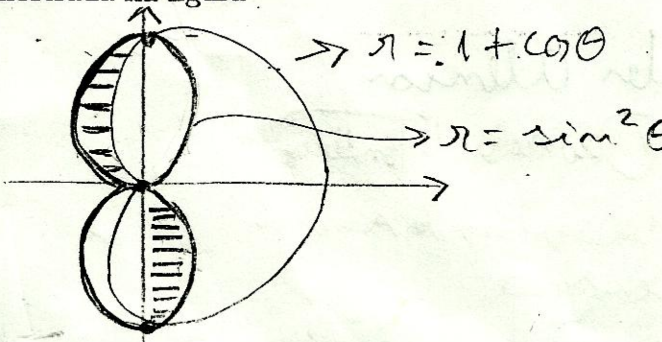
Cálculo B - Prova 2

1. Calcule, caso exista,

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$$

2. Calcule a área da região mostrada na figura



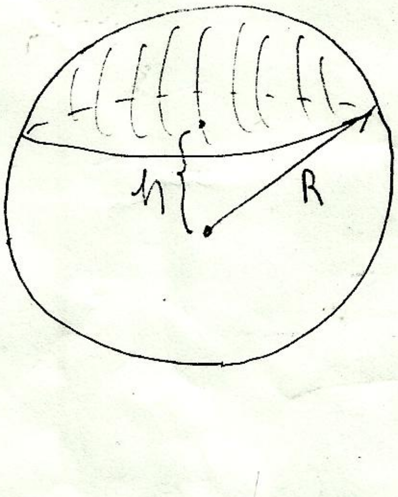
3. (i) Faça um esboço do domínio de $f(x,y) = \ln(xy)$

   (ii) Faça um esboço do gráfico de $z = \sqrt{x^2+y^2-1}$

   (iii) Faça um esboço das curvas de nível ao gráfico de $f(x,y) = x^2 + 4y^2$ para $c = 0$ e $c = 1$ ~~para c = 0 e c = 1~~.

4. Calcule o volume do sólido obtido pela rotação da curva $y = x^2 - 4x + 3$, $0 \leq x \leq 2$ em torno da reta $X = 4$.

5. Calcule a área da calota esférica mostrada na figura



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1. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

Se $x=0$:

$$\lim_{\substack{(x,y)\to(0,0)\\(x=0)}} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \lim_{y\to 0} \frac{y^2}{\sqrt{y^2+1}-1}$$

$$= \lim_{y\to 0} \frac{\cancel{2y}}{\frac{1}{2\sqrt{y^2+1}}\cancel{2y}} = \lim_{y\to 0} 2\sqrt{y^2+1}$$

$$= 2$$

Se o limite existir, seu valor é 2.

Temos:

$$\left| f(x,y) - 2 \right| = \left| \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} - 2 \right| =$$

$$= \left| \frac{x^2+y^2 - 2\sqrt{x^2+y^2+1} + 2}{\sqrt{x^2+y^2+1}-1} \right|$$

$$= \left| \frac{\underbrace{x^2+y^2+1} - 2\sqrt{x^2+y^2+1} + 1}{\sqrt{x^2+y^2+1}-1} \right|$$

$$= \left| \frac{\left(\sqrt{x^2+y^2+1} - 1\right)^{\cancel{2}}}{\cancel{\left(\sqrt{x^2+y^2+1} - 1\right)}} \right|$$

$$= \left| \sqrt{x^2+y^2+1} - 1 \right| \equiv \left| g(x,y) \right|$$

Seja então

$$g(x,y) = \sqrt{x^2+y^2+1} - 1$$

Temos

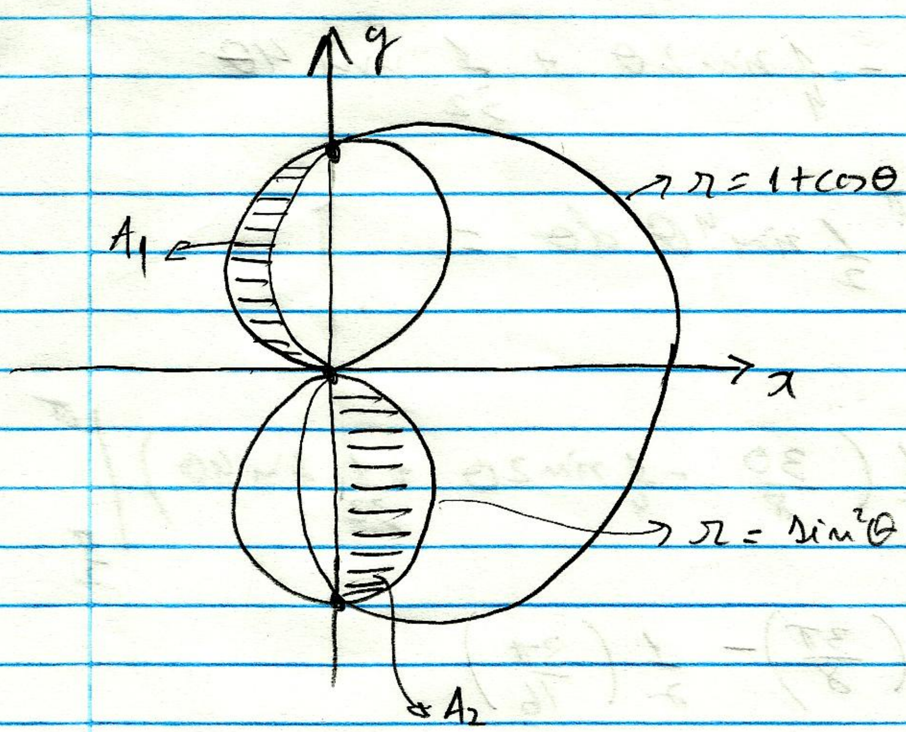
(**) $\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2+1} - 1 = 0$

De (*) : $|f(x,y) - 2| \leq |g(x,y)|$ (***)

De (**) e (***) sabemos então

$$\lim_{(x,y)\to(0,0)} f(x,y) = 2$$

2.



$$\rightarrow A_1 = \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}\sin^4\theta \, d\theta - \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}(1+\cos\theta)^2 \, d\theta$$

Now $\int \sin^4\theta \, d\theta = \int \sin^2\theta \sin^2\theta \, d\theta$

$$= \int \frac{1-\cos 2\theta}{2} \, \frac{1-\cos 2\theta}{2} \, d\theta$$

$$= \frac{1}{4}\int (1 - 2\cos 2\theta + \cos^2 2\theta) \, d\theta$$

$$= \frac{1}{4}\int d\theta - \frac{1}{2}\int \cos 2\theta \, d\theta + \frac{1}{4}\int \cos^2 2\theta \, d\theta$$

$$= \frac{1}{4}\theta - \frac{1}{2}\,\frac{\sin 2\theta}{2} + \frac{1}{4}\int \frac{1+\cos 4\theta}{2} \, d\theta$$

$$= \frac{1}{4}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{8}\int d\theta + \frac{1}{8}\int \cos 4\theta \, d\theta$$

$$= \frac{1}{4}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{8}\theta + \frac{1}{8}\,\frac{\sin 4\theta}{4}$$

$$= \frac{3\theta}{8} - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta$$

$$\therefore \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}\sin^4\theta \, d\theta =$$

$$= \frac{1}{2}\left(\frac{3\theta}{8} - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right)\Bigg|_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{1}{2}\left(\frac{3\pi}{8}\right) - \frac{1}{2}\left(\frac{3\pi}{16}\right)$$

$$= \frac{3\pi}{16} - \frac{3\pi}{32} = \frac{3\pi}{32}$$

$$\int (1+\cos\theta)^2 \, d\theta =$$

$$= \int (1 + 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \theta + 2\sin\theta + \int \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \theta + 2\sin\theta + \frac{1}{2}\theta + \frac{\sin 2\theta}{4}$$

$$= \frac{3\theta}{2} + 2\sin\theta + \frac{1}{4}\sin 2\theta$$

$$\therefore -\int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}(1+\cos\theta)^2 \, d\theta =$$

$$= -\frac{1}{2}\left(\frac{3\theta}{2} + 2\sin\theta + \frac{1}{4}\sin 2\theta\right)\Bigg]_{\frac{\pi}{2}}^{\pi}$$

$$= -\frac{1}{2}\left(\frac{3\pi}{2}\right) + \frac{1}{2}\left(\frac{3\pi}{4} + 2\right)$$

$$= -\frac{3\pi}{4} + \frac{3\pi}{8} + 1$$

$$= -\frac{3\pi}{8} + 1 \quad //$$

$$// A_1 = \frac{3\pi}{32} - \frac{3\pi}{8} + 1$$

$$= \frac{3\pi - 12\pi + 32}{32}$$

$$= \frac{32 - 9\pi}{32} \quad //$$

$$\rightarrow // A_2 = \int_{\frac{3\pi}{2}}^{2\pi} \frac{1}{2} \sin^4\theta \, d\theta$$

$$= \frac{1}{2}\left(\frac{3\theta}{8} - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right)\Bigg|_{\frac{3\pi}{2}}^{2\pi}$$

$$= \frac{1}{2}\left(\frac{3\pi}{4}\right) - \frac{1}{2}\left(\frac{9\pi}{16}\right)$$

$$= \frac{3\pi}{8} - \frac{9\pi}{32}$$

$$= \frac{12\pi - 9\pi}{32} = \frac{3\pi}{32} \quad //$$

Dari $A = A_1 + A_2 = \frac{32 - 9\pi}{32} + \frac{3\pi}{32} = \frac{32 - 6\pi}{32}$ //

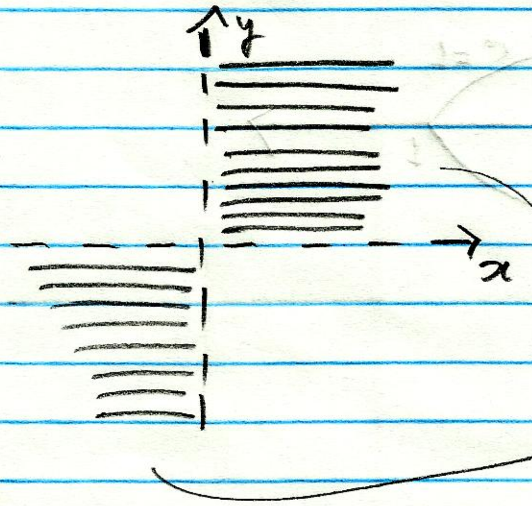
3.-

(i) $f(x,y) = \ln(xy)$

$xy > 0 \implies x > 0, y > 0$

ou

$x < 0, y < 0$

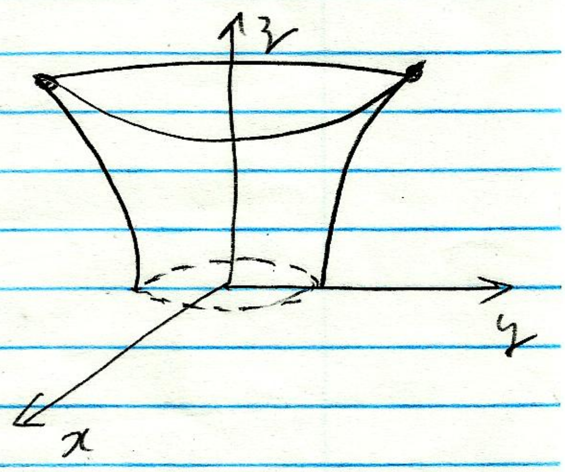


Domínio da f

(ii) $z = \sqrt{x^2+y^2-1}$

$\therefore \quad z^2 = x^2+y^2-1$

$\therefore \quad x^2+y^2-z^2 = 1$



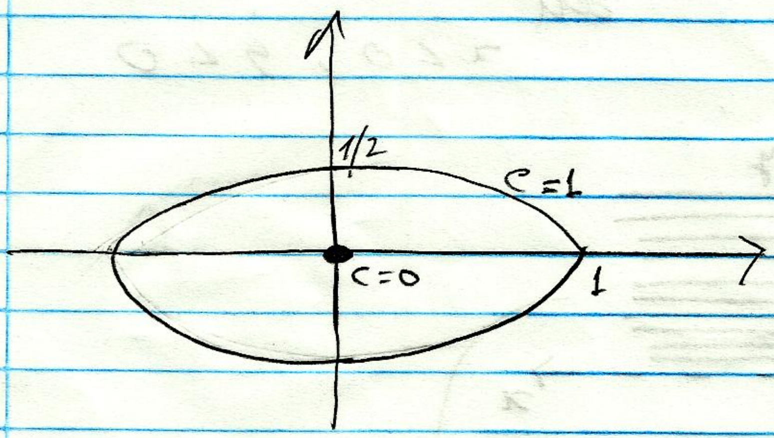
(iii) $f(x,y) = x^2+4y^2$

$\rightarrow \underline{\underline{c=0}}: \quad 0 = x^2+4y^2$

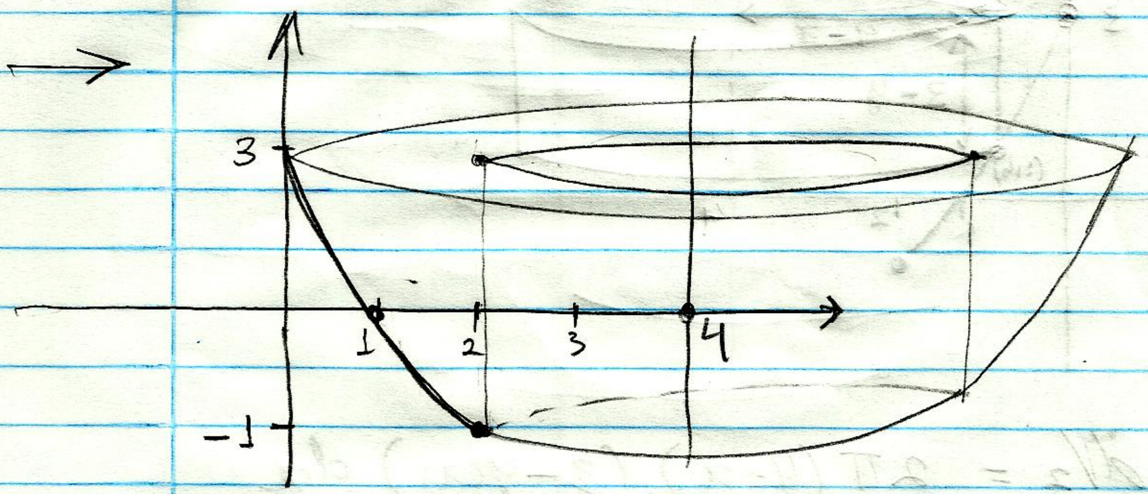
$\therefore \quad (x,y) = (0,0)$

$\underline{\underline{c=1}}$: $x^2+4y^2=1$

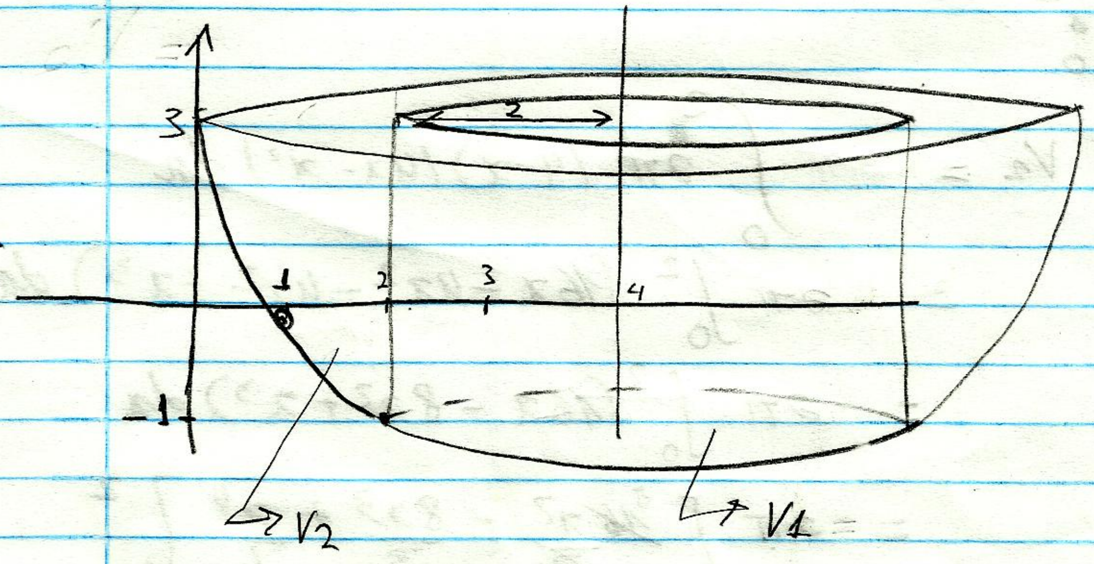
$$x^2+\frac{y^2}{\frac{1}{4}}=1$$



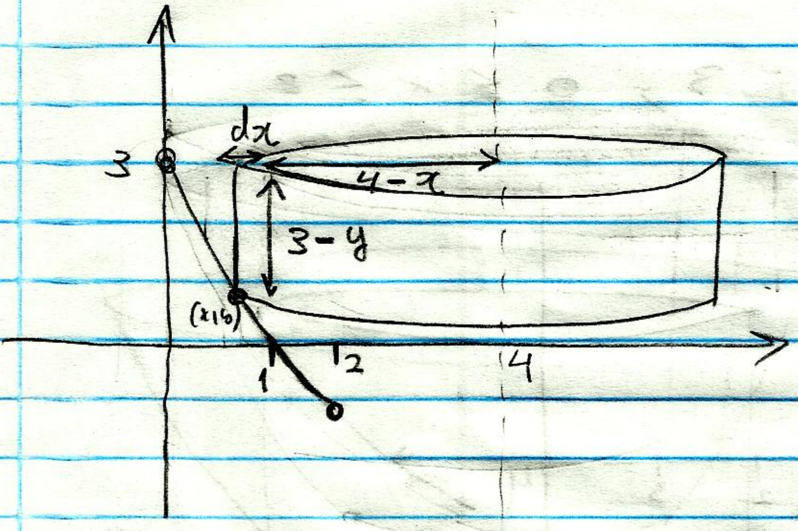
4. $y = x^2 - 4x + 3, \quad 0 \leq x \leq 2$



1ª mitade ; Cascas cilíndricas



$V_1 = \pi \cdot 2^2 \cdot 4 = 16\pi$



$$dV_2 = 2\pi(4-x)(3-y(x))\,dx$$

$$= 2\pi(4-x)(\cancel{3} - x^2+4x-\cancel{3})\,dx$$

$$= 2\pi(4-x)(4x-x^2)\,dx$$

$\therefore$

$$V_2 = \int_0^2 2\pi(4-x)(4x-x^2)\,dx$$

$$= 2\pi\int_0^2 (16x-4x^2-4x^2+x^3)\,dx$$

$$= 2\pi\int_0^2 (16x-8x^2+x^3)\,dx$$

$$= 2\pi\left[\frac{\overset{8}{\cancel{16}}x^2}{\cancel{2}} - \frac{8x^3}{3} + \frac{x^4}{4}\right]_0^2$$

$$= 2\pi\left[8\cdot 4 - 8\cdot\frac{8}{3} + \frac{16}{4}\right]$$

$$= 2\pi\left[32 - \frac{64}{3} + \frac{16}{4}\right]$$

$$= 2\pi\left[36 - \frac{64}{3}\right] = 2\pi\left[\frac{108-64}{3}\right]$$

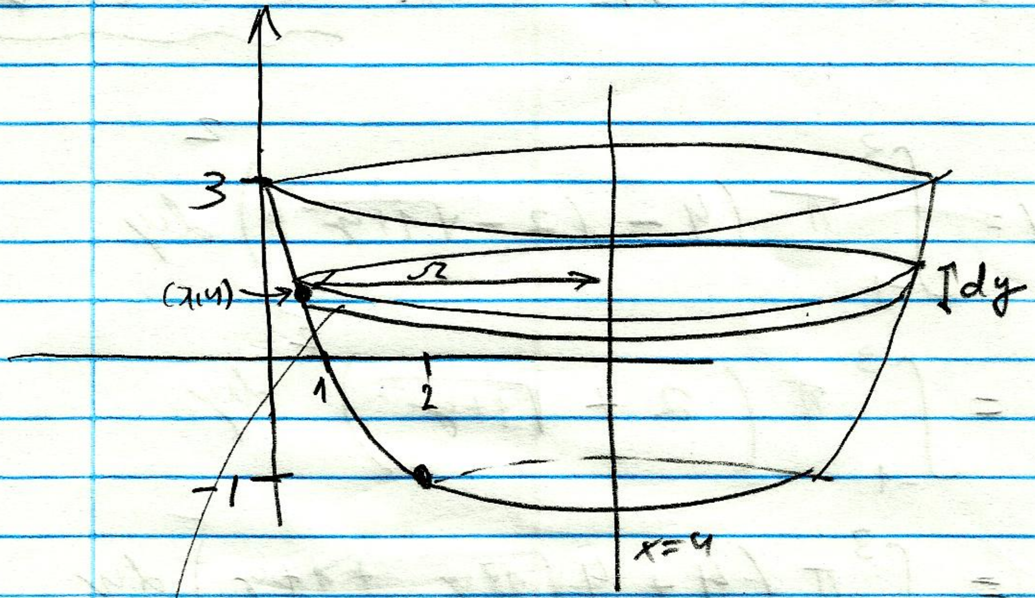
$$= 2\pi\left(\frac{44}{3}\right) = \frac{88\pi}{3}$$ //

$$V = V_1 + V_2 = 16\pi + \frac{88\pi}{3}$$

$$= \frac{48\pi + 88\pi}{3}$$

$$\boxed{V = \frac{136}{3}\pi}$$

2º método : Discos



$$dV = \pi\, r^2(y)\, dy$$

$$= \pi\,(4 - x(y))^2\, dy$$

$$V = \int_{-1}^{3} \pi\,(4 - x(y))^2\, dy$$

Mas

$$y = x^2 - 4x + 3$$

$$x^2 - 4x + 3 - y = 0$$

$$x = \frac{4 \pm \sqrt{16 - 4(3-y)}}{2} = \frac{4 \pm \sqrt{16 - 12 + 4y}}{2}$$

$$= \frac{4 \pm \sqrt{4 + 4y}}{2}$$

$$x = 2 \pm \sqrt{1+y}$$

$(1,0)$: $x = 2 \pm \sqrt{1+y}$

$\therefore 1 = 2 \pm \sqrt{1}$ $\therefore$ $x = 2 - \sqrt{1+y}$

$\therefore$

$$V = \int_{-1}^{3} \pi \left(4 - (2 - \sqrt{1+y})\right)^2 dy$$

$$= \int_{-1}^{3} \pi \left(2 + \sqrt{1+y}\right)^2 dy$$

$$= \int_{-1}^{3} \pi \left(4 + 4\sqrt{1+y} + 1 + y\right) dy$$

$$= \int_{-1}^{3} \pi \left(5 + 4\sqrt{1+y} + y\right) dy$$

$$= \left(5\pi y + 4\pi \frac{2}{3}(1+y)^{3/2} + \frac{\pi y^2}{2}\right)\Big]_{-1}^{3}$$

$$= 5\pi 3 + \frac{8\pi}{3} 4^{3/2} + \frac{\pi 9}{2}$$

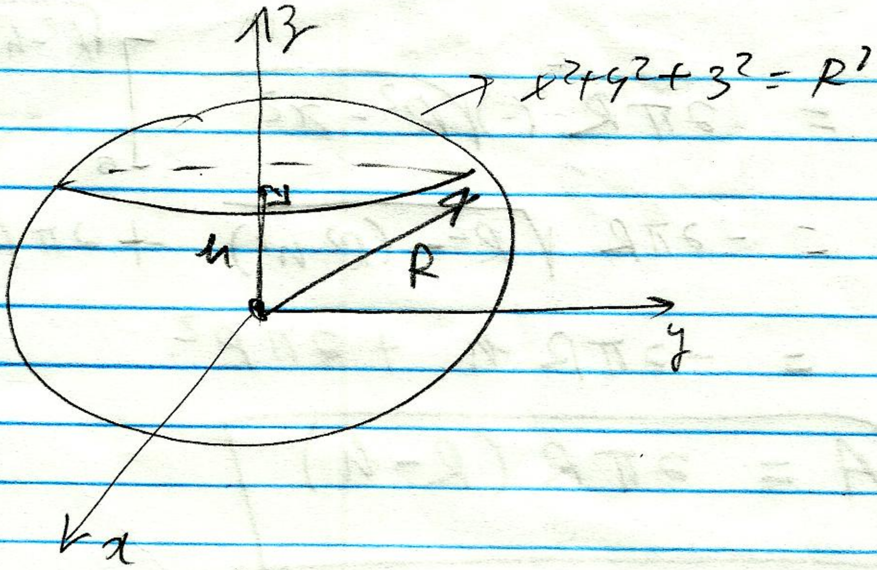
$$- \left(5\pi(-1) + \frac{8\pi}{3} \cdot 0 + \frac{\pi}{2}\right)$$

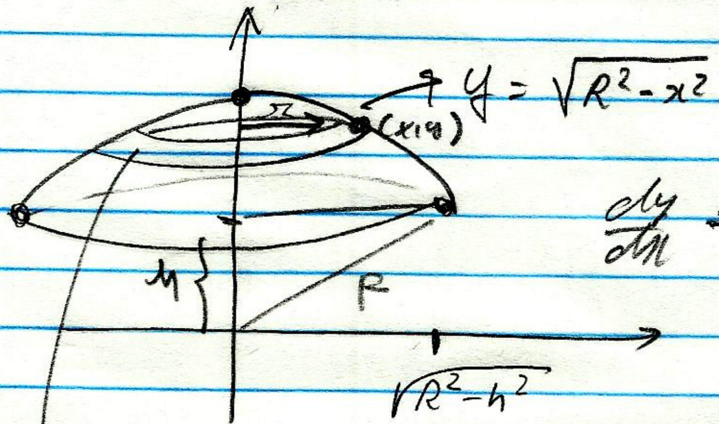
$$= 15\pi + \frac{8\pi}{3} \cdot 8 + \frac{9\pi}{2} + 5\pi - \frac{\pi}{2}$$

$$= 20\pi + \frac{64\pi}{3} + 4\pi = 24\pi + \frac{64\pi}{3}$$

$$= \frac{72\pi + 64\pi}{3} = \frac{136}{3}\pi \;//$$

5.





$$\frac{dy}{dx} = \frac{-x}{\sqrt{R^2-x^2}}$$

$$dA = 2\pi r(x)\, ds(x)$$

$$= 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi x \sqrt{1 + \frac{x^2}{R^2-x^2}}\, dx$$

$$= 2\pi x \sqrt{\frac{R^2-\cancel{x^2}+\cancel{x^2}}{R^2-x^2}}\, dx$$

$$= \frac{2\pi R\, x}{\sqrt{R^2-x^2}}\, dx$$

$$A = \int_0^{\sqrt{R^2-h^2}} 2\pi R \frac{x}{\sqrt{R^2-x^2}}\, dx$$

$$= 2\pi R\,(-)\sqrt{R^2 - x^2}\,\Big]_0^{\sqrt{R^2-h^2}}$$

$$= -2\pi R\sqrt{R^2 - (R^2 - h^2)} + 2\pi R R$$

$$= -2\pi R h + 2\pi R^2$$

$$\boxed{A = 2\pi R(R - h)}$$