Cálculo B - Prova 1

1. $\int \cos^3 x \, dx$ 1.0
2. $\int tg^2 x \; \sec^3 x \, dx$ 1.0
3. $\int \frac{1}{\sqrt{x^2-6x+13}} \, dx$ 1.5
4. $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2} \, dx$ 2.0
5. Calcule ou mostre que é divergente $\int_2^\infty \frac{1}{x\sqrt{x^2-4}} \, dx$ 2.5

Tabela de Integrais

$\int \sin x \, dx = -\cos x$

$\int \cos x \, dx = \sin x$

$\int \tan x \, dx = \ln|\sec x|$

$\int \cot x \, dx = -\ln|\csc x|$

$\int \sec x \, dx = \ln|\sec x + \tan x|$

$\int \csc x \, dx = \ln|\csc x - \cot x|$

$\int \sec^2 x \, dx = \tan x$

$\int \csc^2 x \, dx = -\cot x$

$\int \sec x \tan x \, dx = \sec x$

$\int \csc x \cot x \, dx = -\csc x$

Relações Trigonométricas

$\sin^2 = 1 - \cos^2 x$

$\cos^2 x = 1 - \sin^2 x$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\sin 2x = 2\sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

$\sin mx \sin nx = \frac{1}{2}\cos(m-n)x - \frac{1}{2}\cos(m+n)x$

$\cos mx \cos nx = \frac{1}{2}\cos(m+n)x + \frac{1}{2}\cos(m-n)x$

$\sin mx \cos nx = \frac{1}{2}\sin(m-n)x + \frac{1}{2}\sin(m+n)x$

Substituição útil: $z = \tan\frac{x}{2}$, $\sin x = \frac{2z}{1+z^2}$, $\cos x = \frac{1-z^2}{1+z^2}$

arc sec

1. $\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx$

$u = \cos^2 x \rightarrow du = -2\cos x \sin x \, dx$

$dv = \cos x \, dx \rightarrow v = \sin x$

$= \cos^2 x \sin x + 2\int \cos x \sin^2 x \, dx$

$= \cos^2 x \sin x + 2\int \cos x (1-\cos^2 x) \, dx$

$\int \cos^3 x \, dx = \cos^2 x \sin x + 2\int \cos x \, dx - 2\int \cos^3 x \, dx$

$\therefore \int \cos^3 x \, dx + 2\int \cos^3 x \, dx = \cos^2 x \sin x + 2\sin x$

$3\int \cos^3 x \, dx = \cos^2 x \sin x + 2\sin x$

$\therefore \int \cos^3 x \, dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x$

outra solução

$\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx$

$= \int (1-\sin^2 x) \cos x \, dx$

$= \int \cos x \, dx - \int \sin^2 x \cos x \, dx$

$= \sin x - \frac{1}{3}\sin^3 x$

2. $\int \operatorname{tg}^2 x \sec^3 x \, dx =$

$= \int (\sec^2 x - 1) \sec^3 x \, dx$

$= \int \sec^5 x \, dx - \int \sec^3 x \, dx$

Now

$\int \sec^n x \, dx = \int \sec^{n-2} x \sec^2 x \, dx$

$u = \sec^{n-2} x \rightarrow du = (n-2) \sec^{n-3} x \sec x \operatorname{tg} x \, dx$

$dv = \sec^2 x \, dx \rightarrow v = \operatorname{tg} x$

$= \sec^{n-2} x \operatorname{tg} x - \int \operatorname{tg} x (n-2) \sec^{n-3} x \sec x \operatorname{tg} x \, dx$

$= \sec^{n-2} x \operatorname{tg} x - (n-2) \int \sec^{n-2} x \operatorname{tg}^2 x \, dx$

$= \sec^{n-2} x \operatorname{tg} x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx$

$\int \sec^n x \, dx = \sec^{n-2} x \operatorname{tg} x - (n-2) \int \sec^n x \, dx +$

$+ (n-2) \int \sec^{n-2} x \, dx$

$\therefore$

$\underbrace{\int \sec^n x \, dx + (n-2) \int \sec^n x \, dx} = \sec^{n-2} x \operatorname{tg} x +$

$(n-2) \int \sec^{n-2} x \, dx$

$(n-1) \int \sec^n x \, dx = \sec^{n-2} x \operatorname{tg} x + (n-2) \int \sec^{n-2} x \, dx$

$\therefore \int \sec^n x \, dx = \frac{1}{(n-1)} \sec^{n-2} x \operatorname{tg} x + \frac{(n-2)}{(n-1)} \int \sec^{n-2} x \, dx$

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x\, tg\, x + \frac{1}{2}\int \sec x\,dx$$

$$= \frac{1}{2}\sec x\, tg\, x + \frac{1}{2}\ln|\sec x + tg\, x|$$

$$\int \sec^5 x\,dx = \frac{1}{4}\sec^3 x\, tg\, x + \frac{3}{4}\int \sec^3 x\,dx$$

$$= \frac{1}{4}\sec^3 x\, tg\, x + \frac{3}{4}\left[\frac{1}{2}\sec x\, tg\, x + \frac{1}{2}\ln|\sec x + tg\, x|\right]$$

$$= \frac{1}{4}\sec^3 x\, tg\, x + \frac{3}{8}\sec x\, tg\, x + \frac{3}{8}\ln|\sec x + tg\, x|$$

Daí

$$\int tg^2 x\,\sec^3 x\,dx = \frac{1}{4}\sec^3 x\, tg\, x + \frac{3}{8}\sec x\, tg\, x +$$

$$+ \frac{3}{8}\ln|\sec x + tg\, x|$$

$$- \frac{1}{2}\sec x\, tg\, x - \frac{1}{2}\ln|\sec x + tg\, x|$$

$$= \frac{1}{4}\sec^3 x\, tg\, x - \frac{1}{8}\sec x\, tg\, x - \frac{1}{8}\ln|\sec x + tg\, x|$$

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3. $\displaystyle\int \frac{dx}{\sqrt{x^2-6x+13}} = \int \frac{dx}{\sqrt{(x-3)^2+4}}$

$\left(x^2-6x+13 = (x-3)^2+4 \right)$ $\quad x-3=u$

$$= \int \frac{du}{\sqrt{u^2+4}} \qquad \downarrow 0.5$$

$u = 2\,\mathrm{tg}\,\theta$

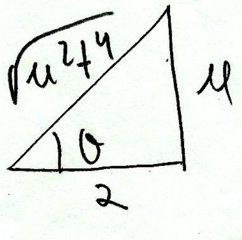
$du = 2\sec^2\theta\, d\theta$

$$= \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\,\mathrm{tg}^2\theta+4}}$$

$$= \int \frac{\cancel{2}\sec^2\theta\, d\theta}{\cancel{2}\sec\theta}$$

$$= \int \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta + \mathrm{tg}\,\theta\right| \qquad \downarrow 1.0$$



$\sec\theta = \dfrac{\sqrt{u^2+4}}{2}$

$$= \ln\left|\frac{\sqrt{u^2+4}}{2} + \frac{u}{2}\right|$$

$$= \ln\left|\frac{\sqrt{(x-3)^2+4}}{2} + \frac{x-3}{2}\right|$$

$$= \ln\left|\frac{\sqrt{x^2-6x+13}+x-3}{2}\right| \qquad (1.5)$$

4. $\displaystyle\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$

Temos

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$$

0.5

$$= \frac{A(x^2+1)^2 + (Bx+C)\overbrace{(x^2+1)\,x}^{x^3+x} + (Dx+E)\,x}{x(x^2+1)^2}$$

$$= \frac{\left(A(x^4+2x^2+1) + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex\right)}{x(x^2+1)^2}$$

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{\left(x^4(A+B) + Cx^3 + x^2(2A+B+D) + x(C+E) + A\right)}{x(x^2+1)^2}$$

$$\Rightarrow \begin{cases} A+B=0 \xrightarrow{A=1} \|B=-1\| \\ \|C=-1\| \\ 2A+B+D=2 \Longrightarrow 2+-1+D=2 \therefore \|D=1\| \\ C+E=-1 \xrightarrow{C=-1} \|E=0\| \\ \|A=+1\| \end{cases}$$

Daí $$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{1}{x} + \frac{-x-1}{x^2+1} + \frac{x}{(x^2+1)^2}$$

(1.0)

$\therefore$

$$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx =$$

$$= \int \frac{1}{x}\,dx \;-\; \underbrace{\int \frac{x+1}{x^2+1}\,dx}_{} \;+\; \int \frac{x}{(x^2+1)^2}\,dx$$

(0.5) (0.5)

$$= \ln|x| - \underset{(*)}{\int \frac{x}{x^2+1}\,dx} - \underset{(**)}{\int \frac{1}{x^2+1}\,dx} + \underset{(***)}{\int \frac{x}{(x^2+1)^2}\,dx}$$

$$(*) = -\int \frac{x}{x^2+1}\,dx = -\int \frac{\frac{1}{2}\,du}{u+1} = -\frac{1}{2}\ln|u+1|$$

$$= -\frac{1}{2}\ln|x^2+1|$$

$u = x^2$

$du = 2x\,dx$

$x\,dx = \frac{1}{2}\,du$

$$(**) = -\int \frac{1}{x^2+1}\,dx = -\int \frac{\cancel{\sec^2\theta}\,d\theta}{\cancel{\sec^2\theta}} = -\int d\theta$$

$x = \operatorname{tg}\theta \;,\; dx = \sec^2\theta\,d\theta$

$$= -\theta$$

$$= -\operatorname{arctg} x$$

$$(***) = \int \frac{x}{(x^2+1)^2}\,dx = \int \frac{\frac{1}{2}\,du}{u^2} = \frac{1}{2}\int \frac{1}{u^2}\,du$$

$u = x^2+1 \rightarrow du = 2x\,dx$

$$= -\frac{1}{2}\frac{1}{u} = -\frac{1}{2(x^2+1)}$$

$\therefore$
$$\boxed{\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx = \ln|x| - \frac{1}{2}\ln(x^2+1) - \operatorname{arctg} x - \frac{1}{2(x^2+1)}}$$

5. $\displaystyle\int_2^{\infty} \frac{dx}{x\sqrt{x^2-4}} = \underbrace{\lim_{t\to 2^+} \int_t^3 \frac{dx}{x\sqrt{x^2-4}}}_{(*)} + \underbrace{\lim_{t\to +\infty} \int_3^t \frac{dx}{x\sqrt{x^2-4}}}_{(**)}$

0.5

Mas

$$\int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{\cancel{2\sec\theta}\,\operatorname{tg}\theta\, d\theta}{\cancel{2\sec\theta}\,\sqrt{4\sec^2\theta-4}} =$$

$x = 2\sec\theta$

$dx = 2\sec\theta\,\operatorname{tg}\theta\, d\theta$

$$= \int \frac{\cancel{\operatorname{tg}\theta}}{2\cancel{\operatorname{tg}\theta}}\, d\theta$$

$$= \frac{1}{2}\theta = \frac{1}{2}\operatorname{arc\,sec}\frac{x}{2}$$

1.0

$$(*) = \lim_{t\to 2^+} \int_t^3 \frac{dx}{x\sqrt{x^2-4}}$$

$$= \lim_{t\to 2^+} \frac{1}{2}\operatorname{arc\,sec}\frac{x}{2}\Big]_{x=t}^{x=3}$$

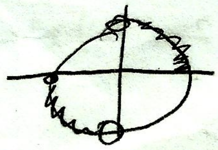
$$= \frac{1}{2}\lim_{t\to 2^+}\left(\operatorname{arc\,sec}\frac{3}{2} - \operatorname{arc\,sec}\frac{t}{2}\right)$$

Mas $\displaystyle\lim_{t\to 2^+} \operatorname{arc\,sec}\frac{t}{2} = \lim_{t\to 2^+} y(t)$

onde $y = \operatorname{arc\,sec}\frac{t}{2}$ $\therefore$ $\sec y = \frac{t}{2}$

$\sec y \to 1^+$ $\therefore$ $\cos y = 1^-$

com $y \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$



$\underline{y = 0}$

0.5

$\therefore \quad (*) = \frac{1}{2} \operatorname{arc\,sec} \frac{3}{2}$

$$(**) = \lim_{t \to \infty} \frac{1}{2} \operatorname{arc\,sec} \frac{x}{2} \Big]_{x=3}^{t}$$

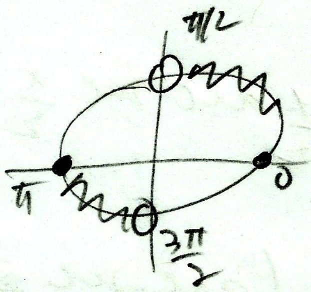
$$= \lim_{t \to \infty} \left(\frac{1}{2} \operatorname{arc\,sec} \frac{t}{2} - \frac{1}{2} \operatorname{arc\,sec} \frac{3}{2} \right)$$

Mas $\quad \lim_{t \to \infty} \underbrace{\operatorname{arc\,sec} \frac{t}{2}}_{y(t)} = \lim_{t \to \infty} y(t)$

$y(t) = \operatorname{arc\,sec} \frac{t}{2} \quad \therefore \quad \sec y(t) = \frac{t}{2}$

$t \to \infty, \quad \sec y(t) \to +\infty$

$\therefore \quad \cos y(t) \to 0^{+}$



$y \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$

$y = \frac{\pi}{2}$

$$\therefore \quad \lim_{t \to \infty} \operatorname{arc\,sec} \frac{t}{2} = \frac{\pi}{2}$$

$$\therefore \quad (**) = \frac{\pi}{4} - \frac{1}{2} \operatorname{arc\,sec} \frac{3}{2}$$

Daí $\quad \int_{2}^{+\infty} \frac{1}{x\sqrt{x^2-4}}\, dx = (*) + (**) = \frac{\pi}{4}$ //

0.5