Cálculo B - Prova 1

1. $\int \frac{1}{(x^2-4x+4)(x^2-4x+5)} dx$ [2.5]
2. $\int \frac{\sin^2 x}{\cos^6 x}\, dx$ [1.0]
3. $\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\, dx$ [2.0]
4. $\int \cos^4 x\, dx$ [1.5]
5. $\int \frac{1}{\sin^2 x \sqrt[4]{\cot x}}\, dx$ [1.0]
6. $\int \frac{1}{\sin x + \tan x}\, dx$ [2.0]

Tabela de Integrais

$\int \sin x\, dx = -\cos x$

$\int \cos x\, dx = \sin x$

$\int \tan x\, dx = \ln|\sec x|$

$\int \cot x\, dx = -\ln|\csc x|$

$\int \sec x\, dx = \ln|\sec x + \tan x|$

$\int \csc x\, dx = \ln|\csc x - \cot x|$

$\int \sec^2 x\, dx = \tan x$

$\int \csc^2 x\, dx = -\cot x$

$\int \sec x\, \tan x\, dx = \sec x$

$\int \csc x\, \cot x\, dx = -\csc x$

Relações Trigonométricas

$\sin^2 = 1 - \cos^2 x$

$\cos^2 x = 1 - \sin^2 x$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\sin 2x = 2\sin x\, \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

$\sin mx \sin nx = \frac{1}{2}\cos(m-n)x - \frac{1}{2}\cos(m+n)x$

$\cos mx \cos nx = \frac{1}{2}\cos(m+n)x + \frac{1}{2}\cos(m-n)x$

$\sin mx \cos nx = \frac{1}{2}\sin(m-n)x + \frac{1}{2}\sin(m+n)x$

Substituição útil: $z = \tan\frac{x}{2}$, $\sin x = \frac{2z}{1+z^2}$, $\cos x = \frac{1-z^2}{1+z^2}$

1. $$\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)}$$

$$\begin{cases} x^2-4x+4 : & \Delta = 16-16 = 0 \\ x^2-4x+5 : & \Delta = 16-20 < 0 \end{cases}$$

$$\frac{1}{(x^2-4x+4)(x^2-4x+5)} = \frac{1}{(x-2)^2(x^2-4x+5)}$$

$$= \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C(2x-4)+D}{x^2-4x+5}$$

(0.5)

$\therefore$

$$1 = A(x-2)(x^2-4x+5) + B(x^2-4x+5) +$$
$$+ [C(2x-4)+D](x-2)^2$$

$\therefore$

$$1 = A(x^3-2x^2-4x^2+8x+5x-10) +$$
$$+ B(x^2-4x+5) + [C(2x-4)+D](x^2-4x+4)$$

$$= A(x^3-6x^2+13x-10) +$$
$$+ B(x^2-4x+5) +$$
$$+ 2C(x^3-4x^2+4x)$$
$$-4C(x^2-4x+4)$$
$$+ D(x^2-4x+4)$$

$$= x^3(A+2C) + x^2(-6A+B-8C-4C+D)$$
$$+ x(13A-4B+8C+16C-4D)$$
$$+ (-10A+5B-16C+4D)$$

$$1 = x^3(A+2C) + x^2(-6A+B-12C+D)$$
$$+x(13A-4B+24C-4D)$$
$$+(-10A+5B-16C+4D)$$

$$\therefore \begin{cases} A+2C=0 \quad \therefore \ \| A=-2C \| \\ -6A+B-12C+D=0 \quad (*) \\ 13A-4B+24C-4D=0 \quad (**) \\ -10A+5B-16C+4D=1 \quad (***) \end{cases}$$

$\therefore$

$(*)$: $$-6(-2C)+B-12C+D=0$$
$$\cancel{12C}+B-\cancel{12C}+D=0$$
$$\| B=-D \|$$

$(**)$: $$13(-2C)-4(-D)+24C-4D=0$$
$$-26C+\cancel{4D}+24C-\cancel{4D}=0$$
$$-2C=0$$
$$\therefore \ \| C=0 \| \quad \therefore \ \| A=0 \|$$

$(***)$: $$5(-D)+4D=1$$
$$-D=1 \quad \therefore \ \boxed{D=-1}$$
$$\boxed{B=1}$$

$$\frac{1}{(x^2-4x+4)(x^2-4x+5)} = \frac{1}{(x-2)^2} - \frac{1}{x^2-4x+5}$$

(1.0)

$$\int \frac{1}{(x^2-4x+4)(x^2-4x+5)}\,dx =$$

$$= \int \frac{1}{(x-2)^2}\,dx - \int \frac{1}{x^2-4x+5}\,dx$$

(*) (**)

Mas

$$(*) = \int \frac{1}{(x-2)^2}\,dx = -\frac{1}{(x-2)}$$

(1.5)

$$(**) = -\int \frac{1}{x^2-4x+5}\,dx$$

$$= -\int \frac{1}{(x-2)^2+1}\,dx$$

$$(x-2 = u \rightarrow dx = du)$$

$$= -\int \frac{1}{u^2+1}\,du$$

$$u = tg\,\theta \rightarrow du = \sec^2\theta\,d\theta$$

$$= -\int \frac{\cancel{\sec^2}\theta\,d\theta}{\cancel{\sec^2}\theta} = -\theta = -\operatorname{arctg} u$$ (2.0)

$$= -\operatorname{arctg}(x-2)$$

∴

$$\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)} =$$

$$= -\frac{1}{(x-2)} - \text{arc tg}(x-2)$$

Obs.: Caso tenha cometido um erro no início na decomposição de fração parcial dar no máximo 1 ponto pela resolução de uma integral por substituição trigonométrica

Outra Solução (Questão 1)

$$\int \frac{dx}{(x^2-4x+4)\,\underline{(x^2-4x+5)}} =$$

$$= \int \frac{dx}{(x-2)^2\left((x-2)^2+1\right)}$$

$$(u = x-2 \,,\quad du = dx)$$

$$= \int \frac{du}{u^2(u^2+1)}$$

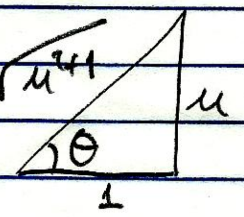
$$(u = tg\,\theta \,,\quad du = \sec^2\theta\, d\theta)$$

$$= \int \frac{\cancel{\sec^2\theta}\, d\theta}{tg^2\theta\, \cancel{\sec^2\theta}}$$

$$= \int \frac{1}{tg^2\theta}\, d\theta$$

$$= \int cotg^2\theta\, d\theta$$

$$= \int (cosec^2\theta - 1)\, d\theta$$



$$= -cotg\,\theta - \theta$$

$$= -\frac{1}{u} - arctg\, u$$

$$= -\frac{1}{x-2} - arctg(x-2)$$

$$2. \quad \int \frac{\sin^2 x}{\cos^6 x}\, dx =$$

$$= \int \frac{\sin^2 x}{\cos^2 x} \, \frac{1}{\cos^4 x}\, dx$$

$$= \int tg^2 x \, sec^4 x \, dx$$

$$= \int tg^2 x \, sec^2 x \, sec^2 x \, dx$$

$$= \int tg^2 x \, (tg^2 x + 1) \, sec^2 x \, dx$$

$$= \int t^4 x \, sec^2 x \, dx + \int tg^2 x \, sec^2 x \, dx$$

$$= \frac{t^5 x}{5} + \frac{tg^3 x}{3} .$$

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Outra Solução (Questão 2)

$$\int \frac{\sin^2 x}{\cos^6 x}\, dx = \int \frac{\sin^2 x}{\cos^2 x} \frac{1}{\cos^4 x}\, dx =$$

$$= \int tg^2 x \sec^4 x\, dx$$

$$= \int (\sec^2 x - 1) \sec^4 x\, dx$$

$$= \int \sec^6 x\, dx - \int \sec^4 x\, dx$$

Mas

$$\int \sec^n x\, dx = \int \sec^{n-2} x \sec^2 x\, dx$$

$$\left(\begin{array}{l} u = \sec^{n-2} x \rightarrow du = (n-2) \sec^{n-3} x \sec x\, tg\, x\, dx \\ dv = \sec^2 x\, dx \rightarrow v = tg\, x \end{array} \right)$$

$$= \sec^{n-2} x\, tg\, x - (n-2) \int \sec^{n-3} x \sec x\, tg\, x\, tg\, x\, dx$$

$$= \sec^{n-2} x\, tg\, x - (n-2) \int \sec^{n-2} x\, tg^2 x\, dx$$

$$= \sec^{n-2} x\, tg\, x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1)\, dx$$

$\therefore$

$$\int \sec^n x\, dx = \sec^{n-2} x\, tg\, x - (n-2) \int \sec^n x\, dx +$$

$$+ (n-2) \int \sec^{n-2} x\, dx$$

$\therefore$

$$\underbrace{\int \sec^n x\, dx + (n-2) \int \sec^n x\, dx} = \sec^{n-2} x\, tg\, x + (n-2) \int \sec^{n-2} x\, dx$$

$$(n-1)\int \sec^n x\,dx = \sec^{n-2}x\,\operatorname{tg}x + (n-2)\int \sec^{n-2}x\,dx$$

$$\int \sec^n x\,dx = \frac{1}{(n-1)}\sec^{n-2}x\,\operatorname{tg}x + \frac{(n-2)}{(n-1)}\int \sec^{n-2}x\,dx$$

Daí:

(*)
$$\int \sec^4 x\,dx = \frac{1}{3}\sec^2 x\,\operatorname{tg}x + \frac{2}{3}\int \sec^2 x\,dx$$
$$= \frac{1}{3}\sec^2 x\,\operatorname{tg}x + \frac{2}{3}\operatorname{tg}x$$

(**)
$$\int \sec^6 x\,dx = \frac{1}{5}\sec^4 x\,\operatorname{tg}x + \frac{4}{5}\int \sec^4 x\,dx$$
$$= \frac{1}{5}\sec^4 x\,\operatorname{tg}x + \frac{4}{5}\left(\frac{1}{3}\sec^2 x\,\operatorname{tg}x + \frac{2}{3}\operatorname{tg}x\right)$$
$$= \frac{1}{5}\sec^4 x\,\operatorname{tg}x + \frac{4}{15}\sec^2 x\,\operatorname{tg}x + \frac{8}{15}\operatorname{tg}x$$

$$\int \frac{\sin^2 x}{\cos^6 x}\,dx = (**) - (*)$$
$$= \frac{1}{5}\sec^4 x\,\operatorname{tg}x + \frac{4}{15}\sec^2 x\,\operatorname{tg}x + \frac{8}{15}\operatorname{tg}x$$
$$- \frac{1}{3}\sec^2 x\,\operatorname{tg}x - \frac{2}{3}\operatorname{tg}x$$

$$\boxed{\int \frac{\sin^2 x}{\cos^6 x}\,dx = \frac{1}{5}\sec^4 x\,\operatorname{tg}x - \frac{1}{15}\sec^2 x\,\operatorname{tg}x - \frac{2}{15}\operatorname{tg}x}$$

3. $\displaystyle\int_1^{+\infty} \frac{1}{x\sqrt{x^2-1}}\,dx =$

$= \displaystyle\lim_{t\to 1^-} \int_t^2 \frac{1}{x\sqrt{x^2-1}}\,dx + \lim_{b\to+\infty}\int_2^b \frac{1}{x\sqrt{x^2-1}}\,dx$

0.5

Mas

$\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx =$

$(x = \sec\theta \to dx = \sec\theta\, tg\theta\, d\theta)$

$= \displaystyle\int \frac{\cancel{\sec\theta}\, tg\theta\, d\theta}{\cancel{\sec\theta}\sqrt{\sec^2\theta - 1}}$

$= \displaystyle\int \frac{tg\theta}{\sqrt{tg^2\theta}}\,d\theta = \int \frac{tg\theta}{tg\theta}\,d\theta$

$= \displaystyle\int d\theta = \theta = \text{arc sec } x$ (0.5)

Assim,

$(*) = \displaystyle\lim_{t\to 1^-} \int_t^2 \frac{1}{x\sqrt{x^2-1}}\,dx =$

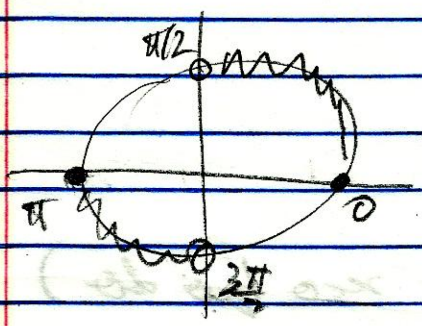
$= \displaystyle\lim_{t\to 1^-} \text{arc sec } x \Big]_t^2 =$

$= \displaystyle\lim_{t\to 1^-} (\text{arc sec } 2 - \text{arc sec } t)$

$= \text{arc sec } 2 - \displaystyle\lim_{t\to 1^-} \text{arc sec } t$

Seja $y = \operatorname{arcsec} t \therefore \begin{cases} \sec y = t \\ y \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2}) \end{cases}$

Quando $t \to 1^-$ temos $\sec y \to 1^-$



$\therefore \dfrac{1}{\cos y} \to 1^-$

$\therefore \cos y \to 1^-$

$\therefore y \to 0$

Daí $\lim\limits_{t \to 1^-} \operatorname{arcsec} t = 0$

$\therefore$ // (*) $= \operatorname{arcsec} 2 - 0 = \operatorname{arcsec} 2$ //

(0.5)

Também

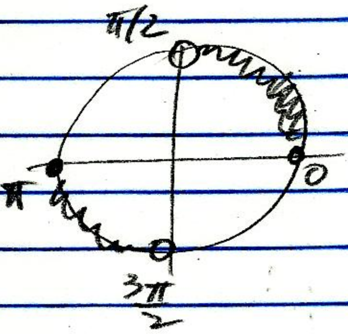
(**) $= \lim\limits_{b \to +\infty} \displaystyle\int_2^b \frac{1}{x\sqrt{x^2-1}}\, dx =$

$= \lim\limits_{b \to +\infty} \operatorname{arcsec} x \Big]_2^b$

$= \lim\limits_{b \to +\infty} (\operatorname{arcsec} b - \operatorname{arcsec} 2)$

$= \lim\limits_{b \to +\infty} \operatorname{arcsec} b - \operatorname{arcsec} 2$

Seja $y = \operatorname{arc\,sec} b \;\therefore\; \left\{\begin{array}{l} \sec y = b \\ y \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2}) \end{array}\right.$



Quando $b \to +\infty$, $\sec y \to +\infty$

$\therefore \quad \frac{1}{\cos y} \to +\infty$

$\therefore \quad \cos y \to 0^+$

$\therefore \quad y \to \frac{\pi}{2}$

$\therefore$

$$\lim_{b \to +\infty} \operatorname{arc\,sec} b = \frac{\pi}{2}$$

Daí

$$// \; (**) = \frac{\pi}{2} - \operatorname{arc\,sec} 2 \; //$$ (0.5)

Temos então

$$\int_1^{+\infty} \frac{1}{x\sqrt{x^2-1}}\,dx = (*) + (**)$$

$$= \cancel{\operatorname{arc\,sec} 2} + \frac{\pi}{2} - \cancel{\operatorname{arc\,sec} 2}$$

$$= \pi/2$$

$$\therefore \boxed{\int_1^{+\infty} \frac{1}{x\sqrt{x^2-1}}\,dx = \frac{\pi}{2}}$$

4. $\int \cos^4 x \, dx$

$(n \geq 2)$: $\int \cos^n x \, dx = \int \cos^{n-1} x \cos x \, dx$

$$\begin{pmatrix} u = \cos^{n-1} x \rightarrow du = -(n-1)\cos^{n-2} x \sin x \, dx \\ dv = \cos x \, dx \rightarrow v = +\sin x \end{pmatrix}$$

$$= +\sin x \cos^{n-1} x - \int -\sin x (n-1) \cos^{n-2} x \sin x \, dx$$

$$= +\sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \sin^2 x \, dx$$

$\therefore$ $$= +\sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) \, dx$$

$$\int \cos^n x \, dx = +\sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$$

$\therefore$

$$\int \cancel{\cos^n x \, dx} + (n-\cancel{1}) \int \cos^n x \, dx = \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \, dx$$

$\therefore$ $$n \int \cos^n x \, dx = \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \, dx$$

$\therefore$ $$\left\| \int \cos^n x \, dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{(n-1)}{n} \int \cos^{n-2} x \, dx \right\|$$

(1.0)

Dai :

$$\int \cos^2 x \, dx = \frac{1}{2} \sin x \cos x + \frac{1}{2} \int dx$$
$$= \frac{1}{2} \sin x \cos x + \frac{1}{2} x$$

$\therefore$

$$\int \cos^4 x \, dx = \frac{1}{4} \sin x \cos^3 x + \frac{3}{4} \int \cos^2 x \, dx$$
$$= \frac{1}{4} \sin x \cos^3 x +$$
$$+ \frac{3}{4} \left(\frac{1}{2} \sin x \cos x + \frac{1}{2} x \right)$$

$\therefore$

$$\boxed{\int \cos^4 x \, dx = \frac{1}{4} \sin x \cos^3 x + \frac{3}{8} \sin x \cos x + \frac{3}{8} x}$$

1.5

Outra solução (Questão 4)

$$\int \cos^4 x\,dx = \int (\cos^2 x)^2\,dx$$

$$= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{4}\int dx + \frac{1}{2}\int \cos 2x\,dx + \frac{1}{4}\int \cos^2 2x\,dx$$

$$= \frac{1}{4}x + \frac{1}{2}\,\frac{\sin 2x}{2} + \frac{1}{4}\int \frac{1+\cos 4x}{2}\,dx$$

$$= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}\int dx + \frac{1}{8}\int \cos 4x\,dx$$

$$= \underbrace{\frac{1}{4}x} + \frac{1}{4}\sin 2x + \underbrace{\frac{1}{8}x} + \frac{1}{8}\,\frac{1}{4}\sin 4x$$

$$= \frac{3x}{8} + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x$$

5. $$\int \frac{1}{\sin^2 x \sqrt[4]{\cot g\, x}} \, dx =$$

$$= \int \frac{1}{\sqrt[4]{\cot g\, x}} \cdot \operatorname{cosec}^2 x \, dx$$

$$= \int (\cot g\, x)^{-\frac{1}{4}} \cdot \operatorname{cosec}^2 x \, dx$$

$$= \frac{-(\cot g\, x)^{-\frac{1}{4}+1}}{-\frac{1}{4}+1}$$

$$= \frac{-(\cot g\, x)^{3/4}}{\frac{3}{4}} =$$

$$= -\frac{4}{3}(\cot g\, x)^{3/4}$$

6. $\displaystyle\int \frac{1}{\sin x + tg\, x}\, dx =$ $\quad \left(z = tg\,\frac{x}{2}\right)$

$$= \int \frac{\frac{2dz}{1+z^2}}{\frac{2z}{1+z^2} + \frac{2z}{1-z^2}} \qquad tg\, x = \frac{\sin x}{\cos x} = \frac{\frac{2z}{\cancel{1+z^2}}}{\frac{1-z^2}{\cancel{1+z^2}}} = \frac{2z}{1-z^2}$$

$$= \int \frac{\frac{2dz}{\cancel{1+z^2}}}{\frac{2z(1-z^2)+2z(1+z^2)}{\cancel{(1+z^2)}(1-z^2)}}$$

$$= \int \frac{2(1-z^2)\, dz}{2z - \cancel{2z^3} + 2z + \cancel{2z^3}}$$

$$= \int \frac{\cancel{2}(1-z^2)\, dz}{\cancel{4}_{2} z}$$

$$= \frac{1}{2} \int \frac{1-z^2}{z}\, dz$$ 10

$$= \frac{1}{2} \int \frac{1}{z}\, dz - \frac{1}{2} \int z\, dz$$

$$= \frac{1}{2} \ln|z| - \frac{1}{4} z^2$$

$$= \frac{1}{2} \ln\left|tg\,\frac{x}{2}\right| - \frac{1}{4}\, tg^2\,\frac{x}{2}$$ 10