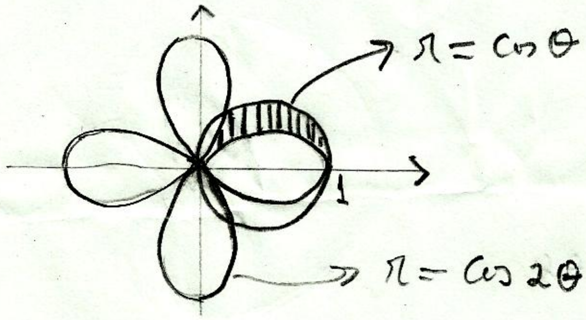
Cálculo B - Prova 2

1. Calcule, caso exista,

$$\lim_{(x,y)\to(1,0)} \frac{xy - y}{x^2 + y^2 - 2x + 1}$$

2. Calcule a área da região mostrada na figura

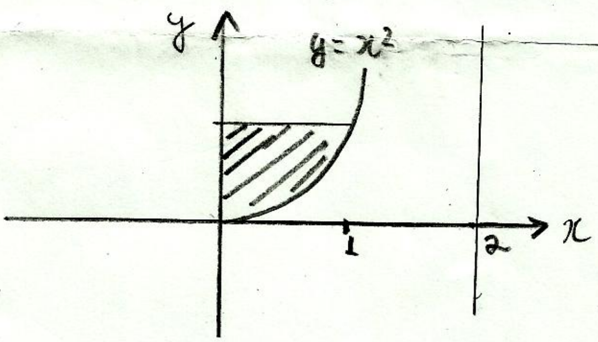


3. (i) Faça um esboço do domínio de $f(x,y) = \frac{1}{\sqrt{x^2 - y}}$

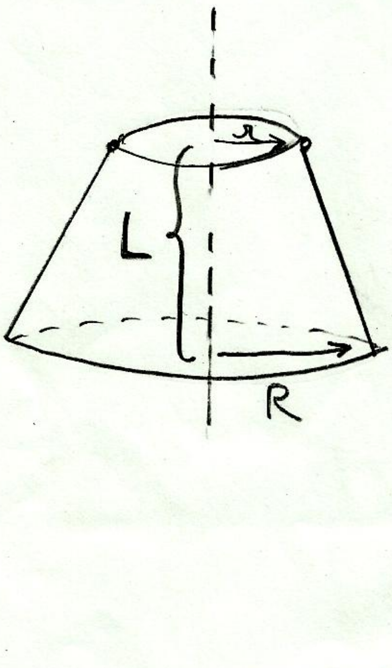
   (ii) Faça um esboço do gráfico de $z = \sqrt{x^2 + y^2 + 1}$

   (iii) Faça um esboço das curvas de nível ao gráfico de $f(x,y) = \frac{\sqrt{x^2 - y^2}}{2}$ para $c = 0$ e $c = 1$.

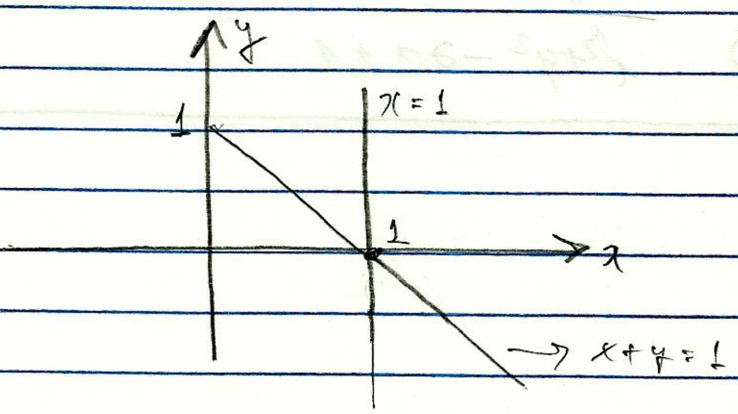
4. Calcule o volume do sólido obtido pela rotação da região mostrada na figura em torno da reta $x = 2$.



5. Calcule a área da superfície do tronco de cone mostrado na figura



1. $\displaystyle\lim_{(x,y)\to(1,0)} \frac{xy-y}{x^2+y^2-2x+1}$



→ Seja $(x,y)\to(1,0)$ com $x=1$.

Temos

$$\lim_{\substack{(x,y)\to(1,0)\\(x=1)}} \frac{xy-y}{x^2+y^2-2x+1} = \lim_{(x,y)\to(1,0)} \frac{0}{y^2}$$

$$= 0 \quad (*)$$

→ Seja $(x,y)\to(1,0)$ com $x+y=1$.

Temos

$$\lim_{\substack{(x,y)\to(1,0)\\(x+y=1)}} \frac{xy-y}{x^2+y^2-2x+1} = \lim_{\substack{(x,y)\to(1,0)\\(x=1-y)}} \frac{(1-y)y-y}{(1-y)^2+y^2-2(1-y)+1}$$

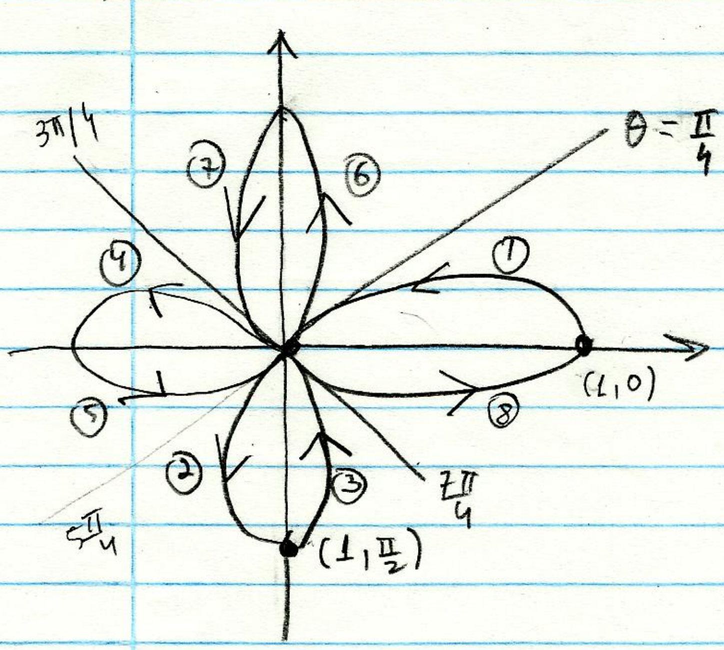
$$= \lim_{(x,y)\to(1,0)} \frac{\cancel{y}-y^2-\cancel{y}}{\cancel{1}-\cancel{2y}+y^2+y^2-\cancel{2}+\cancel{2y}+\cancel{1}}$$

$$= \lim_{(x,y)\to(1,0)} \frac{-y^2}{2y^2} = -\frac{1}{2} \quad (**)$$

De (*) e (**) temos que:

$$\boxed{\lim_{(x,y)\to(1,0)} \frac{xy-y}{x^2+y^2-2x+1} \quad \text{não existe}}$$

2. Análise da curva : $r = \cos 2\theta$



1) $0 \leq \theta \leq \frac{\pi}{4}$: $1 \geq \cos\theta \geq 0$

2) $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$: $0 \geq \cos\theta \geq -1$

3) $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$: $-1 \leq \cos\theta \leq 0$

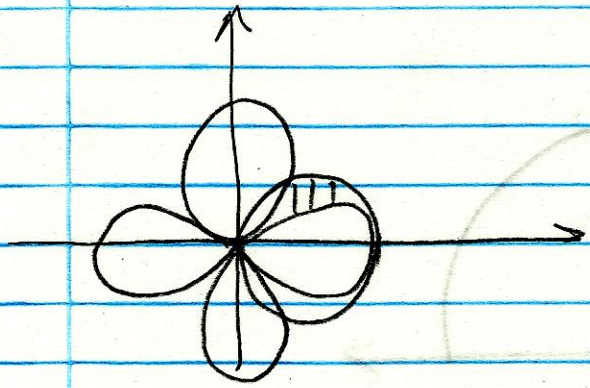
4) $\frac{3\pi}{4} \leq \theta \leq \pi$: $0 \leq \cos\theta \leq 1$

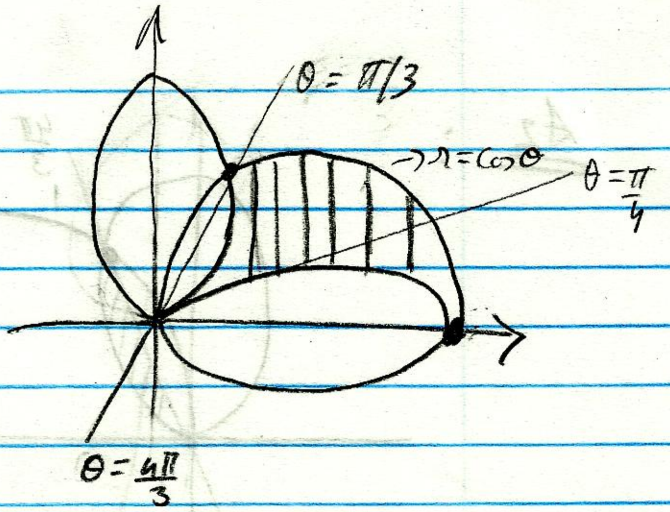
5) $\pi \leq \theta \leq \frac{5\pi}{4}$: $1 \leq \cos\theta \leq 0$

6) $\frac{5\pi}{4} \leq \theta \leq \frac{3\pi}{2}$: $0 \geq \cos\theta \geq -1$

7) $\frac{3\pi}{2} \leq \theta \leq \frac{7\pi}{4}$: $-1 \leq \cos\theta \leq 0$

8) $\frac{7\pi}{4} \leq \theta \leq 2\pi$: $0 \leq \cos\theta \leq 1$





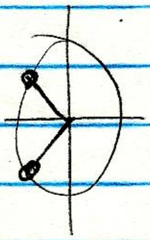
$$\left.\begin{array}{l} r = \cos\theta \\ r = \cos 2\theta \end{array}\right\} \quad \begin{aligned} \cos\theta &= \cos 2\theta \\ \cos\theta &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1-\cos^2\theta) \\ &= 2\cos^2\theta - 1 \end{aligned}$$

$\therefore$

$$2\cos^2\theta - \cos\theta - 1 = 0$$

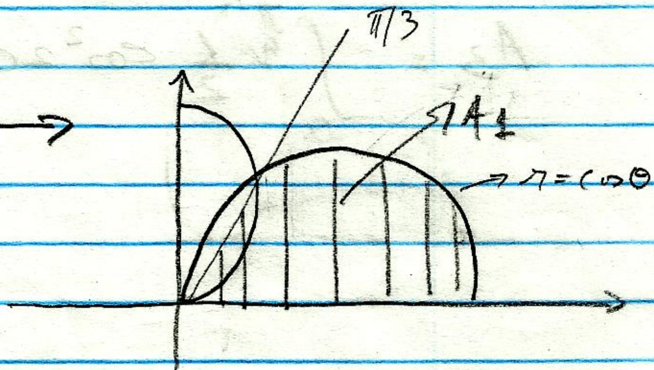
$$\cos\theta = \frac{1 \pm \sqrt{1+8}}{4}$$

$$= \frac{1 \pm 3}{4} \nearrow 1, \quad \searrow -\frac{1}{2}$$

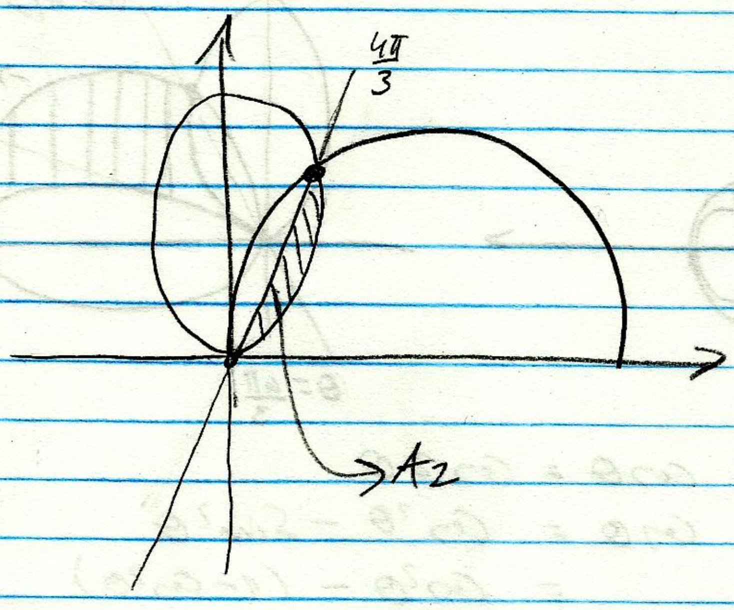


$$\begin{cases} \cos\theta = 1 \to \theta = 0 \\ \cos\theta = -\frac{1}{2} \to \theta = \frac{2\pi}{3}, \frac{4\pi}{3} \end{cases}$$

$$A_1 = \int_0^{\frac{\pi}{3}} \frac{1}{2}\cos^2\theta \, d\theta \quad \to$$

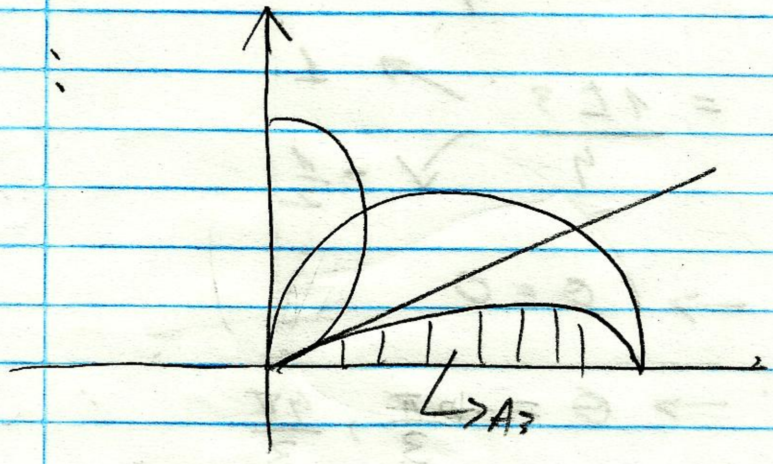


$\underline{A_2}$:



$$A_2 = \int_{\frac{5\pi}{4}}^{\frac{4\pi}{3}} \frac{1}{2} \cos^2 2\theta \, d\theta$$

$\underline{A_3}$:



$$A_3 = \int_{0}^{\frac{\pi}{4}} \frac{1}{2} \cos^2 2\theta \, d\theta$$

$$\| A = A_1 - A_2 - A_3 \|$$

Μος :

$$\rightarrow \int \frac{1}{2}\cos^2\theta \, d\theta = \frac{1}{2}\int \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= \frac{1}{4}\left(\theta + \frac{\sin 2\theta}{2}\right)$$

$$\therefore \quad \| A_1 = \int_0^{\frac{\pi}{3}} \frac{1}{2}\cos^2\theta \, d\theta =$$

$$= \frac{1}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/3}$$

$$= \frac{1}{4}\left(\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right)$$

$$= \frac{\pi}{12} + \frac{1}{8}\frac{\sqrt{3}}{2} = \frac{\pi}{12} + \frac{\sqrt{3}}{16} \|$$

$$\rightarrow \int \frac{1}{2}\cos^2 2\theta \, d\theta = \frac{1}{2}\int \frac{1+\cos 4\theta}{2} \, d\theta$$

$$= \frac{1}{4}\left(\theta + \frac{\sin 4\theta}{4}\right)$$

$$\therefore \quad A_2 = \int_{\frac{5\pi}{4}}^{\frac{4\pi}{3}} \frac{1}{2}\cos^2 2\theta \, d\theta = \frac{1}{4}\left(\theta + \frac{\sin 4\theta}{4}\right)\Bigg|_{\frac{5\pi}{4}}^{\frac{4\pi}{3}}$$

$$= \frac{1}{4}\left(\frac{4\pi}{3} + \frac{1}{4}\sin\frac{16\pi}{3}\right) - \frac{1}{4}\left(\frac{5\pi}{4}\right) =$$

$$= \frac{1}{4}\left(\frac{4\pi}{3} + \frac{1}{4}\underbrace{\sin\frac{4\pi}{3}}_{-\sin\frac{\pi}{3}}\right) - \frac{5\pi}{16}$$

$$= \frac{\pi}{3} - \frac{1}{16}\frac{\sqrt{3}}{2} - \frac{5\pi}{16}$$

$$= \frac{16\pi - 15\pi}{48} - \frac{\sqrt{3}}{32}$$

$\therefore$

$$A_2 = \frac{\pi}{48} - \frac{\sqrt{3}}{32}$$

$$A_3 = \int_0^{\frac{\pi}{4}} \frac{1}{2}\cos^2 2\theta \, d\theta$$

$$= \frac{1}{4}\left(\theta + \frac{1}{4}\sin 4\theta\right)\Bigg]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left(\frac{\pi}{4} + \frac{1}{4}\cancelto{0}{\sin\pi}\right) = \frac{\pi}{16}$$

$\therefore$

$$A = A_1 - A_2 - A_3$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{16} - \frac{\pi}{48} + \frac{\sqrt{3}}{32} - \frac{\pi}{16}$$

$$= \cancel{\frac{4\pi - \pi - 3\pi}{48}} + \frac{3\sqrt{3}}{32}$$

$$\boxed{A = \frac{3\sqrt{3}}{32}}$$

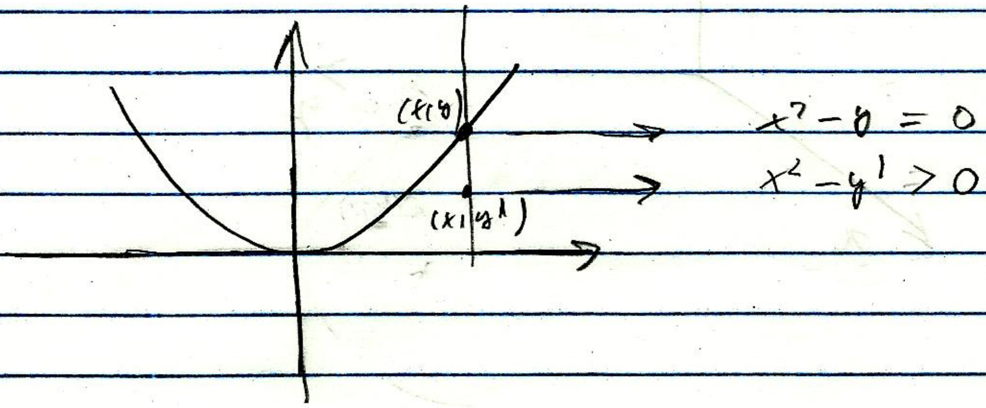
3.

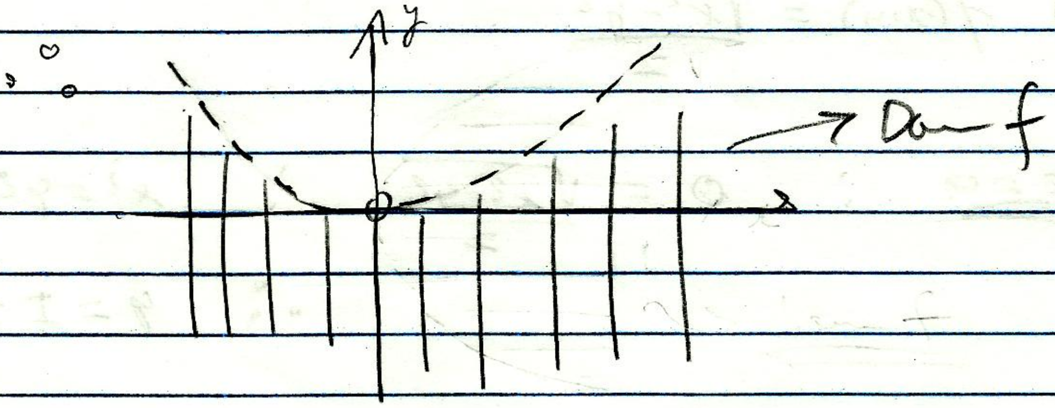
(i) $f(x,y) = \dfrac{1}{\sqrt{x^2-y}}$

$D = \{ (x,y) \in \mathbb{R}^2 \; ; \; x^2-y > 0 \}$

$x^2 - y > 0$:

$x^2 - y = 0 \quad \therefore \quad x^2 = y$





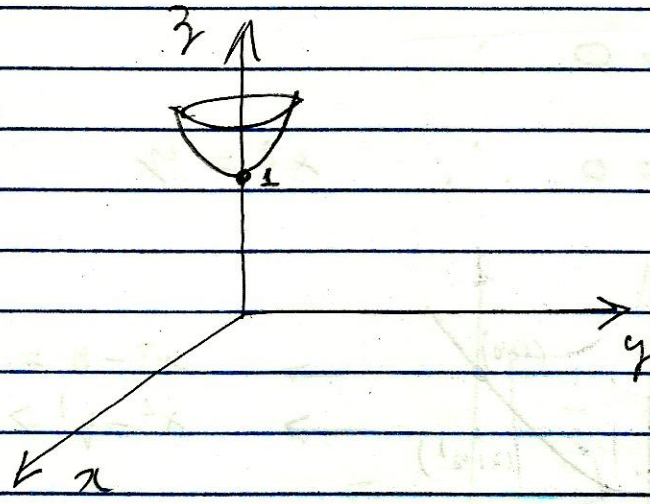
ii) $z = \sqrt{x^2+y^2+1}$ $(z > 0)$

$\therefore$

$$z^2 = x^2+y^2+1$$

$\therefore$

$$z^2 - x^2 - y^2 = 1$$



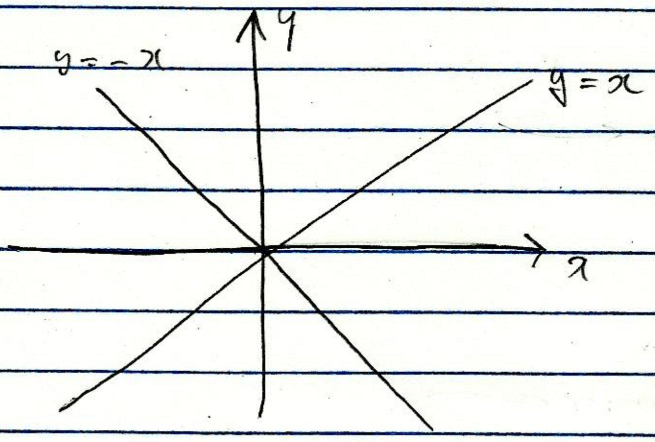
iii) $f(x,y) = \dfrac{\sqrt{x^2-y^2}}{2}$

$\underline{\underline{c=0}}$: $0 = \dfrac{\sqrt{x^2-y^2}}{2}$ $\therefore$ $x^2 = y^2$

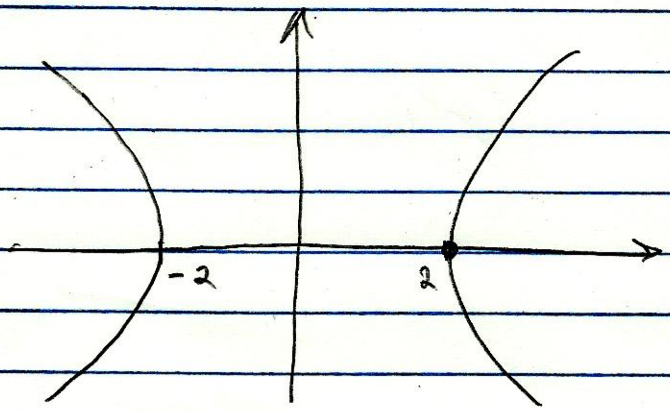
$\therefore$ $y = \pm x$

$\underline{\underline{c=1}}$: $1 = \dfrac{\sqrt{x^2-y^2}}{2}$ $\therefore$ $1 = \dfrac{x^2}{4} - \dfrac{y^2}{4}$

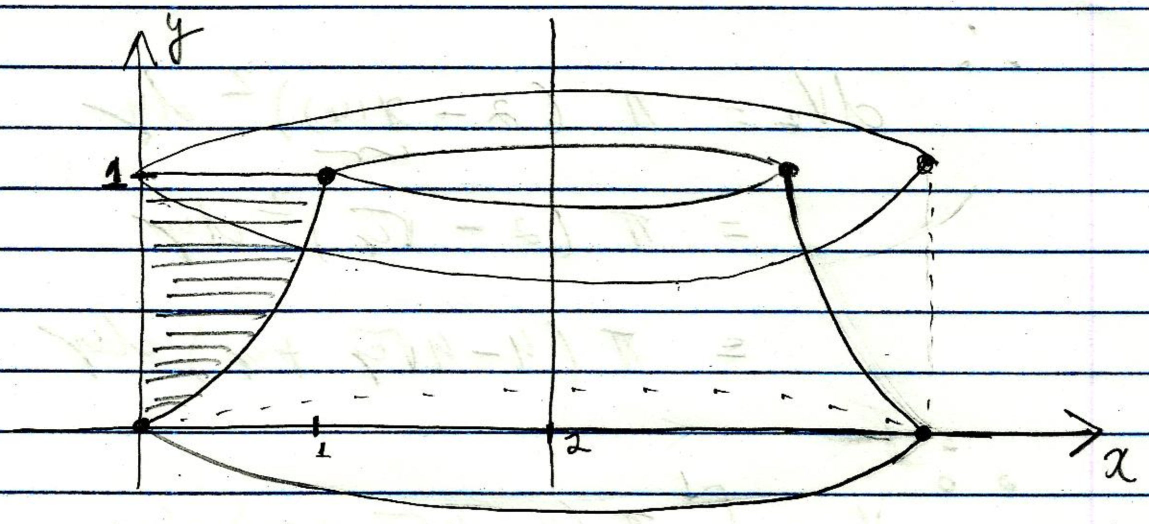
C = 0

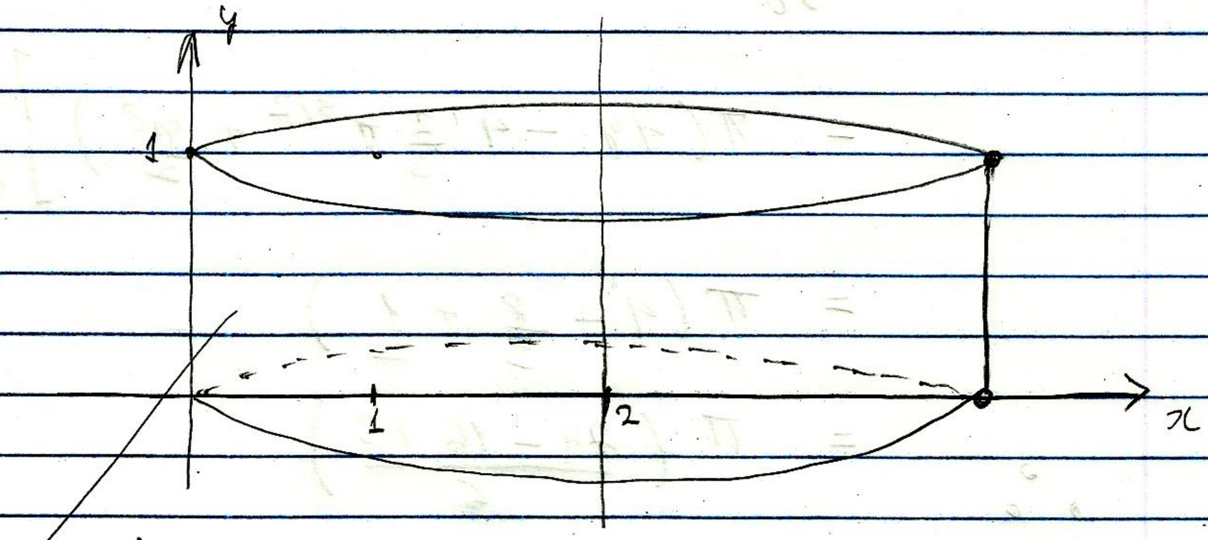


C = 1

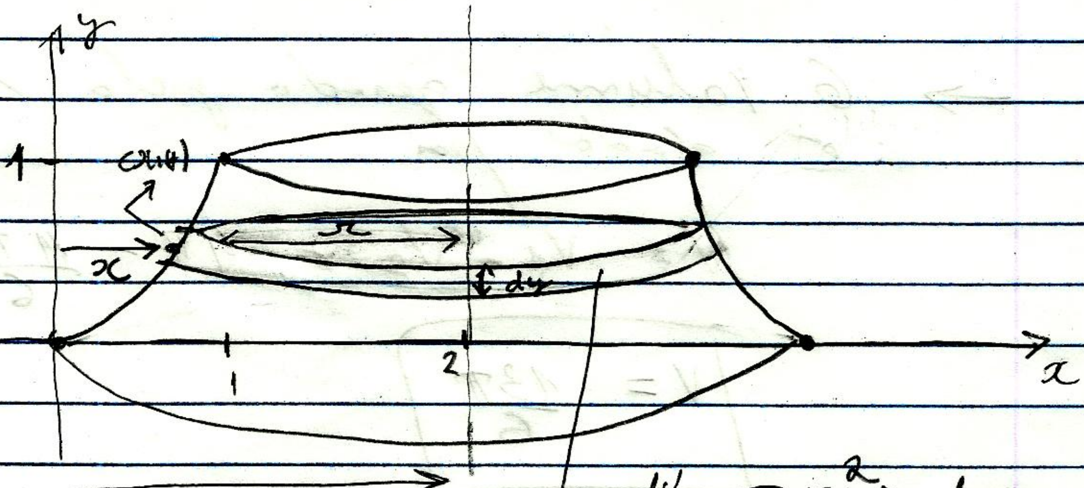


4.





$$\| V_1 = \pi\, 2^2 \cdot 1 = 4\pi \|$$



$$\begin{cases} y = x^2 \\ \therefore \\ x = \sqrt{y} \end{cases}$$

$$dV_2 = \pi r^2(y)\, dy$$

$\therefore$

$$dV_2 = \pi\,(2 - \underbrace{x(y)}_{\sqrt{y}})^2\,dy$$

$$= \pi\,(2 - \sqrt{y})^2\,dy$$

$$= \pi\,(4 - 4\sqrt{y} + y)\,dy$$

$\therefore$

$$V_2 = \int_0^1 \pi\,(4 - 4\sqrt{y} + y)\,dy$$

$$= \pi\left(4y - 4\,\frac{2}{3}\,y^{3/2} + \frac{y^2}{2}\right)\Big]_0^1$$

$$= \pi\left(4 - \frac{8}{3} + \frac{1}{2}\right)$$

$$= \pi\left(\frac{24 - 16 + 3}{6}\right)$$

$\therefore$

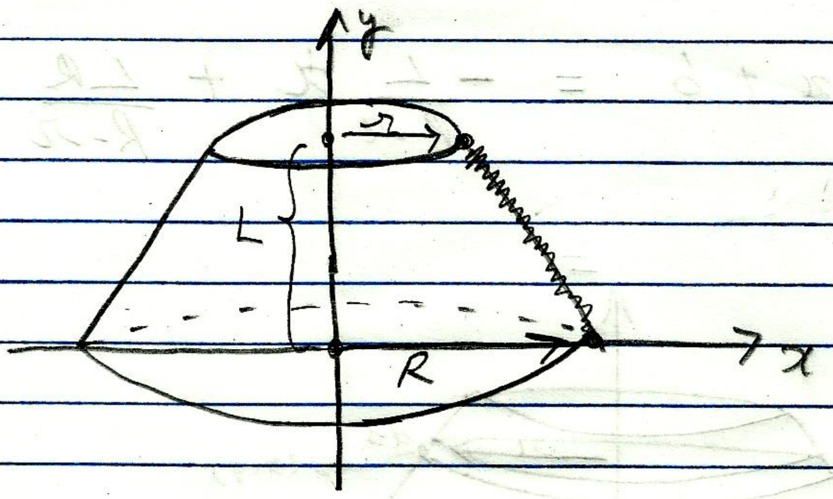
$$\| V_2 = \frac{11\pi}{6} \|$$

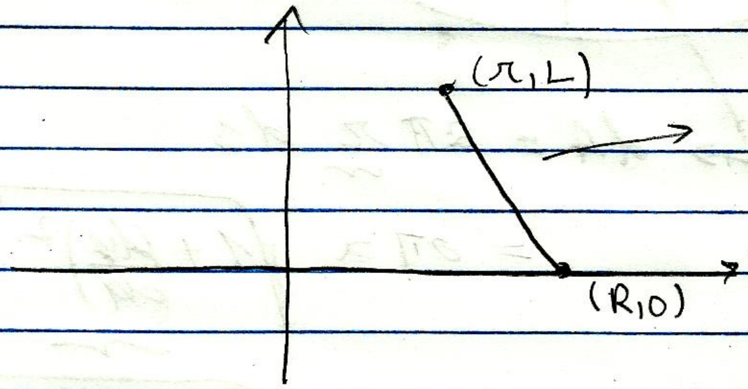
$\rightarrow$ O volume gerado pela região é dado por

$$V = V_1 - V_2 = 4\pi - \frac{11\pi}{6}$$

$$\boxed{V = \frac{13\pi}{6}}$$

5.





(r, L) : $L = ar + b$

$(R, 0)$: $0 = aR + b \quad \therefore \quad b = -aR$

Daí,

$$L = ar + b = ar - aR = a(r - R)$$

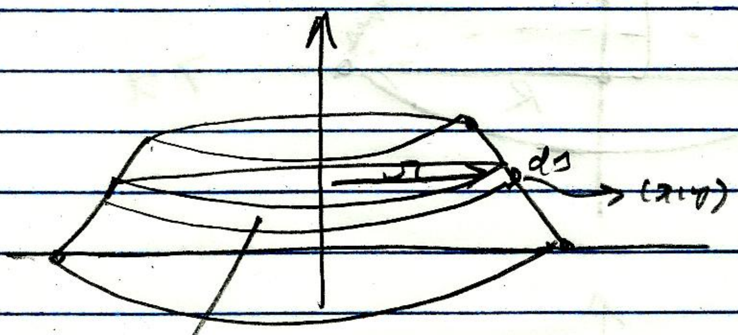
$$\therefore \quad \| a = \frac{L}{r - R} = -\frac{L}{R - r} \|$$

$$\therefore \quad b = -aR = -\left(-\frac{L}{R-r}\right)R$$

$$\| b = \frac{LR}{R - r} \|$$

$$y = ax + b = -\frac{L}{R-r}x + \frac{LR}{R-r}$$

Daí



$$dA = 2\pi r\, ds$$

$$= 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi x \sqrt{1+\left(-\frac{L}{R-r}\right)^2}\, dx$$

$$= 2\pi x \sqrt{1+\frac{L^2}{(R-r)^2}}\, dx$$

$\therefore$

$$A = \int_r^R 2\pi x \sqrt{1+\frac{L^2}{(R-r)^2}}\, dx$$

$$= 2\pi \frac{\sqrt{(R-r)^2+L^2}}{(R-r)} \int_r^R x\, dx$$

$$= \cancel{2}\pi \frac{\sqrt{(R-r)^2+L^2}}{(R-r)} \frac{1}{\cancel{2}}(R^2-r^2)$$

$$\boxed{A = \pi (R+r)\sqrt{(R-r)^2+L^2}}$$