to 1^+} 3(x-1)^{1/3}\Big]_t^2$$

$$= \lim_{t\to 1^-}\left(3(t-1)^{1/3} - 3(-1)^{1/3}\right) + \lim_{t\to 1^+}\left(3\cdot 1^{1/3} - 3(t-1)^{1/3}\right)$$

$$= \lim_{t\to 1^-}\left(3\underbrace{(t-1)^{1/3}}_{\to 0} + 3\right) + \lim_{t\to 1^+}\left(3 - 3\underbrace{\sqrt[3]{t-1}}_{\to 0}\right)$$

$$= 3 + 3 = 6$$

$$\therefore \boxed{\int_0^2 \frac{dx}{(x-1)^{2/3}} = 6}$$

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13. $\displaystyle\int_0^4 \frac{dx}{x^2-2x-3}$ $\quad \left\{ \begin{array}{l} x^2-2x-3=0 \\ x = 2 \pm \sqrt{4+12}\,/2 \\ = \frac{2\pm 4}{2} \to 3 \to \text{não está definida} \\ \qquad\to -1 \end{array}\right.$

$$= \lim_{t\to 3^-} \int_0^t \frac{dx}{x^2-2x-3} + \lim_{t\to 3^+} \int_t^4 \frac{dx}{x^2-2x-3}$$

Mas

$\quad / \quad /$

$$\int \frac{dx}{x^2-2x-3} = \int \frac{dx}{(x-1)^2-4} \overset{y=x-1}{=} \int \frac{dy}{y^2-4}$$

$$\int \frac{du}{u^2-a^2} = \frac{1}{2a} \ln\left|\frac{u-a}{u+a}\right| \qquad = \frac{1}{2\times 2} \ln\left|\frac{y-2}{y+2}\right| \quad y=x-1$$

$$= \frac{1}{4} \ln\left|\frac{x-3}{x+1}\right|$$

$\therefore$

$$\lim_{t\to 3^-} \int_0^t \frac{dx}{x^2-2x-3} = \lim_{t\to 3^-} \frac{1}{4} \ln\left|\frac{x-3}{x+1}\right|\Big]_0^t$$

$$= \frac{1}{4} \lim_{t\to 3^-} \left(\ln\left|\frac{t-3}{t+1}\right| - \ln\left|\frac{-3}{1}\right| \right)$$

$$= \frac{1}{4} \lim_{t\to 3^-} \left(\underbrace{\ln\left|\frac{t-3}{t+1}\right|}_{\ln\left|\frac{0}{4}\right| \to -\infty} - \ln 3 \right)$$

$$= -\infty$$

$$\lim_{t\to 3^+} \int_t^4 \frac{dx}{x^2-2x-3} = \lim_{t\to 3^+} \frac{1}{4} \ln\left|\frac{x-3}{x+1}\right|\Big|_t^4$$

$$= \frac{1}{4} \lim_{t\to 3^+} \left(\ln \frac{1}{5} - \underbrace{\ln\left|\frac{t-3}{t+1}\right|}_{t\to 3^+} \overset{-\infty}{} \right)$$

$$= +\infty$$

$\therefore$ $\boxed{\int_0^4 \frac{dx}{x^2-2x-3} = \text{divergente}}$

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14. $\int_2^{+\infty} \frac{dx}{x\sqrt{x^2-4}}$ f(x) não está definida em $x = 2$.

$$= \lim_{t\to 2^+}\int_t^3 \frac{dx}{x\sqrt{x^2-4}} + \lim_{b\to+\infty}\int_3^b \frac{dx}{x\sqrt{x^2-4}}$$

Mas

$$\int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{\cancel{2\sec\theta}\, tg\theta\, d\theta}{\cancel{2\sec\theta}\sqrt{4(\sec^2\theta-1)}}$$

$x = 2\sec\theta$

$dx = 2\sec\theta\, tg\theta\, d\theta$

$1+tg^2\theta = \sec^2\theta$

$$= \int \frac{tg\theta\, d\theta}{\sqrt{4\,tg^2\theta}}$$

$$= \int \frac{\cancel{tg\theta}\, d\theta}{2\,\cancel{tg\theta}}$$

$$= \frac{1}{2}\int d\theta = \frac{1}{2}\theta$$

$$= \frac{1}{2}\cos^{-1}\frac{2}{x}$$

$x = 2\sec\theta$

$\frac{x}{2} = \sec\theta$

$\Leftrightarrow$

$\frac{2}{x} = \cos\theta$

$\theta = \cos^{-1}\frac{2}{x}$

D

$$\therefore \lim_{t\to 2^+}\int_t^3 \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to 2^+} \frac{1}{2}\cos^{-1}\frac{2}{x}\Big]_t^3$$

$\lim_{t\to 2^+}\cos^{-1}\frac{2}{t} = \cos^{-1}1 = 0$

$\cos^{-1}$: Im $[0,\pi]$

$$= \lim_{t\to 2^+}\left(\frac{1}{2}\cos^{-1}\frac{2}{3} - \frac{1}{2}\overbrace{\cos^{-1}\frac{2}{t}}^{0}\right)$$

$$= \frac{1}{2}\cos^{-1}\frac{2}{3}$$

1/2

$$\lim_{b\to+\infty}\int_3^b \frac{dx}{x\sqrt{x^2-4}} = \lim_{b\to+\infty}\left(\frac{1}{2}\cos^{-1}\frac{2}{x}\Big]_3^b\right) =$$

$$\lim_{b\to+\infty} \frac{1}{2}\cos^{-1}\frac{2}{x}\Big]_3^b =$$

$$= \lim_{b\to+\infty}\left(\frac{1}{2}\cos^{-1}\frac{2}{b} - \frac{1}{2}\cos^{-1}\frac{2}{3}\right)$$

$$\lim_{b\to\infty}\cos^{-1}\frac{2}{b} = \cos^{-1}0 = \frac{\pi}{2}$$

$$= \frac{1}{2}\frac{\pi}{2} - \frac{1}{2}\cos^{-1}\frac{2}{3}$$

$$= \frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{2}{3}$$

Daí,

$$\int_2^{+\infty}\frac{dx}{x\sqrt{x^2-4}} = \cancel{\frac{1}{2}\cos^{-1}\frac{2}{3}} + \frac{\pi}{4} - \cancel{\frac{1}{2}\cos^{-1}\frac{2}{3}}$$

$$= \frac{\pi}{4}$$

15. $\int_0^{+\infty} \ln x \, dx =$ ln x não está definida em $x = 0$

$$= \lim_{t \to 0^+} \int_t^1 \ln x \, dx + \lim_{b \to +\infty} \int_1^b \ln x \, dx$$

Mas $\int \ln x \, dx = x \ln x - \int dx$

$u = \ln x \to du = \frac{1}{x} dx$

$dv = dx \to v = x$

$= x \ln x - x$

$= x(\ln x - 1)$

$$\therefore \lim_{t \to 0^+} \int_t^1 \ln x \, dx = \lim_{t \to 0^+} x(\ln x - 1) \Big]_t^1$$

$$= \lim_{t \to 0^+} \left[1 \cdot (\cancel{\ln 1}^{\,0} - 1) - t(\ln t - 1) \right]$$

$$= \lim_{t \to 0^+} \left(-1 - t \ln t + t \right)$$

Mas $\lim_{t \to 0^+} -t \ln t = -\lim_{t \to 0^+} \frac{\ln t}{\frac{1}{t}} \overset{\text{L'Hopital}}{=} -\lim_{t \to 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}}$

$= -\lim_{t \to 0^+} -t$

$= 0$

$\therefore$

$= -1$

$\therefore$ $\lim_{t \to 0^+} \int_t^1 \ln x \, dx = -1$

$$\lim_{b\to+\infty}\int_1^b \ln x\,dx =$$

$$= \lim_{b\to+\infty} x(\ln x-1)\Big]_1^b$$

$$= \lim_{b\to+\infty}\left[b(\ln b-1) - (\ln 1-1)\right]$$

$$= \lim_{b\to\infty}\left[b(\ln b-1) - (-1)\right]$$

Mas,

$$\lim_{b\to\infty} b(\ln b-1) = \lim_{b\to\infty} \infty\cdot\infty = \infty$$

$$\therefore \lim_{b\to+\infty}\int_1^b \ln x\,dx = \infty$$

$$\therefore \boxed{\int_0^{+\infty} \ln x\,dx = +\infty \quad \text{divergente}}$$