Cálculo B - Prova 2

Nome:

1. Mostre que o círculo de raio R tem comprimento $2\pi R$ 1.5

2. Calcular a área da região limitada pelas curvas $y = x^3$, $x + y = 0$, $y = x + 6$. 1.5

3. Encontre a área da superfície de um cone circular de raio r e altura h. 1.5

4. Calcule o volume do sólido obtido pela rotação da região limitada por $y = x$ e $y = x^2$ em torno da reta $y = 1$. 2.0

5. Seja D_1 a região no interior da curva $r = \cos\theta$. Seja D_2 a região no interior da curva $r = \sin\theta$. Determine a área de $D_1 \cap D_2$. 2.0

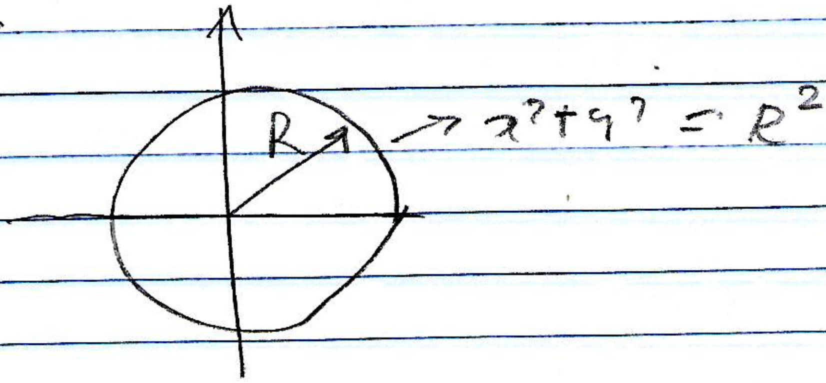
6. Verifique se existe ou não o limite 1.5

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + 2y^2}{x^2 + y^2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

1.



Consideremos a parte superior do círculo, denotado por $\gamma(t)$.

Temos a seguinte parametrização

$$\gamma(t):\begin{cases} x = x & (x \text{ é o parâmetro}) \\ y(x) = \sqrt{R^2-x^2} \; ; & -R \leq x \leq R \end{cases}$$

0.5

Daí

$$L_{\gamma(t)} := \int_{-R}^{R} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad \frac{dy}{dx} = \frac{1(-2x)}{2\sqrt{R^2-x^2}} = \frac{-x}{\sqrt{R^2-x^2}}$$

$$= \int_{-R}^{R} \sqrt{1+\left(\frac{-x}{\sqrt{R^2-x^2}}\right)^2}\, dx$$

$$= \int_{-R}^{R} \sqrt{1+\frac{x^2}{R^2-x^2}}\, dx$$

$$= \int_{-R}^{R} \sqrt{\frac{R^2-\cancel{x^2}+\cancel{x^2}}{R^2-x^2}}\, dx$$

$$= \int_{-R}^{R} \frac{R}{\sqrt{R^2-x^2}}\, dx$$

Mas

$$\int \frac{R}{\sqrt{R^2-x^2}}\, dx =$$

$(x = R\sin\theta \;,\; dx = R\cos\theta\, d\theta)$

$$= \int \frac{R \; R\cos\theta\, d\theta}{\sqrt{R^2-R^2\sin^2\theta}}$$

$$= \int \frac{R^{\cancel{2}} \cancel{\cos\theta}}{\cancel{R\cos\theta}}\, d\theta = \int R\, d\theta = R\theta$$

$$= R \arcsin \frac{x}{R}$$

Daí

$$L\gamma^{(+)} = \int_{-R}^{R} \frac{R}{\sqrt{R^2-x^2}}\, dx = R\arcsin\frac{x}{R}\Big]_{x=-R}^{R}$$

$$= R\arcsin 1 - R\arcsin(-1)$$

$$= R\frac{\pi}{2} - R\left(-\frac{\pi}{2}\right)$$

$$\therefore \quad L\gamma^{(+)} = \pi R$$ (comprimento do semi-círculo superior)

Daí,

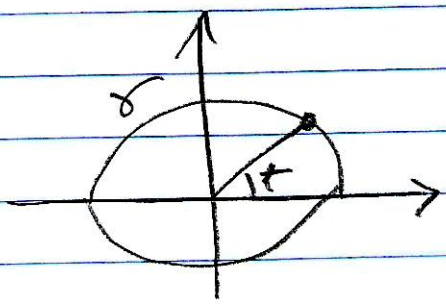
$$L\gamma = 2L\gamma^{(+)} = 2\pi R$$

↳ comprimento do círculo de raio R

(1.5)

Outra solução

Parametrizemos o círculo de raio R na forma



$$\begin{cases} x = R\cos t \\ y = R\sin t \end{cases}$$

Então, temos para o comprimento do círculo

$$L_\gamma = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

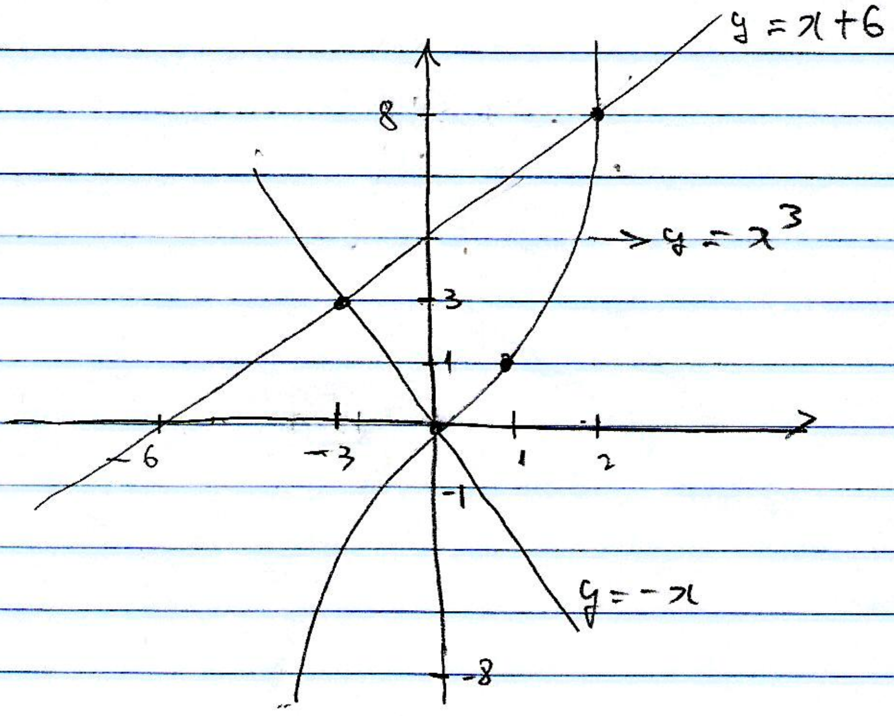
$$= \int_0^{2\pi} \sqrt{(R(-\sin t))^2 + (R\cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{\underbrace{R^2\sin^2 t + R^2\cos^2 t}}\, dt$$

$$= \int_0^{2\pi} R\, dt = Rt\Big]_{t=0}^{2\pi}$$

$$= 2\pi R$$

2.



$$\left.\begin{array}{l} y = x+6 \\ y = -x \end{array}\right\} \quad \begin{array}{l} x+6 = -x \\ \therefore\ 2x = -6 \\ x = -3 \end{array}$$

$$\left.\begin{array}{l} y = x+6 \\ y = x^3 \end{array}\right\} \quad \begin{array}{l} x^3 = x+6 \\ \therefore\ x = 2 \end{array}$$

$$A = \int_{-3}^{0} (x+6-(-x))\,dx + \int_{0}^{2} (x+6-x^3)\,dx$$

0.5

$$= \int_{-3}^{0} (2x+6)\,dx + \int_{0}^{2} (-x^3+x+6)\,dx$$

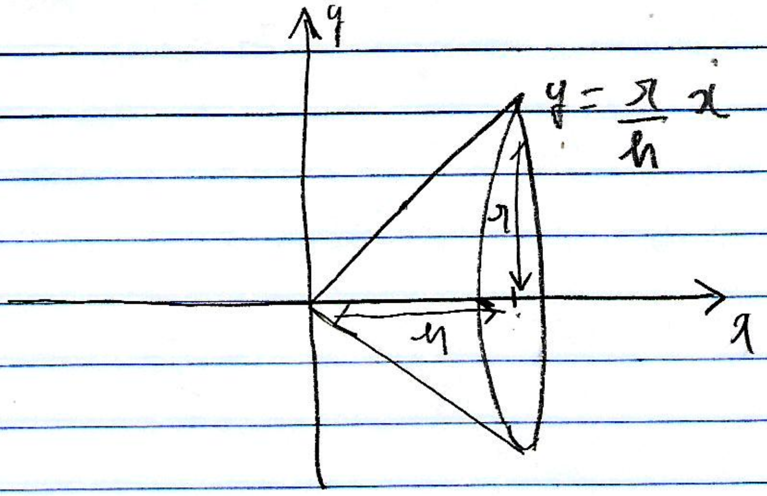
$$= \frac{\cancel{2}x^2}{\cancel{2}} + 6x \Big]_{x=-3}^{0} + \left(-\frac{x^4}{4} + \frac{x^2}{2} + 6x\right)\Big|_{x=0}^{2}$$

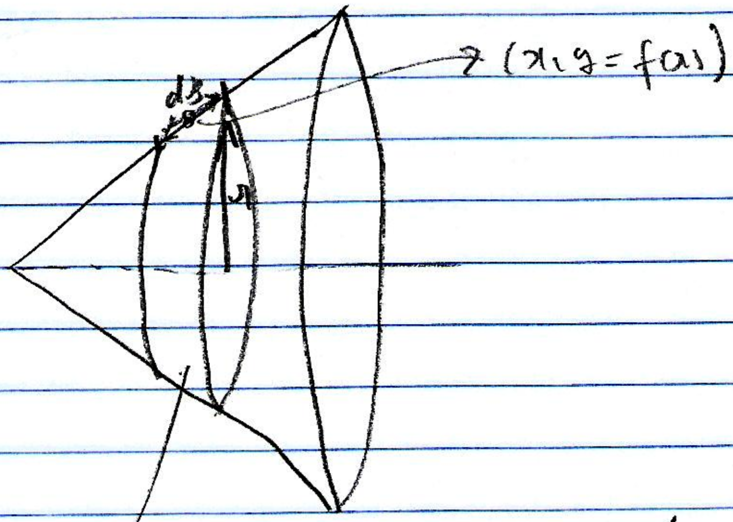
$$= -(-3)^2 - 6(-3) + -\frac{1}{4}16 + \frac{4}{2} + 12$$

$$= -9 + 18 - 4 + 2 + 12 = 19 //$$

1.0

3.





$$dS = 2\pi r \, ds$$
$$= 2\pi y(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

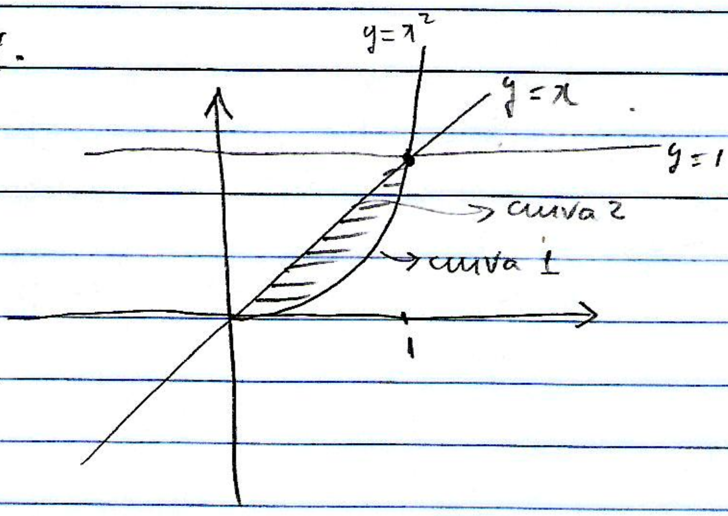
$$S = \int_0^h 2\pi y(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$
$$= \int_0^h 2\pi \frac{r}{h} x \sqrt{1 + \frac{r^2}{h^2}} \, dx$$
$$= \int_0^h 2\pi \frac{r}{h} \frac{\sqrt{r^2 + h^2}}{h} x \, dx =$$

$$= \frac{\cancel{2}\pi r}{h^2} \sqrt{r^2+h^2} \left. \frac{x^2}{\cancel{2}} \right]_{x=0}^{h}$$

$$= \frac{\pi r}{\cancel{h^2}} \sqrt{r^2+h^2}\, \cancel{h^2}$$

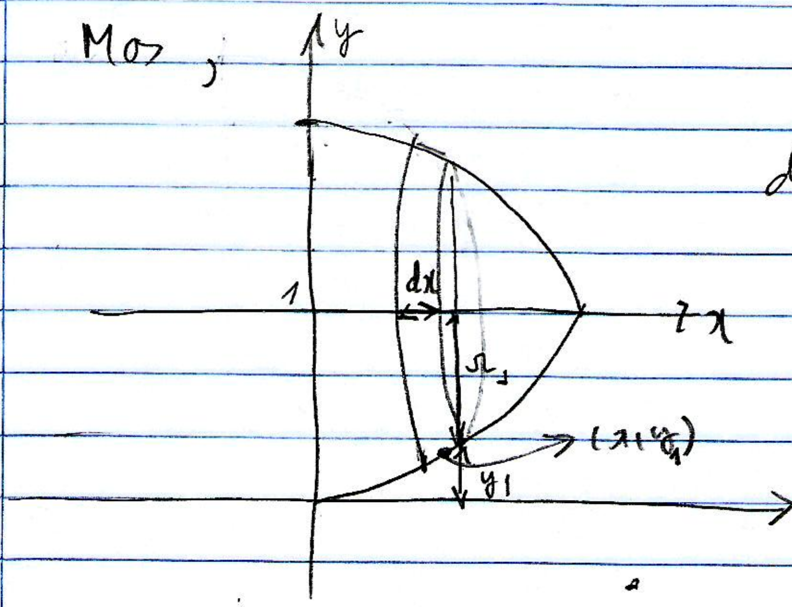
$$= \pi r \sqrt{r^2+h^2}$$

4.



$$dV = dV_1 - dV_2$$

Mas,



$$dV_1 = \pi x_1^2 \, dx$$

$$= \pi \left(1 - y_1(x)\right)^2 dx$$

com $y_1(x) = x^2$

0,25

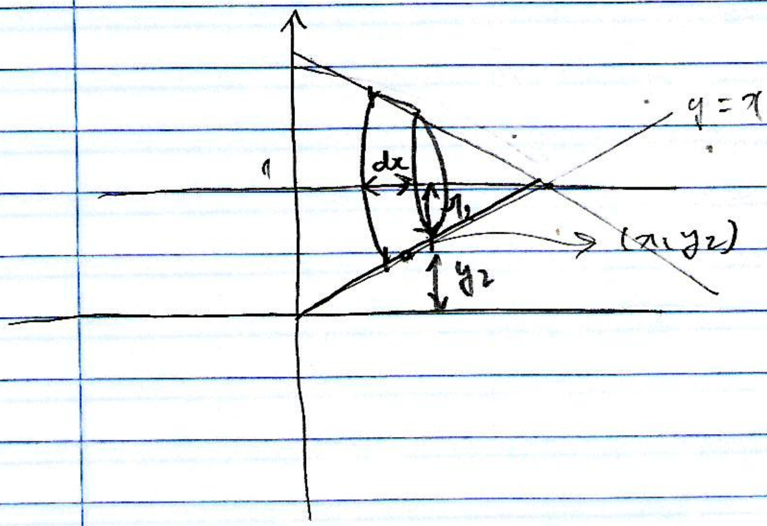
$$V_1 = \int_0^1 \pi (1 - x^2)^2 \, dx$$

$$= \int_0^1 \pi (1 - 2x^2 + x^4) \, dx$$

$$= \pi x - \frac{2\pi x^3}{3} + \frac{\pi x^5}{5} \Big]_{x=0}^{1}$$

$$= \pi - \frac{2\pi}{3} + \frac{\pi}{5} =$$

0,25 $$= \frac{15\pi - 10\pi + 3\pi}{15} = \frac{8\pi}{15} \,//$$



$$dV_2 = \pi r_2^2 \, dx$$

$$= \pi \left(1 - y_2(x)\right)^2 dx$$

com $y_2(x) = x$ 0.25

$$\therefore \quad dV_2 = \pi (1-x)^2 \, dx$$

$$\therefore \quad V_2 = \int_0^1 \pi (1-x)^2 \, dx$$

$$= \frac{\pi (1-x)^3}{3}(-1) \Big]_{x=0}^{1}$$

$$= -\frac{\pi}{3}(1-x)^3 \Big]_{x=0}^{1}$$

$$= \frac{\pi}{3}$$ 0.75

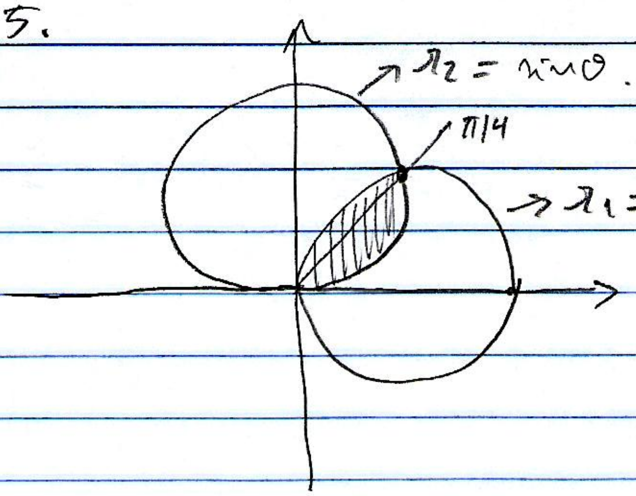
Daí

$$V = \underbrace{\int_0^1 dV_1} - \underbrace{\int_0^1 dV_2}$$

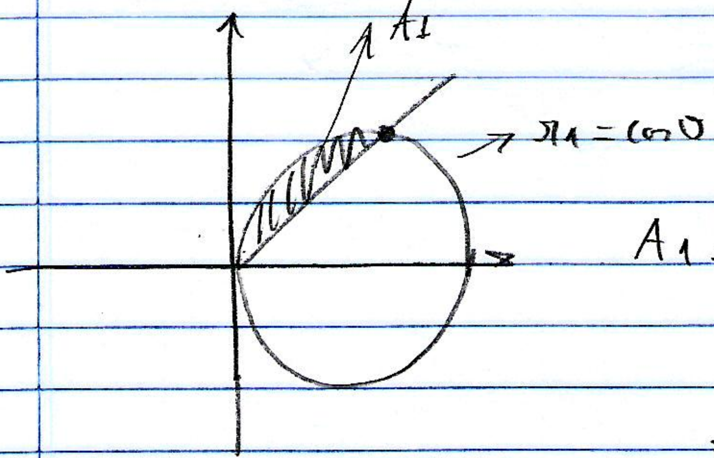
$$= \frac{8\pi}{15} - \frac{\pi}{3} = \frac{8\pi - 5\pi}{15}$$

$$= \frac{3\pi}{15} = \frac{\pi}{5} \; //$$

5.



$$\begin{cases} r_1 = r_2 \\ \cos\theta = \sin\theta \end{cases} \Rightarrow \theta = \frac{\pi}{4} + k\pi \, , \, k \in \mathbb{Z}.$$



$$A_1 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2} r_1^2(\theta)\, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2} \cos^2\theta \, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2} \frac{(1+\cos 2\theta)}{2} \, d\theta$$

$$= \frac{1}{4} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1+\cos 2\theta)\, d\theta$$

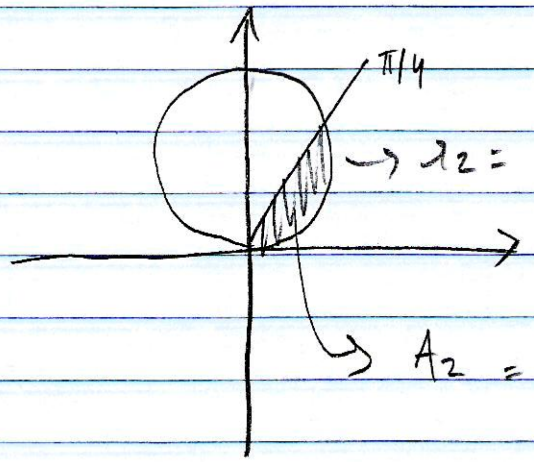
$$= \frac{1}{4}\left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left(\frac{\pi}{2} + \frac{1}{2}\underbrace{\sin\pi}_{0}\right) - \frac{1}{4}\left(\frac{\pi}{4} + \frac{1}{2}\underbrace{\sin\frac{\pi}{2}}_{1}\right)$$

$$= \frac{\pi}{8} - \frac{1}{4}\left(\frac{\pi}{4} + \frac{1}{2}\right)$$

$$= \frac{\pi}{8} - \frac{\pi}{16} - \frac{1}{8}$$

$$// \; A_1 = \frac{\pi}{16} - \frac{1}{8} \; //$$



$$\to A_2 = \int_0^{\frac{\pi}{4}} \frac{1}{2} r_2^2(\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin^2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Bigg|_{\theta=0}^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - 0$$

$$\therefore \; // \; A_2 = \frac{\pi}{16} - \frac{1}{8} \; //$$

1,0

Daí

$$A = A_1 + A_2.$$

$$= \frac{\pi}{16} - \frac{1}{8} + \frac{\pi}{16} - \frac{1}{8}$$

$$= \frac{\pi}{8} - \frac{1}{4}$$