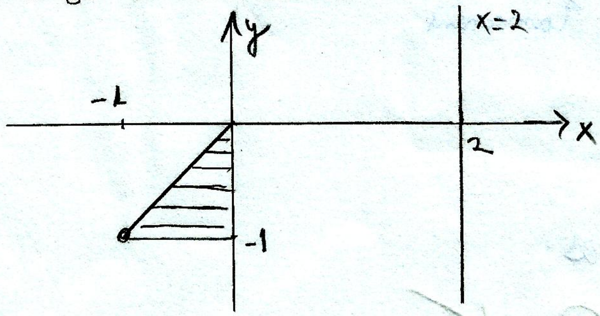
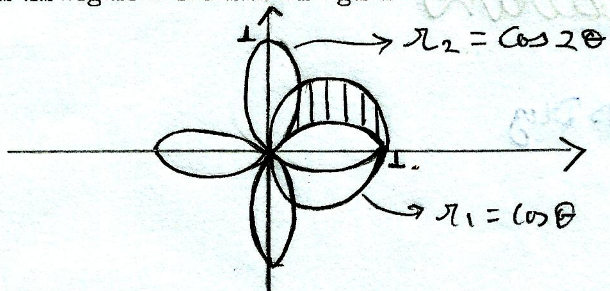
Cálculo B - Prova 2

1. Usando o método das cascas cilíndricas, calcule o volume obtido pela rotação da região mostrada na figura em torno da reta $x = 2$. 2pts.



2. Calcule a área da região mostrada na figura 2pts.

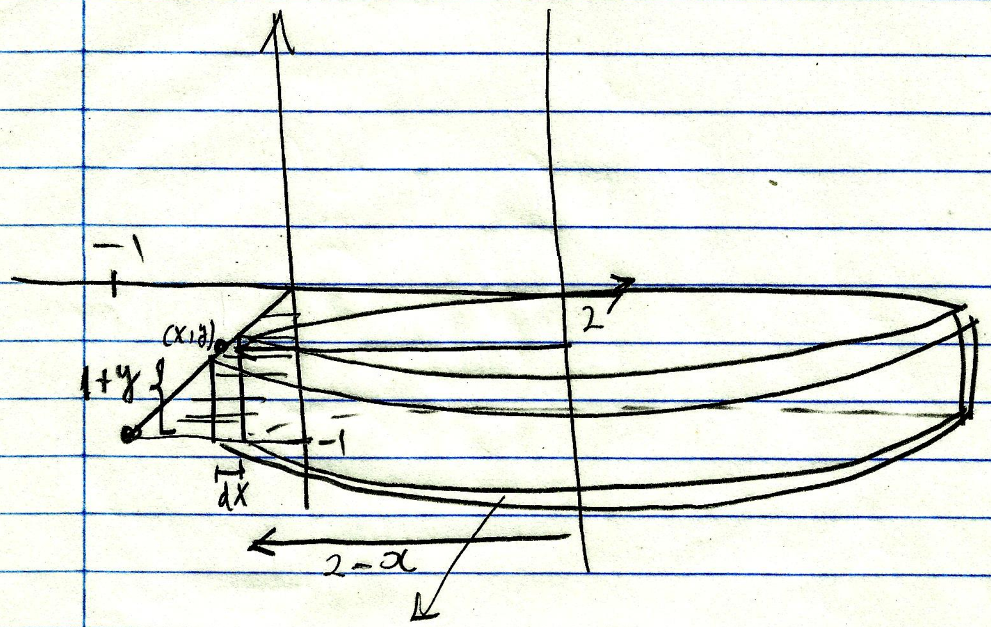


3. Se o limite existir calcule seu valor, do contrário, mostre que o limite não existe.
   (i) $\lim_{(x,y)\to(0,0)} \frac{x^3}{\sqrt{x^2+y^2}}$ 1.0 pt.
   (ii) $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ 1.0 pt

4. Seja $f(x,y) = \sqrt{\frac{y}{x^2} - 1}$.
   (i) Determine o domínio da f e faça um esboço da região que corresponde a este domínio. 1.0
   (ii) Faça um esboço das cruvas de nível associado a $k = 0$ e $k = 1$. ~~1.0~~ 0.5

5. Faça um esboço da quádrica de equação $1 = x^2 - y^2$. 0.5

1.



$$dV = (1+\underbrace{y(x)}_{+x})\,2\pi\,(2-x)\,dx$$

1.0

∴

$$V = \int_{-1}^{0} 2\pi\,(2-x)(1+x)\,dx$$

$$= \int_{-1}^{0} 2\pi(2+2x-x-x^2)\,dx$$

$$= \int_{-1}^{0} 2\pi(2+x-x^2)\,dx$$

$$= 2\pi\left(2x+\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_{-1}^{0}$$

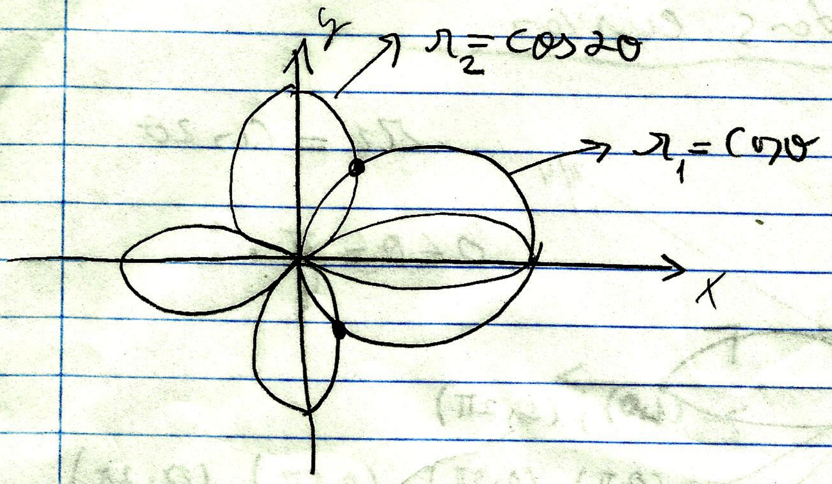
$$= -2\pi\left(2(-1)+\frac{(-1)^2}{2}-\frac{(-1)^3}{3}\right)$$

$$= -2\pi\left(-2+\frac{1}{2}+\frac{1}{3}\right) = -2\pi\left(\frac{-12+3+2}{6}\right)$$

$$= -2\pi\left(\frac{-7}{6}\right) = \frac{7\pi}{3}$$

1.0

2.



Pontos de intersecção

$$r_2(\theta) = r_1(\theta)$$

$$\cos 2\theta = \cos\theta$$

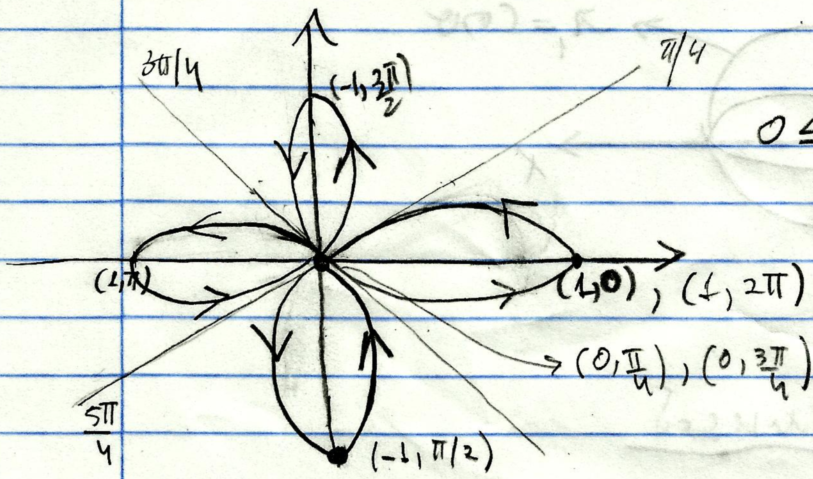
$$2\cos^2\theta - 1 = \cos\theta \quad \therefore \quad 2\cos^2\theta - \cos\theta - 1 = 0$$

$$\cos\theta = \frac{1 \pm \sqrt{1+8}}{4} = \frac{1 \pm 3}{4} \begin{matrix} \nearrow 1 \\ \searrow -\frac{1}{2} \end{matrix}$$

$$\begin{matrix} \cos\theta = 1 \quad \therefore \\ \cos\theta = -\frac{1}{2} \quad \therefore \end{matrix} \left\{ \begin{array}{l} \theta = 2k\pi \quad , k \in \mathbb{Z} \\ \theta = \frac{2\pi}{3} + 2k\pi \quad , k \in \mathbb{Z} \\ \theta = \frac{4\pi}{3} + 2k\pi \quad , k \in \mathbb{Z} \end{array} \right.$$

0.5

Análise das curvas



$r^2 = \cos 2\theta$

$0 \leq \theta \leq \frac{\pi}{4}$ ∴

$0 \leq \theta \leq \frac{\pi}{4}$ ∴ $0 \leq 2\theta \leq \frac{\pi}{2}$ ∴ $1 \geq \cos 2\theta \geq 0$

$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$ ∴ $\frac{\pi}{2} \leq 2\theta \leq \pi$ ∴ $0 \geq \cos 2\theta \geq -1$

$\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$ ∴ $\pi \leq 2\theta \leq \frac{3\pi}{2}$ ∴ $-1 \leq \cos 2\theta \leq 0$

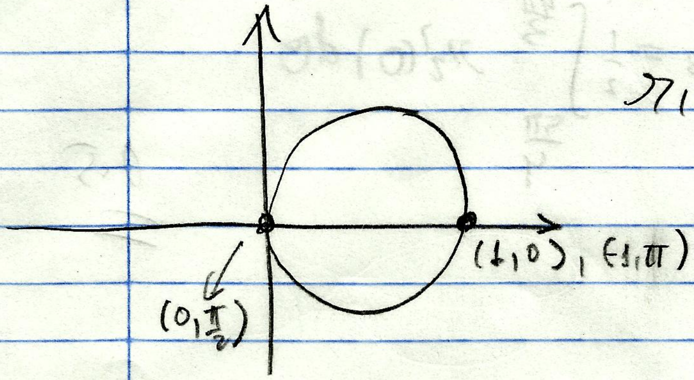
$\frac{3\pi}{4} \leq \theta \leq \pi$ ∴ $\frac{3\pi}{2} \leq 2\theta \leq 2\pi$ ∴ $0 \leq \cos 2\theta \leq 1$

$\pi \leq \theta \leq \frac{5\pi}{4}$ ∴ $2\pi \leq 2\theta \leq \frac{5\pi}{2}$ ∴ $1 \geq \cos 2\theta \geq 0$

$\frac{5\pi}{4} \leq \theta \leq \frac{3\pi}{2}$ ∴ $\frac{5\pi}{2} \leq 2\theta \leq 3\pi$ ∴ $0 \geq \cos 2\theta \geq -1$

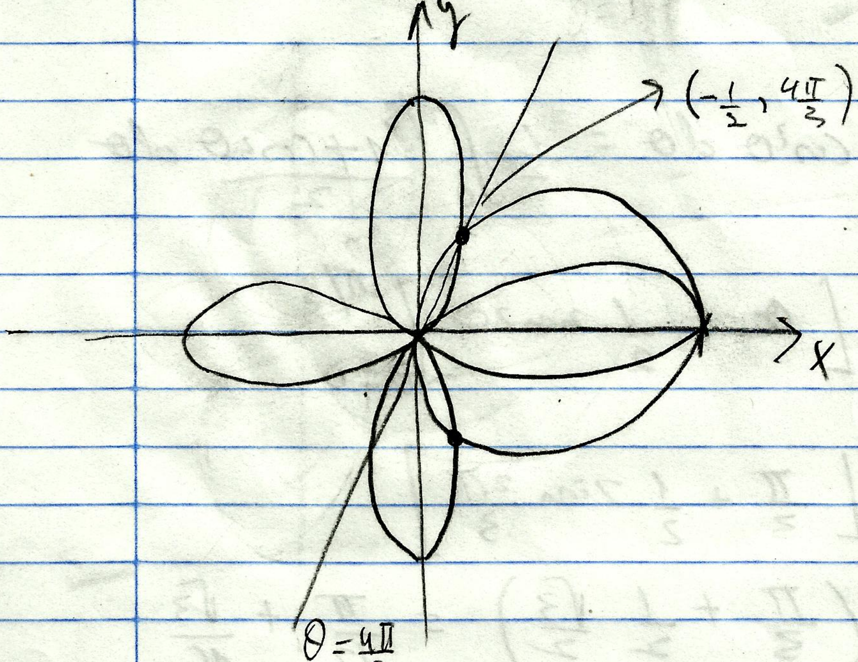
$\frac{3\pi}{2} \leq \theta \leq \frac{7\pi}{4}$ ∴ $3\pi \leq 2\theta \leq \frac{7\pi}{2}$ ∴ $-1 \leq \cos 2\theta \leq 0$

$\frac{7\pi}{4} \leq \theta \leq 2\pi$ ∴ $\frac{7\pi}{2} \leq 2\theta \leq 4\pi$ ∴ $0 \leq \cos 2\theta \leq 1$



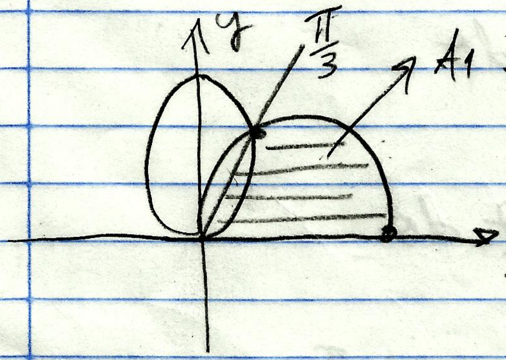
$r_1 = \cos\theta$

Daí:



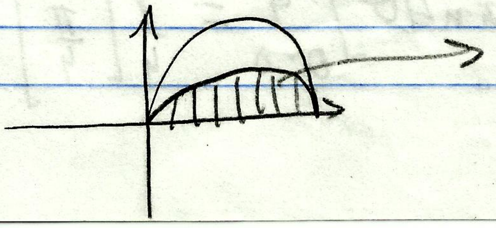
$$r_2\left(\frac{4\pi}{3}\right) = \cos\left(\frac{8\pi}{3}\right) =$$
$$= \cos\left(2\pi + \frac{2\pi}{3}\right)$$
$$= \cos\frac{2\pi}{3} = -\cos\frac{\pi}{3}$$
$$= -\frac{1}{2}$$

Seja então:



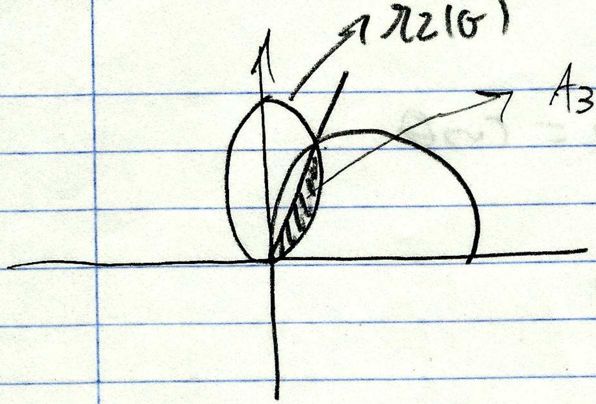
$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} r_1^2(\theta)\, d\theta$$

0.5



$$A_2 = \frac{1}{2}\int_0^{\frac{\pi}{4}} r_2^2(\theta)\, d\theta$$

0.5



$$A_3 = \frac{1}{2}\int_{\frac{5\pi}{4}}^{\frac{4\pi}{3}} r_2^2(\theta)\,d\theta$$

0.5

$$A = A_1 - A_2 - A_3$$

Mas

$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} \cos^2\theta\, d\theta = \frac{1}{2}\int_0^{\pi/3} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\theta=0}^{\pi/3}$$

$$= \frac{1}{4}\left[\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right]$$

$$= \frac{1}{4}\left(\frac{\pi}{3} + \frac{1}{2}\frac{\sqrt{3}}{2}\right) = \frac{\pi}{12} + \frac{\sqrt{3}}{16}$$

$$A_2 = \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos^2 2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{1+\cos 4\theta}{2}\, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{4}\sin 4\theta\right]_{\theta=0}^{\frac{\pi}{4}} = \frac{1}{4}\left[\frac{\pi}{4}\right] = \frac{\pi}{16}$$

$$A_3 = \frac{1}{2}\int_{\frac{5\pi}{4}}^{\frac{4\pi}{3}} \cos^2 2\theta \, d\theta$$

$$= \frac{1}{2}\int_{\frac{5\pi}{4}}^{\frac{4\pi}{3}} \frac{1+\cos 4\theta}{2} \, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{4}\sin 4\theta\right]_{\frac{5\pi}{4}}^{\frac{4\pi}{3}}$$

$$= \frac{1}{4}\left[\frac{4\pi}{3} + \frac{1}{4}\sin\frac{16\pi}{3} - \frac{5\pi}{4} - \frac{1}{4}\sin \cancel{5\pi}\right]$$

$\sin\frac{16\pi}{3} =$

$= \sin\frac{4\pi}{3} = -\sin\frac{\pi}{3}$

$$= \frac{1}{4}\left[\frac{\pi}{12} + \frac{1}{4}\left(-\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{\pi}{48} - \frac{\sqrt{3}}{32}$$

Dai

$$A = A_1 - A_2 - A_3$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{16} - \frac{\pi}{16} - \frac{\pi}{48} + \frac{\sqrt{3}}{32}$$

$$= \underbrace{\frac{4\pi - 3\pi - \pi}{48}}_{0} + \frac{\sqrt{3}}{16} + \frac{\sqrt{3}}{32}$$

$$A = \frac{3\sqrt{3}}{32}$$

3.

i) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3}{\sqrt{x^2+y^2}}$

Temos que

$$0 \leq \left| \frac{x^3}{\sqrt{x^2+y^2}} \right| \leq \left| \frac{x^3}{|x|} \right| = x^2 \qquad (*)$$

e

$$\lim_{(x,y)\to(0,0)} x^2 = 0 \qquad (**).$$

De $(*)$ e $(**)$ segue-se que

$$\lim_{(x,y)\to(0,0)} \left| \frac{x^3}{\sqrt{x^2+y^2}} \right| = 0$$

$$\therefore \boxed{\lim_{(x,y)\to(0,0)} \frac{x^3}{\sqrt{x^2+y^2}} = 0}$$

1,0

ii) $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

$\rightarrow$ Seja $x=y$. Temos

$$\lim_{\substack{(x,y)\to(0,0)\\(x=y)}} \frac{x^2}{x^2+y^2} = \lim_{x\to 0} \frac{\cancel{x^2}}{2\cancel{x^2}} = \frac{1}{2} \quad (*)$$

$\rightarrow$ Seja $x=0$. Temos

$$\lim_{\substack{(x,y)\to(0,0)\\(x=0)}} \frac{x^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} 0 = 0 \quad (**)$$

De $(*)$ e $(**)$ segue-se que

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} \nexists}$$

1,0

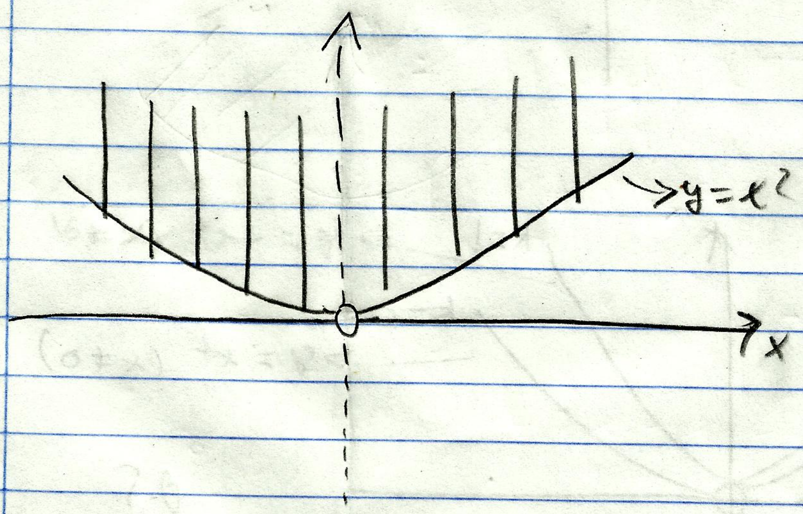
4. $f(x,y) = \sqrt{\dfrac{y}{x^2} - 1}$

i) devemos ter $\dfrac{y}{x^2} - 1 \geq 0$, $x \neq 0$

$\therefore \dfrac{y}{x^2} \geq 1 \quad \therefore y \geq x^2$

$\therefore$

$Dom f = \{ (x,y) \in \mathbb{R}^2 : y \geq x^2 , x \neq 0 \}$

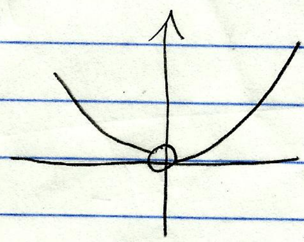


ii) $k=0$: $0 = \sqrt{\dfrac{y}{x^2} - 1}$

$\therefore \dfrac{y}{x^2} - 1 = 0 \quad \therefore y = x^2$

$x \neq 0$

$k=0$

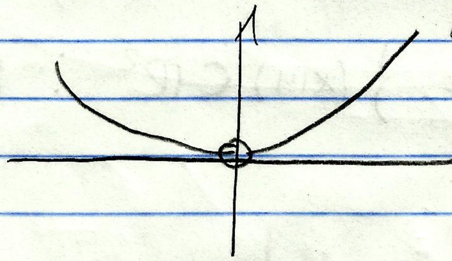


$k=1$: $1 = \sqrt{\frac{y}{x^2} - 1}$

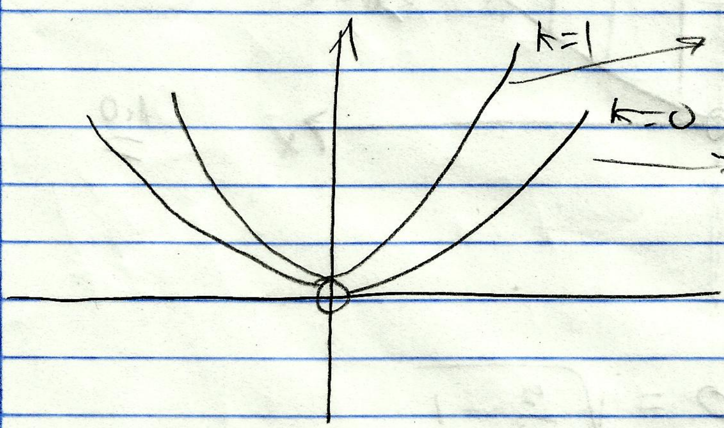
$\therefore \; 1 = \frac{y}{x^2} - 1$

$\therefore \; \frac{y}{x^2} = 2 \; , \; x \neq 0$

$\therefore \; y = 2x^2$



$k=1$

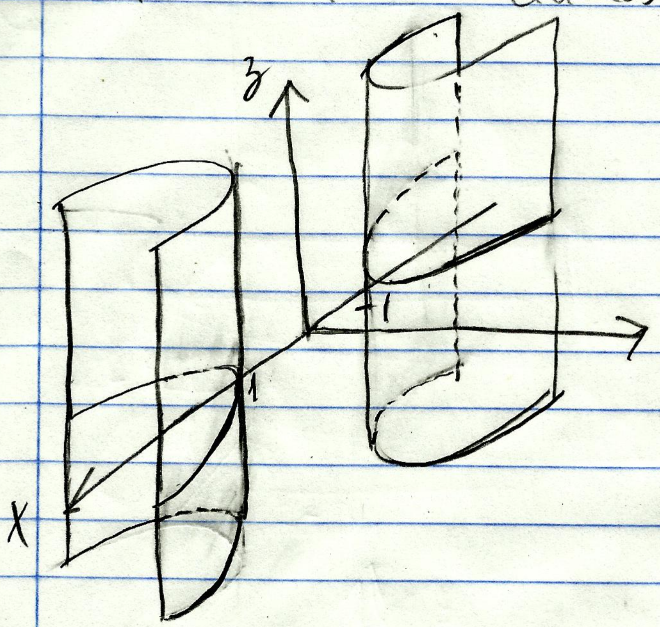


$k=1 \longrightarrow y = 2x^2 \; (x \neq 0)$

$k=0 \longrightarrow y = x^2 \; (x \neq 0)$

0.5

5. $1 = x^2 - y^2$. cilindro hiperbólico



0.5