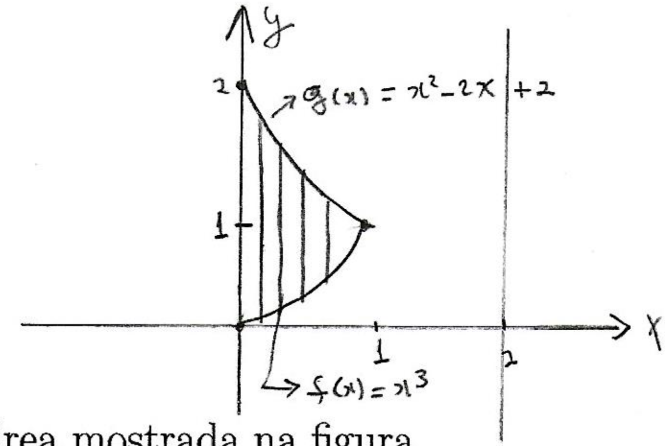
Cálculo B - Prova 2

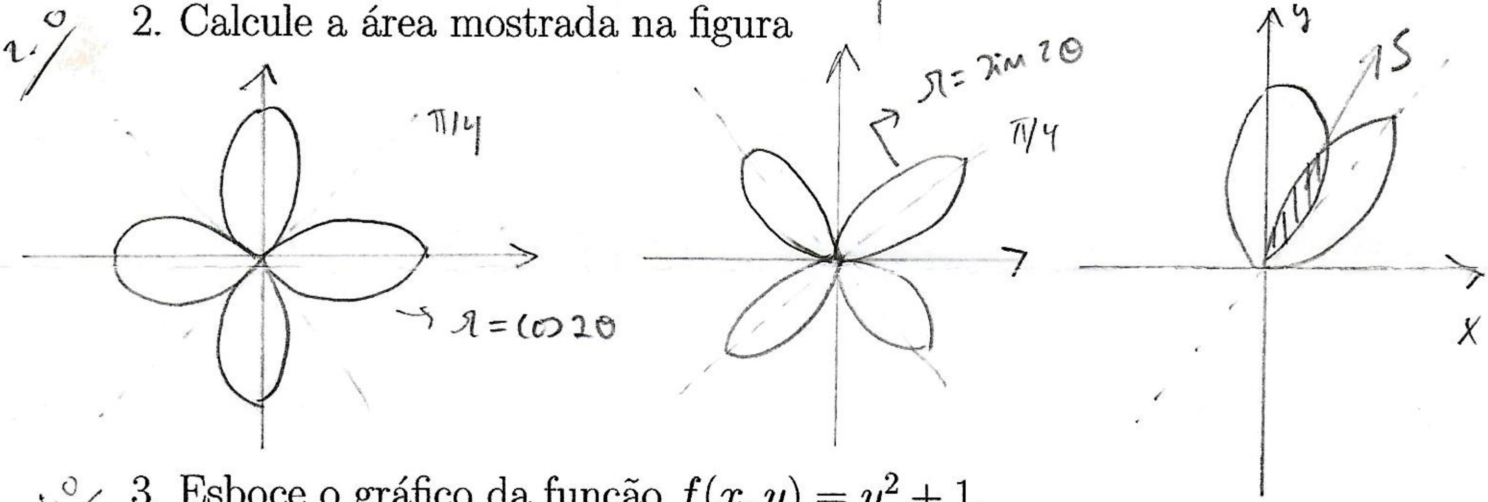
2.0

1. Calcule o volume do sólido de revolução obtido pela rotação da região mostrada em torno da reta $x = 2$



2.0

2. Calcule a área mostrada na figura



1.0

3. Esboce o gráfico da função $f(x, y) = y^2 + 1$.

1.5

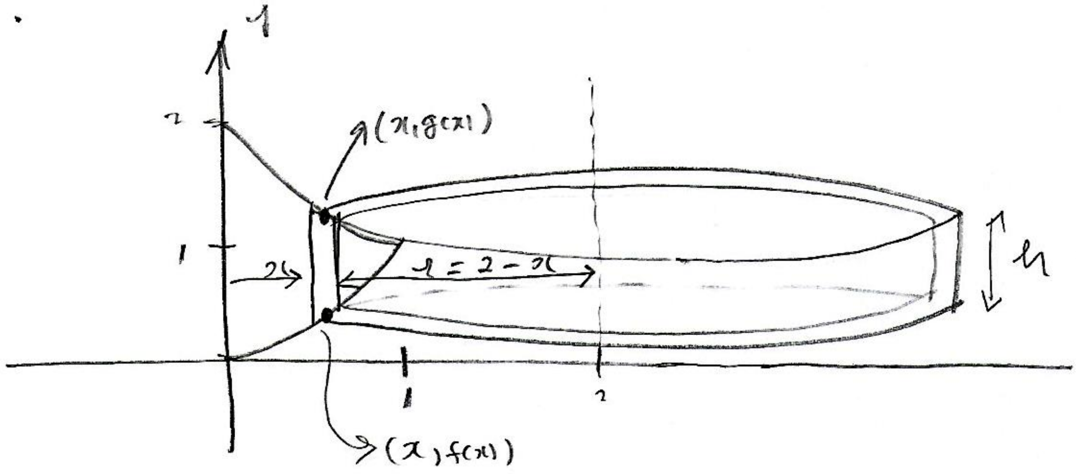
4. Calcule o valor do limite caso ele exista, ou mostre que o limite não existe

$$\lim_{(x,y)\to(1,0)} \frac{xy - y}{(x-1)^2 + y^2}$$

1.5

5. Seja $x^2 - y^2 + z^2 - 2z = 4$ uma relação que determina z como função implícita $z(x, y)$. Use derivação implícita para calcular ~~frac\partial z~~ $\frac{\partial z}{\partial x}$ e $\frac{\partial z}{\partial y}$

1.



$$dV = 2\pi r h \, dx$$

$$= 2\pi (2-x)(g(x) - f(x)) \, dx$$

$$= 2\pi (2-x)(x^2 - 2x + 2 - x^3) \, dx$$ 0.75

1.0

$$V = \int_0^1 2\pi (2-x)(x^2 - 2x + 2 - x^3) \, dx$$

$$= \int_0^1 2\pi (2x^2 - 4x + 4 - 2x^3 - x^3 + 2x^2 - 2x + x^4) \, dx$$

$$= \int_0^1 2\pi (4 - 6x + 4x^2 - 3x^3 + x^4) \, dx$$

$$= 2\pi \left[4x - 6\frac{x^2}{2} + 4\frac{x^3}{3} - 3\frac{x^4}{4} + \frac{x^5}{5} \right]_{x=0}^{1}$$

$$= 2\pi \left[\cancel{4} - \cancel{3} + \frac{4}{3} - \frac{3}{4} + \frac{1}{5} \right]$$

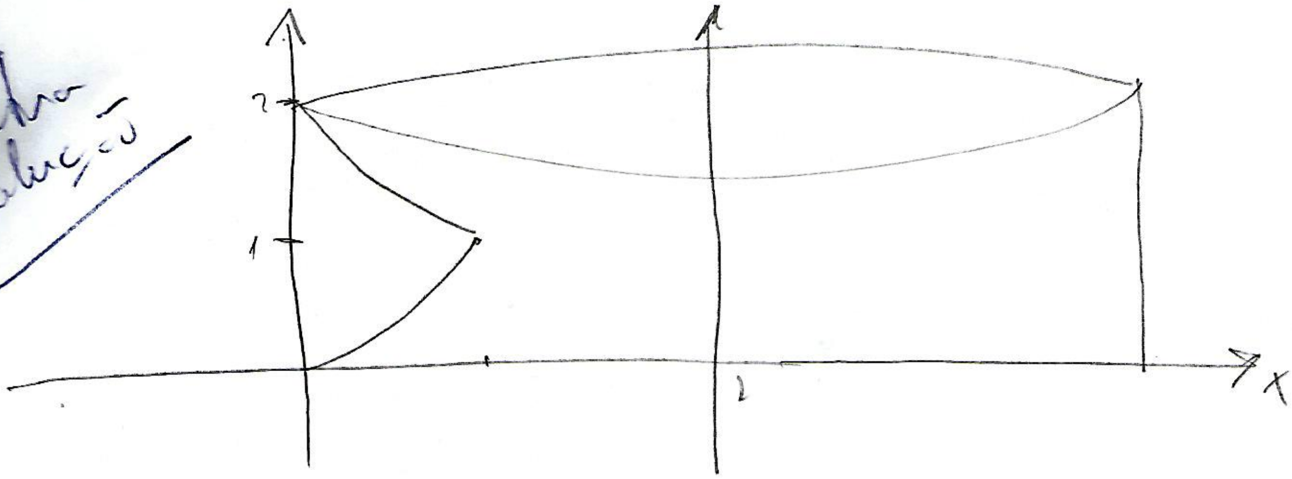
$$= 2\pi \left[1 + \frac{7}{12} + \frac{1}{5} \right] = 2\pi \left(\frac{60 + 35 + 12}{60} \right)$$

$$= 2\pi \left[\frac{107}{60} \right] = \frac{107\pi}{30}$$

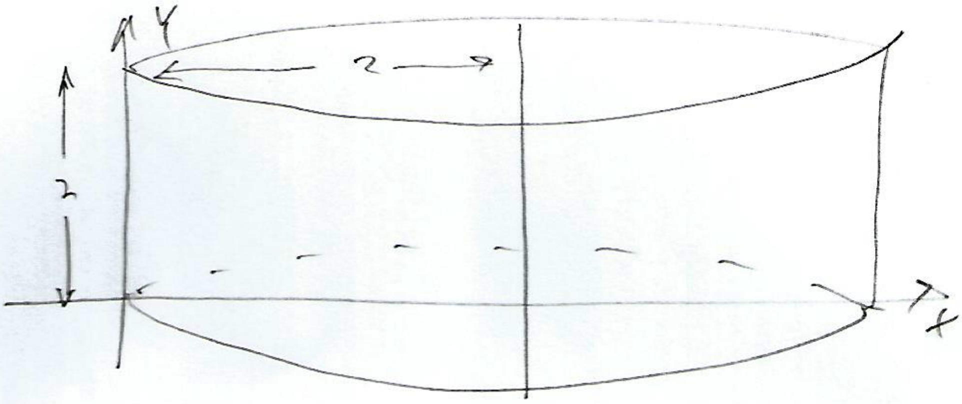
2.0

1.

Outra solução

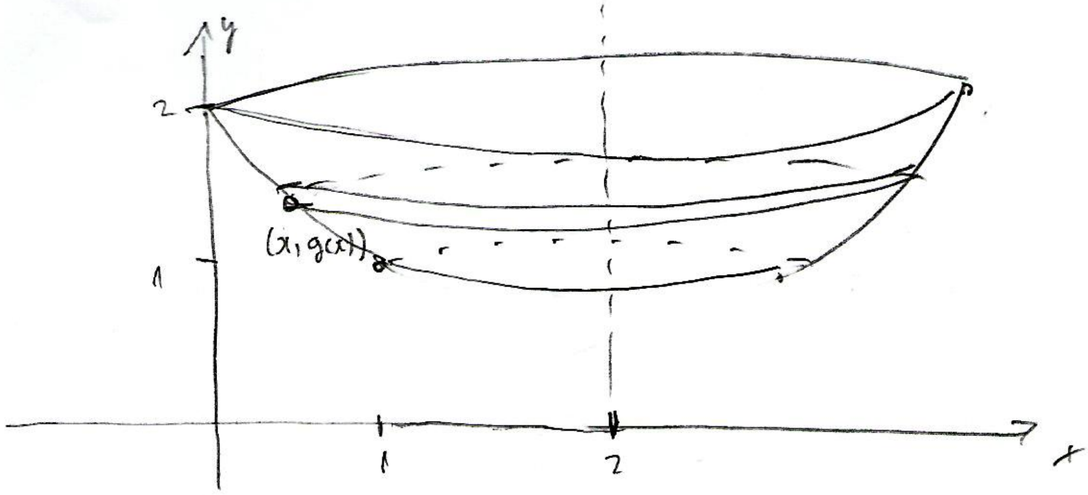


→ $\underline{\underline{V_1}}$:



$$V_1 = \pi \cdot 2^2 \cdot 2 = 8\pi$$

→ $\underline{\underline{V_2}}$



$$dV_2 = \pi r^2 dy = \pi\,(2 - x(y))^2\, dy$$

Mas $y = x^2 - 2x + 2$

$\therefore \quad x^2 - 2x + 2 - y = 0$

$$x = \frac{2 \pm \sqrt{4 - 4(2-y)}}{2} = \frac{2 \pm \sqrt{4 - 8 + 4y}}{2} = \frac{2 \pm \sqrt{4y - 4}}{2}$$

$$x = \frac{2 \pm 2\sqrt{y-1}}{2} = 1 \pm \sqrt{y-1}$$

Mas $1 \leq y \leq 2$ e $0 \leq x \leq 1$

Assim, para que tenhamos $0 \leq x \leq 1$ devemos tomar

$$x = 1 - \sqrt{y-1}$$

Daí,

$$dV_2 = \pi\left(2 - (1-\sqrt{y-1})\right)^2 dy$$

$$= \pi\left(1+\sqrt{y-1}\right)^2 dy$$

$$V_2 = \int_1^2 \pi\left(1+\sqrt{y-1}\right)^2 dy$$

$$= \int_1^2 \pi\left(\cancel{1} + 2\sqrt{y-1} + y - \cancel{1}\right) dy$$

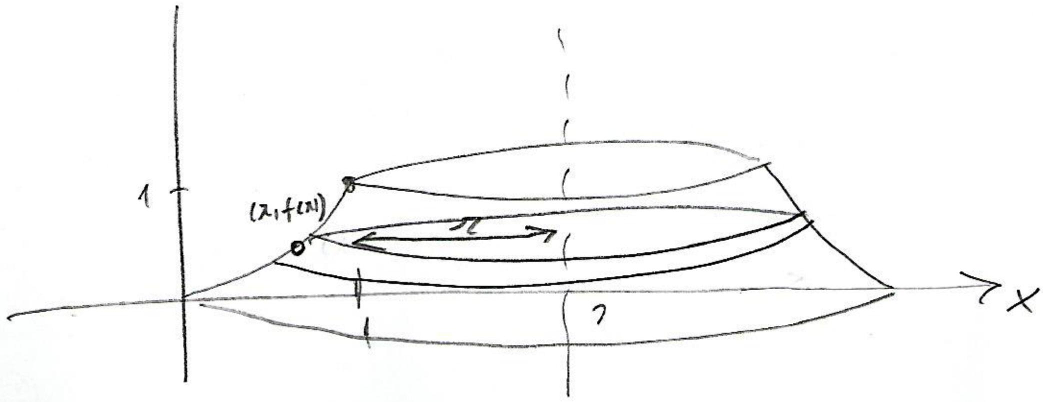
$$= \int_1^2 \pi\left(2\sqrt{y-1} + y\right) dy$$

$$= \left(\pi\, 2\, \frac{2}{3}(y-1)^{3/2} + \pi\frac{y^2}{2}\right)\Big|_{y=1}^{2}$$

$$= \frac{4\pi}{3}(y-1)^{3/2} + \frac{\pi}{2}y^2 \Big|_{y=1}^{2}$$

$$= \frac{4\pi}{3} + 2\pi - \frac{\pi}{2} = \frac{5\pi}{6} + 2\pi = \frac{17\pi}{6}$$

$\rightarrow \underline{V_3}$:



$$dV_3 = \pi r^2 \, dy$$
$$= \pi (2 - x(y))^2 \, dy$$

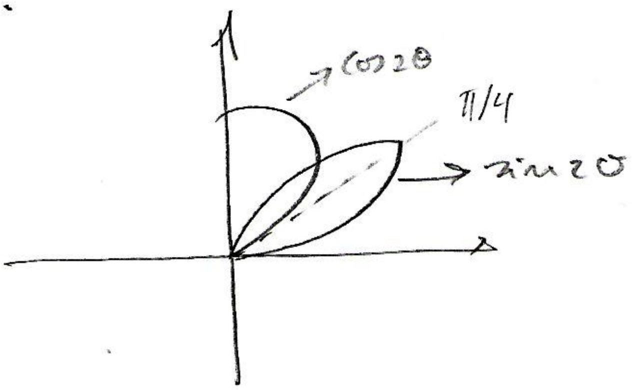
$$\begin{array}{r} 85 \\ 48 \\ \hline 133 \end{array} \qquad \begin{array}{r} 240 \\ 133 \\ \hline 107 \end{array}$$

$y = x^3 \rightarrow x = \sqrt[3]{y}$

$$// \; V_3 = \int_0^1 \pi (2 - \sqrt[3]{y})^2 \, dy$$
$$= \int_0^1 \pi \left(4 - 4\sqrt[3]{y} + y^{2/3}\right) dy$$
$$= \pi \left(4y - \cancel{4}\tfrac{3}{\cancel{4}} y^{4/3} + \tfrac{3}{5} y^{5/3}\right) \Big|_{y=0}^{1}$$
$$= \pi \left(4 - 3 + \tfrac{3}{5}\right) = \pi \left(1 + \tfrac{3}{5}\right)$$
$$= \frac{8\pi}{5} \; //$$

Dai :
$$V = V_1 - V_2 - V_3$$
$$= 8\pi - \frac{17\pi}{6} - \frac{8\pi}{5} = \frac{240\pi - 85\pi - 48\pi}{30}$$
$$= \frac{107\pi}{30}$$

2.

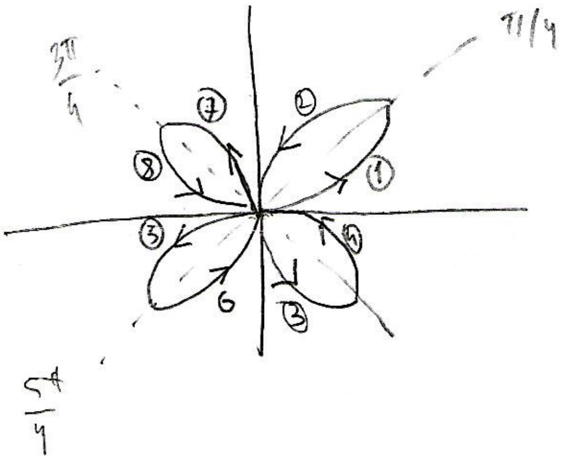


$$\left.\begin{aligned} r_1 &= \sin 2\theta \\ r_2 &= \cos 2\theta \end{aligned}\right\} \Rightarrow$$

$\sin 2\theta = \cos 2\theta$

$2\theta = \frac{\pi}{4} + n\pi \quad ; n \in \mathbb{Z}$

$\theta = \frac{\pi}{8} + \frac{n\pi}{2} \quad ; n \in \mathbb{Z}$



$r_1 = \sin 2\theta$

① : $0 \leq \theta \leq \frac{\pi}{4}$: $0 \leq \sin 2\theta \leq 1$

② : $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$: $1 \geq \sin 2\theta \geq 0$

③ : $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$: $0 \geq \sin 2\theta \geq -1$

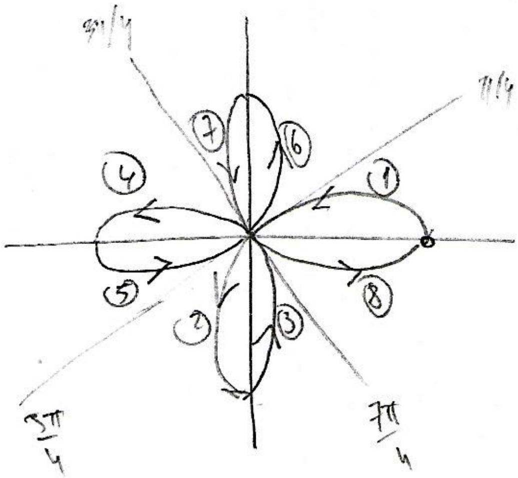
④ : $\frac{3\pi}{4} \leq \theta \leq \pi$: $-1 \leq \sin 2\theta \leq 0$

⑤ : $\pi \leq \theta \leq \frac{5\pi}{4}$: $0 \leq \sin 2\theta \leq 1$

⑥ : $\frac{5\pi}{4} \leq \theta \leq \underbrace{\frac{3\pi}{2}}_{\frac{6\pi}{4}}$: $1 \geq \sin 2\theta \geq 0$

⑦ : $\frac{3\pi}{2} \leq \theta \leq \frac{7\pi}{4}$: $0 \geq \sin 2\theta \geq -1$

⑧ : $\frac{7\pi}{4} \leq \theta \leq 2\pi$: $-1 \leq \sin 2\theta \leq 0$



$r_2 = \cos 2\theta$

① $0 \le \theta \le \frac{\pi}{4}$: $1 \geq \cos 2\theta \geq 0$

② : $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$; $0 \geq \cos 2\theta \geq -1$

③ : $\frac{\pi}{2} \le \theta \le \frac{3\pi}{4}$; $-1 \le \cos 2\theta \le 0$

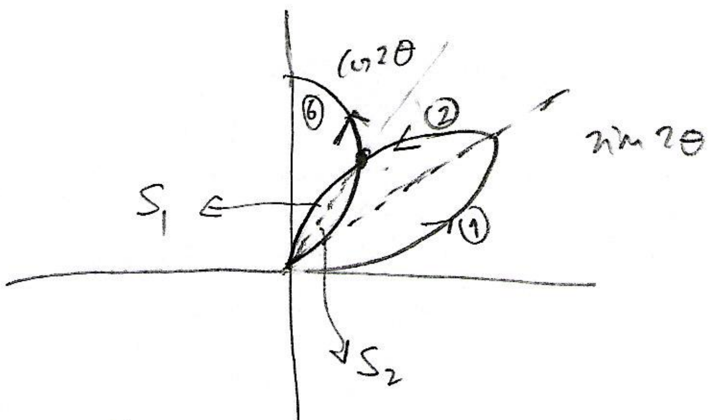
④ : $\frac{3\pi}{4} \le \theta \le \pi$; $0 \le \cos 2\theta \le 1$

⑤ : $\pi \le \theta \le \frac{5\pi}{4}$; $1 \geq \cos 2\theta \geq 0$

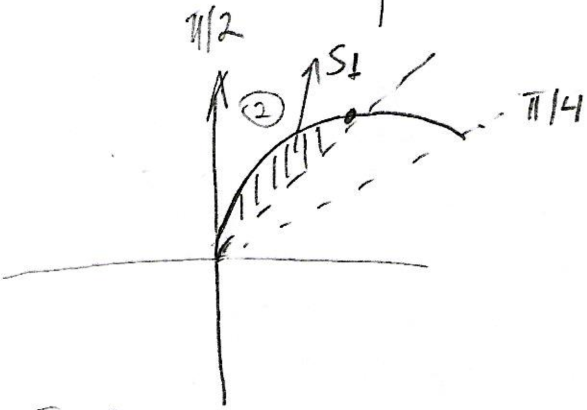
⑥ : $\frac{5\pi}{4} \le \theta \le \frac{3\pi}{2}$; $0 \geq \cos 2\theta \geq -1$

⑦ : $\frac{3\pi}{2} \le \theta \le \frac{7\pi}{4}$; $-1 \le \cos 2\theta \le 0$

⑧ : $\frac{7\pi}{4} \le \theta \le 2\pi$; $0 \le \cos 2\theta \le 1$



$\checkmark$ 0.75



$\frac{\pi}{4} + \frac{\pi}{8} = \frac{3\pi}{8}$

$$S_1 = \frac{1}{2}\int_{\frac{3\pi}{8}}^{\pi/2} \sin^2 2\theta \, d\theta$$

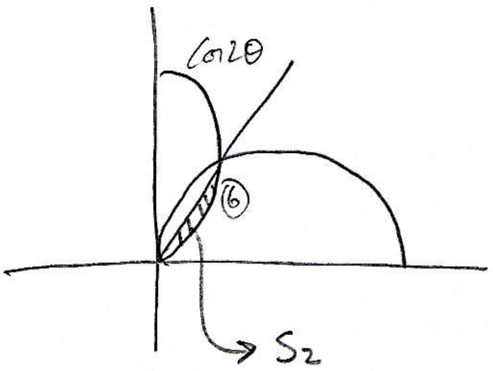
$$= \frac{1}{2}\int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{2} \, d\theta$$

$$= \frac{1}{4}\left(\theta - \frac{1}{4}\sin 4\theta\right)\Bigg|_{\theta=\frac{3\pi}{8}}^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left(\frac{\pi}{2} - \frac{1}{4}\cancel{\sin 2\pi}^{\,0}\right) - \frac{1}{4}\left(\frac{3\pi}{8} - \frac{1}{4}\underbrace{\sin\frac{3\pi}{2}}_{-1}\right)$$

$$= \underbrace{\frac{\pi}{8} - \frac{3\pi}{32}} - \frac{1}{16}$$

$$// \; S_1 = \frac{\pi}{32} - \frac{1}{16} \; //$$ ✓ 0,25



$$\frac{5\pi}{4} + \frac{\pi}{8} = \frac{11\pi}{8}$$

$$\cos 2\,\frac{11\pi}{8} = \cos\frac{11\pi}{4}$$

$$\sin\frac{11\pi}{2} =$$

$$= \sin\left(\frac{12\pi}{2} - \frac{\pi}{2}\right)$$

$$= \sin\left(6\pi - \frac{\pi}{2}\right)$$

$$= -\sin\frac{\pi}{2}\cos 6\pi$$

$$= -1 \cdot 1$$

$$= -1$$

$$S_2 = \frac{1}{2}\int_{\frac{5\pi}{4}}^{\frac{11\pi}{8}} \cos^2 2\theta \, d\theta$$ ✓ 0,75

$$= \frac{1}{2}\int_{\frac{5\pi}{4}}^{\frac{11\pi}{8}} \frac{1+\cos 4\theta}{2}\, d\theta$$

$$= \frac{1}{4}\int_{\frac{5\pi}{4}}^{\frac{11\pi}{8}} (1+\cos 4\theta)\, d\theta$$

$$= \frac{1}{4}\left(\theta + \frac{1}{4}\sin 4\theta\right)\Bigg|_{\theta=\frac{5\pi}{4}}^{\frac{11\pi}{8}}$$

$$= \frac{1}{4}\left(\frac{11\pi}{8} + \frac{1}{4}\sin\frac{11\pi}{2}\right) -$$

$$- \frac{1}{4}\left(\frac{5\pi}{4} + \frac{1}{4}\cancel{\sin 5\pi}^{\,0}\right)$$

$$= \underline{\frac{11\pi}{32}} - \frac{1}{16} - \underline{\frac{5\pi}{16}}$$

$$// \; S_2 = \frac{\pi}{32} - \frac{1}{16} \; //$$ ✓ 0,25

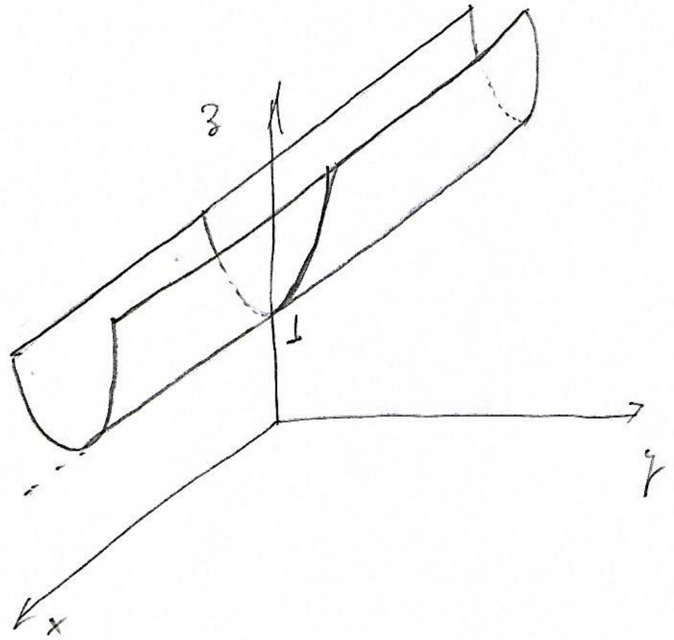
Daí

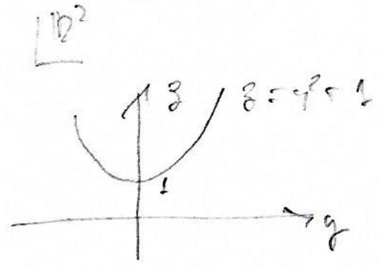
$$S = S_1 + S_2$$

$$= \frac{\pi}{32} - \frac{1}{16} + \frac{\pi}{32} - \frac{1}{16}$$

$$\boxed{S = \frac{\pi}{16} - \frac{1}{8}}$$

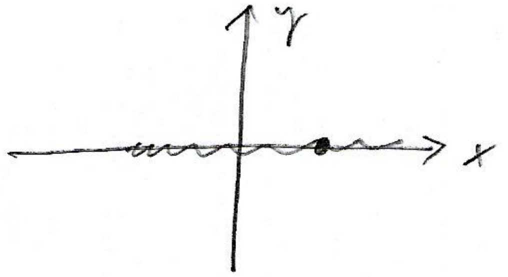
3. $f(x,y) = y^2 + 1$





4. $\displaystyle\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2} \begin{matrix}\to 0\\ \to 0\end{matrix}$

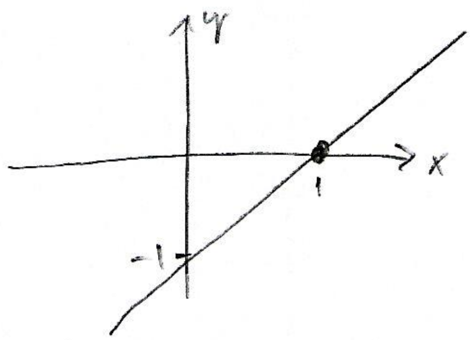
→ Seja o caminho $y=0$



Temos $\displaystyle\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2} = \lim_{\substack{x\to 0\\(y=0)}} \frac{0}{(x-1)^2} =$

$\displaystyle = \lim_{x\to 0} 0 = 0$ // (*)

→ Seja o caminho $y = x-1$



Temos $\displaystyle\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2} = \lim_{\substack{x\to 1\\(y=x-1)}} \frac{x(x-1)-(x-1)}{(x-1)^2+(x-1)^2}$

$\displaystyle = \lim_{\substack{x\to 1\\(y=x-1)}} \frac{(x-1)(x-1)}{2(x-1)^2} = \lim_{x\to 1} \frac{1}{2} = \frac{1}{2}$ // (**)

De (*) e (**) vemos que os limites são distintos, daí

$$\lim_{(x,y)\to(1,0)} \frac{xy - y}{(x-1)^2 + y^2} \nexists$$

5. $$x^2 - y^2 + z^2 - 2z = 4$$

$$z = z(x, y)$$

$$\rightarrow \frac{\partial}{\partial x}\left(x^2 - y^2 + z^2 - 2z\right) = \frac{\partial}{\partial x} 4$$

$$2x + 2z\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial x} = 0$$

$$2x + (2z - 2)\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-2x}{2z - 2}$$

$$\frac{\partial z}{\partial x} = \frac{-x}{z - 1} = \frac{x}{1 - z} \quad //$$

0.75

$$\rightarrow \frac{\partial}{\partial y}\left(x^2 - y^2 + z^2 - 2z\right) = \frac{\partial}{\partial y} 4$$

$$-2y + 2z\frac{\partial z}{\partial y} - 2\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y}(2z - 2) = 2y$$

$$\frac{\partial z}{\partial y} = \frac{2y}{2z - 2} = \frac{y}{z - 1} \quad //$$

0.25