Cálculo B - Prova 3

1. (i) Seja $yz = \ln(x+z)$ uma relação que define z como função implícita de x e y. Determine $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$. 1.0

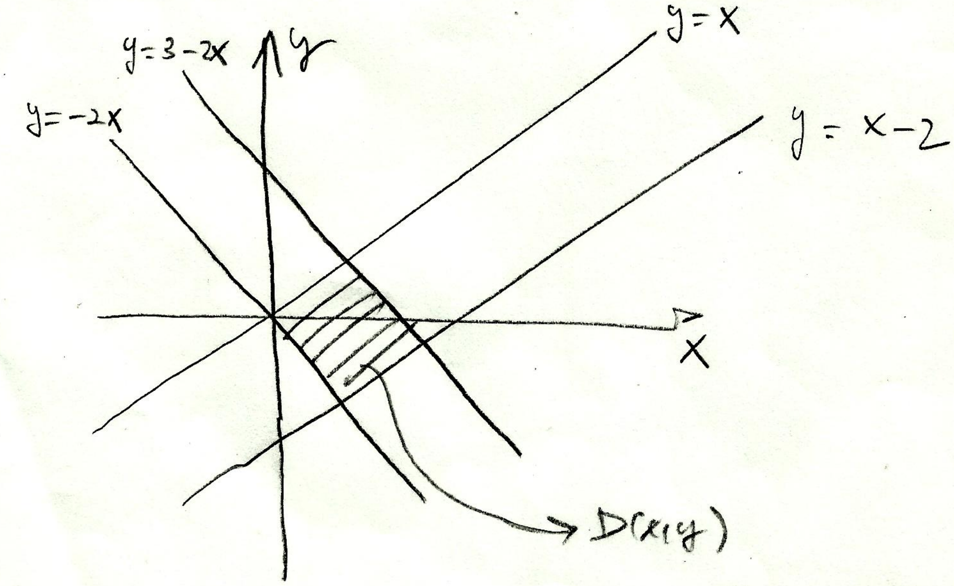
   (ii) Seja $z = \sin x \cos y$. Assuma que $x = \pi t$, $y = \sqrt{t}$. Usando a regra da cadeia determine $\frac{dz}{dt}$. 0.5

2. Determine os extremos absolutas de de $f(x, y) = x^2 - 3y + y^3$ no domínio $D = \{x^2 + 4y^2 \leq 36\}$ 2.5

3. Calcule $\int_D (3x+4y)\, dA$ fazendo uma mudança de variáveis de $(x, y) \to (u, v)$ tal que a região $D(x, y)$ mostrada na figura se transforme num retângulo no plano u, v. 2.0

4. Calcule $\int_\Omega \frac{1}{x^2+y^2+z^2}\, dV$ onde Ω é a região sólida acima do plano xy, limitada pelo cone $z = \sqrt{3x^2 + 3y^2}$ e as esferas $x^2 + y^2 + z^2 = 9$ e $x^2 + y^2 + z^2 = 81$. 2.0

Questão 3



1.

(i) $yz = \ln(x+z)$

$\rightarrow \frac{\partial}{\partial x}(yz) = \frac{\partial}{\partial x}\ln(x+z)$

$$y\frac{\partial z}{\partial x} = \frac{1}{(x+z)}\frac{\partial}{\partial x}(x+z)$$

$$= \frac{1}{(x+z)} \cdot \left(1 + \frac{\partial z}{\partial x}\right)$$

$$y\frac{\partial z}{\partial x} = \frac{1}{x+z} + \frac{1}{(x+z)}\frac{\partial z}{\partial x}$$

$\therefore$

$$\frac{\partial z}{\partial x}\left(y - \frac{1}{x+z}\right) = \frac{1}{x+z}$$

$$\frac{\partial z}{\partial x}\left(\frac{yx+yz-1}{x+z}\right) = \frac{1}{x+z}$$

$$\frac{\partial z}{\partial x} = \frac{1}{y(x+z)-1}$$

0.5

$\rightarrow \frac{\partial}{\partial y}(yz) = \frac{\partial}{\partial y}\ln(x+z)$

$$z + y\frac{\partial z}{\partial y} = \frac{1}{(x+z)} \cdot \frac{\partial z}{\partial y}$$

$\therefore$

$$\frac{\partial z}{\partial y}\left(y - \frac{1}{x+z}\right) = -z$$

$$\frac{\partial z}{\partial y}\left(\frac{y(x+z)-1}{x+z}\right) = -z$$

$$\therefore \quad \frac{\partial z}{\partial y} = \frac{-z(x+z)}{y(x+z)-1}$$

0.5

ii) $\begin{cases} z = \sin x \cos y \\ x = \pi t \; ; \; y = \sqrt{t} \end{cases}$

$$h(t) := z(x(t), y(t)) = \sin x(t) \cos y(t)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \cos x \cos y \, \pi + \sin x(-\sin y)\frac{1}{2\sqrt{t}}$$

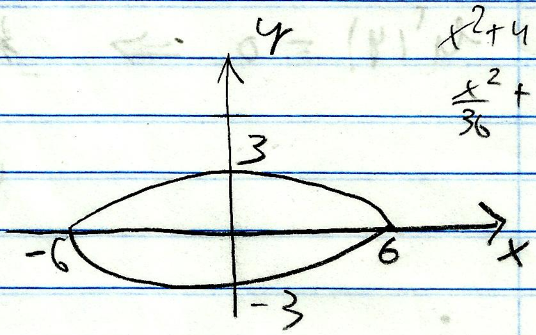
$$\frac{dz}{dt} = \pi \cos x \cos y - \frac{1}{2\sqrt{t}} \sin x \sin y$$

0.5

2. $\begin{cases} f(x,y) = x^2 - 3y + y^3 \\ D: x^2 + 4y^2 \leq 36 \end{cases}$

$x^2 + 4y^2 = 36$

$\frac{x^2}{36} + \frac{y^2}{9} = 1$



→ pontos críticos da f

$\begin{cases} f_x = 2x \rightarrow f_x = 0 \therefore 2x = 0 \\ f_y = -3 + 3y^2 \rightarrow f_y = 0 \therefore -3 + 3y^2 = 0 \end{cases}$

$\therefore \quad x = 0$

$y = \pm 1$

$\underline{\underline{(0,1)}}, \ \underline{\underline{(0,-1)}}$

$\underline{0.5}$

→ Análise da f em pontos da borda

$\begin{cases} f(x,y) = x^2 - 3y + y^3 \\ x^2 + 4y^2 = 36 \rightarrow x^2 = 36 - 4y^2 \end{cases}$

$\therefore$

$$h(y) = f(x,y)\Big|_{x^2 = 36 - 4y^2} = 36 - 4y^2 - 3y + y^3$$

$$h(y) = y^3 - 4y^2 - 3y + 36, \quad -3 \leq y \leq 3$$

$h'(y) = 3y^2 - 8y - 3$

$h'(y) = 0 \Rightarrow 3y^2 - 8y - 3 = 0$

$$y = \frac{8 \pm \sqrt{64+36}}{6}$$

$$= \frac{8 \pm \sqrt{100}}{6} = \frac{8 \pm 10}{6} \begin{matrix} \nearrow 3 \\ \searrow -\frac{1}{3} \end{matrix}$$

$y = 3$: $x^2 + 4y^2 = 36$ $\therefore$ $x^2 + 4 \cdot 9 = 36$

$x^2 = 0$ $\therefore$ $x = 0$

$\therefore$ $\underline{\underline{(0,3)}}$ $\quad$ $\underline{\underline{0.5}}$

$y = -\frac{1}{3}$: $x^2 + 4y^2 = 36$ $\therefore$ $x^2 + 4 \cdot \frac{1}{9} = 36$

$x^2 = 36 - \frac{4}{9} = \frac{320}{9}$

$x = \pm \frac{8\sqrt{5}}{3}$

$$\begin{array}{r|l} 320 & 2 \\ 160 & 2 \\ 80 & 2 \\ 40 & 2 \\ 20 & 2 \\ 10 & 2 \\ 5 & 5 \\ 1 & \end{array} = 2^6 \cdot 5$$

$\therefore$ $\underline{\underline{\left(-\frac{8\sqrt{5}}{3}, -\frac{1}{3}\right)}}, \underline{\underline{\left(+\frac{8\sqrt{5}}{3}; -\frac{1}{3}\right)}}$ $\quad$ $\underline{\underline{0.5}}$

Nos extremos do intervalo $[-3,3]$

$y = 3 \rightarrow x = 0$ $\therefore$ $(0,3)$

$y = -3 \rightarrow x = 0$ $\therefore$ $(0,-3)$ $\quad$ $\underline{\underline{0.5}}$

(x,y)	$f(x,y) = x^2 - 3y + y^3$
$(0,1)$	-2
$(0,-1)$	2
$\left(\frac{-8\sqrt{5}}{3}, -\frac{1}{3}\right)$	$986/27 \geq 36$
$\left(\frac{8\sqrt{5}}{3}, -\frac{1}{3}\right)$	$986/27 \geq 36$
$(0,-3)$	-18
$(0,3)$	18

$$\left\{\begin{array}{l} \text{Máximo } f = 986/27 \\ \text{Mínimo } f = -18 \end{array}\right]\ \underline{\underline{O.T}}$$

Obs: Análise dos pontos da borda usando o método dos multiplicadores de Lagrange

$$\begin{cases} f(x,y) = x^2 - 3y + y^3 \\ \phi(x,y) = x^2 + 4y^2 - 36 \end{cases}$$

$$\begin{cases} f_x = \lambda \phi_x & : \quad \cancel{2}x = \lambda \cancel{2}x \Rightarrow x(1-\lambda) = 0 \\ f_y = \lambda \phi_y & : \quad -3 + 3y^2 = 8\lambda y \\ \phi = 0 & : \quad x^2 + 4y^2 = 36 \end{cases}$$

$$\begin{cases} x = 0 \\ \text{ou} \\ \lambda = 1 \end{cases}$$

Seja $x = 0$:

$$\begin{cases} -3+3y^2 = 8\lambda y \\ x^2+4y^2 = 36 \end{cases} \longrightarrow \begin{aligned} &-3+3y^2 = 8\lambda y \\ &4y^2 = 36 \\ &\therefore y^2 = 9 \\ &// y = \pm 3 // \end{aligned}$$

$\therefore \underline{(0, 3)}, \underline{(0, -3)}$ 0.5

Seja $\lambda = 1$

$$\begin{cases} -3+3y^2 = 8\lambda y \\ x^2+4y^2 = 36 \end{cases} \longrightarrow -3+3y^2 = 8y$$

$$\therefore 3y^2 - 8y - 3 = 0$$

$$y = \frac{8 \pm \sqrt{64+36}}{6}$$

$y = 3$: $x^2+4y^2 = 36$ $\quad = \frac{8 \pm 10}{6} \begin{matrix} \nearrow 3 \\ \searrow -\frac{1}{3} \end{matrix}$

$\therefore x^2 + 36 = 36$

$\therefore x = 0$

$\therefore \underline{(0, 3)}$ 0.5

$y = -\frac{1}{3}$: $x^2 + 4\frac{1}{9} = 36$

$$x^2 = 36 - \frac{4}{9} = \frac{324-4}{9} = \frac{320}{9}$$

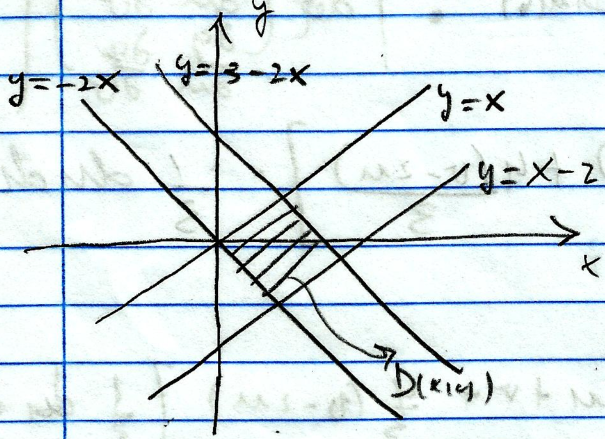
3

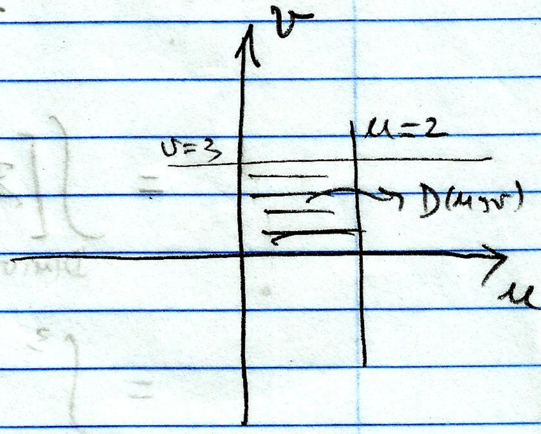
$$x = \pm \frac{8\sqrt{5}}{3}$$

$\therefore \underline{\left(\frac{8\sqrt{5}}{3}, -\frac{1}{3}\right)}, \underline{\left(-\frac{8\sqrt{5}}{3}, -\frac{1}{3}\right)}$

0.5

3. $\displaystyle\int_{D(x,y)} (3x+4y)\, dA$





Seja

$$\begin{cases} u := x-y & \begin{cases} x-y=0 \to u=0 \\ x-y=2 \to u=2 \end{cases} \\ v := y+2x & \begin{cases} y=-2x \to v=0 \\ y=3-2x \to v=3 \end{cases} \end{cases}$$

$\underline{0.5}$ $\begin{cases} u = x-y \\ v = y+2x \end{cases}$ $\therefore$ $u+v = 3x$ $\therefore$ $\left\| x = \dfrac{u+v}{3} \right\|$

$$y = x - u$$
$$= \frac{u+v}{3} - u = \frac{u+v-3u}{3}$$

$$\left\| y = \frac{v-2u}{3} \right\|$$

$$\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$$

$\underline{0.5}$

Dari

$$\int_{D(x,y)} (3x+4y)\, dA = \int_{D(u,v)} \Big(3x(u,v) + 4y(u,v)\Big) \cdot$$

$$\cdot \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| du\,dv$$

$$= \iint_{D(u,v)} \left[\cancel{3}\frac{(u+v)}{\cancel{3}} + 4\frac{(v-2u)}{3} \right] \cdot \frac{1}{3}\, du\,dv$$

$$= \int_{v=0}^{3} \int_{u=0}^{2} \left[u+v+ \frac{4}{3}(v-2u) \right] \frac{1}{3}\, du\, dv$$

$$= \int_{v=0}^{3} \int_{u=0}^{2} \left[u + v + \frac{4}{3}v - \frac{8u}{3} \right] \frac{1}{3}\, du\, dv$$

$$= \int_{v=0}^{3} \int_{u=0}^{2} \left(\frac{-5}{3}u + \frac{7}{3}v \right) \frac{1}{3}\, du\, dv$$

$$= \int_{v=0}^{3} \int_{u=0}^{2} \left(-\frac{5}{9}u + \frac{7}{9}v \right) du\, dv$$

$$= \int_{v=0}^{3} -\frac{5}{9}\frac{u^2}{2} + \frac{7}{9}vu \Big]_{u=0}^{2} dv$$

$$= \int_{v=0}^{3} \left(-\frac{5}{\cancel{18}\,9} \cdot \cancel{4}^{2} + \frac{14}{9}v \right) dv$$

$$= \int_{v=0}^{3} \left(-\frac{10}{9} + \frac{14}{9}v \right) dv$$

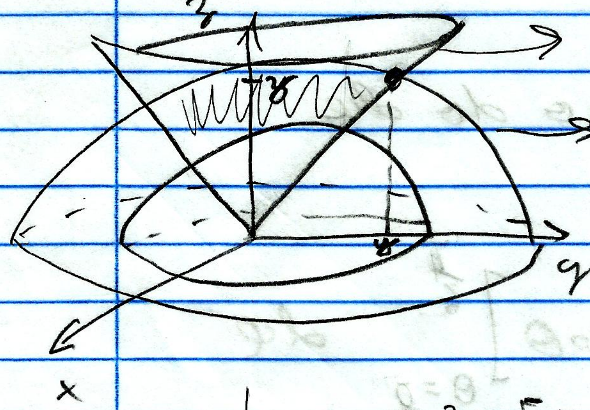
$$= -\frac{10}{9}v + \frac{\cancel{14}^{7}}{9}\frac{v^2}{\cancel{2}}\Bigg]_{v=0}^{3}$$

$$= -\frac{10}{\cancel{9}_3}\cdot\cancel{3} + \frac{7}{\cancel{9}}\cdot\cancel{9}$$

$$= -\frac{10}{3} + 7 = \frac{-10+21}{3} = \frac{11}{3}$$ //

$\underline{\underline{1.0}}$

4. $$\int_{\Omega} \frac{1}{x^2+y^2+z^2}\, dV$$

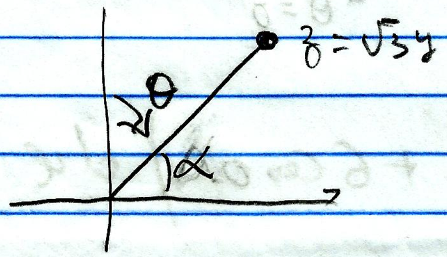


$\rightarrow z^2 = 3x^2+3y^2$

$\rightarrow x^2+y^2+z^2 = 81$

$x=0: \quad z^2 = 3y^2$

$\therefore \quad z = \sqrt{3}\, y$



$tg\, \alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$

$\therefore \quad \theta = \frac{\pi}{2} - \alpha = \frac{\pi}{2} - \frac{\pi}{3}$

$$\| \theta = \frac{\pi}{6} \|$$

Em coordenadas esféricas

$$\Omega : \begin{cases} 0 \leq \theta \leq \frac{\pi}{6} \\ 0 \leq \varphi \leq 2\pi \\ 3 \leq r \leq 9 \end{cases}$$

0.5

Daí

$$\int_{\Omega(x,y,z)} \frac{1}{x^2+y^2+z^2}\, dV = \int_{\Omega(r,\theta,\varphi)} \frac{1}{\cancel{r^2}} \cancel{r^2} \sin\theta \, dr d\theta d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{6}} \left(\int_{r=3}^{9} \sin\theta \, dr \right) d\theta \, d\varphi$$

0.5

$$= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{6}} \sin\theta \, r \Big]_{r=3}^{9} \, d\theta \, d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{6}} 6 \sin\theta \, d\theta \, d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{6}} -6\cos\theta \Big]_{\theta=0}^{\frac{\pi}{6}} \, d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \left(-6\cos\frac{\pi}{6} + 6\cos 0\right) d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \left(-\cancel{6}\,\frac{\sqrt{3}}{\cancel{2}} + 6\right) d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \left(-3\sqrt{3} + 6\right) d\varphi$$

$$= -3\sqrt{3}\,\varphi + 6\varphi \Big]_{\varphi=0}^{2\pi}$$

$$= -6\pi\sqrt{3} + 12\pi$$

$$= 6\pi\left(2 - \sqrt{3}\right)$$

1.0