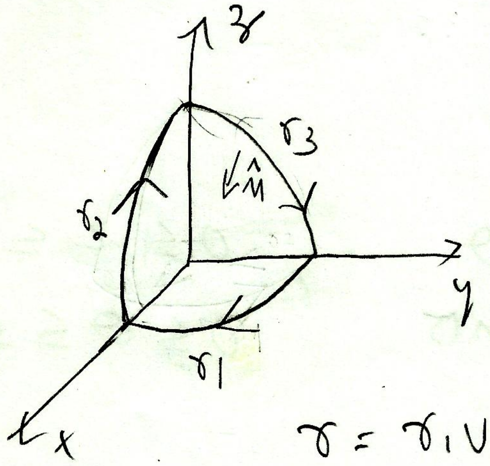
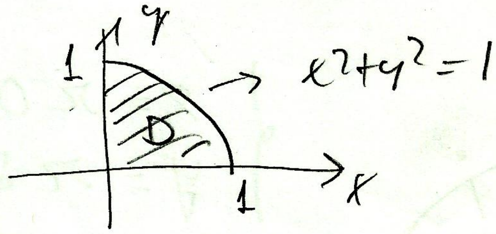
Cálculo C — Lista 8

1. $$\oint_{\gamma} \vec{F}\cdot d\vec{r} = \int_{\Omega} (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$$



$$\Omega : \; z = 1 - x^2 - y^2$$



$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$$

$$d\vec{S} = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right) dA$$

$$= (-2x, -2y, -1)\, dA$$

$$\vec{\nabla}\times\vec{F} = \epsilon_{ijk}\, \hat{e}_i\, \partial_j F_k$$

$$= \hat{e}_1(\partial_y F_3 - \partial_z F_2) + \hat{e}_2(\partial_z F_1 - \partial_x F_3)$$

$$+ \hat{e}_3(\partial_x F_2 - \partial_y F_x)$$

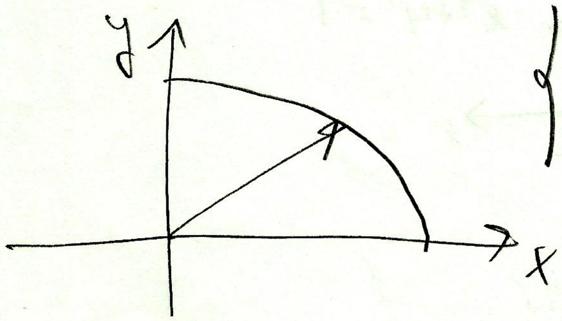
$$= \hat{i}(\partial_y y - \partial_z x) + \hat{j}(\partial_z z - \partial_x y)$$

$$+ \hat{k}(\partial_x x - \partial_y z)$$

$$= \hat{i} + \hat{j} + \hat{k}$$

$$\int_D (\hat{i}+\hat{j}+\hat{k}) \cdot (-2x\hat{i} - 2y\hat{j} - \hat{k})\, dA$$

$$= \int_D (-2x - 2y - 1)\, dx\, dy$$



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} ; \quad \begin{matrix} 0 \le r \le 1 \\ 0 \le \theta \le \frac{\pi}{2} \end{matrix}$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{\frac{\pi}{2}} (-2r\cos\theta - 2r\sin\theta - 1)\, r\, dr\, d\theta$$

$$= \int_{r=0}^{1} \left(-2r\sin\theta + 2r\cos\theta - \theta\right)\Big|_0^{\frac{\pi}{2}} r\, dr$$

$$= \int_{r=0}^{1} \left[-2r - \frac{\pi}{2} - (2r)\right] r\, dr$$

$$= \int_{r=0}^{1} \left(-4r - \frac{\pi}{2}\right) r\, dr$$

$$= \int_{r=0}^{1} -4r^2 - \frac{\pi}{2} r\, dr = \frac{-4r^3}{3}\Big|_0^1 - \frac{\pi}{2}\frac{r^2}{2}\Big|_0^1$$

$$= -\frac{4}{3} - \frac{\pi}{4}$$

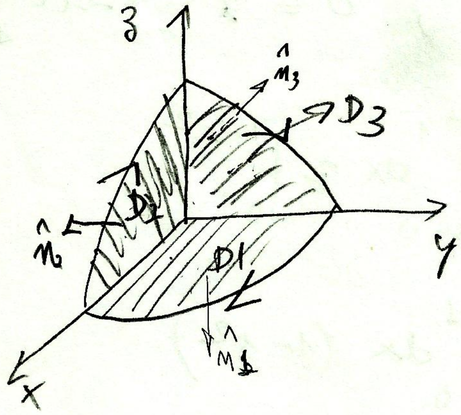
1. Cont.

Forma alternativa

Vimos $\gamma = \partial\Omega$ (paraboloide)

Podemos considerar γ a borda de

$$\gamma = D_1 \cup D_2 \cup D_3$$



$$\oint_\gamma \vec{F}\cdot d\vec{r} = \int_{D_1} \vec{\nabla}\times\vec{F}\cdot d\vec{S}_1 + \int_{D_2} \vec{\nabla}\times\vec{F}\cdot d\vec{S}_2 + \int_{D_3} \vec{\nabla}\times\vec{F}\cdot d\vec{S}_3$$

$$\int_{D_1} \vec{\nabla}\times\vec{F}\cdot d\vec{S}_1 = \int_{D_1} (\hat{\imath}+\hat{\jmath}+\hat{k})\cdot(0,0,-1)\, dS_1$$

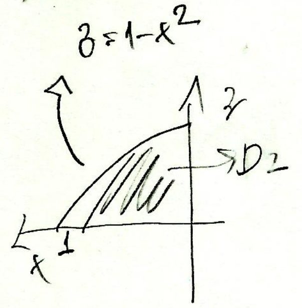
$$= \int_{D_1} -\,dS_1 = -\underbrace{\int_{D_1} dA_1}_{\frac{1}{4}\text{ Área do disco de raio 1}} = -\frac{1}{4}\pi \quad (*)$$

$D_1 = \{(x,y,z)\in\mathbb{R}^3 \mid z=0;\ x\geq 0,\ y\geq 0 \text{ e } x^2+y^2\leq 1\}$

$$dS_1 = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy = dx\,dy$$

$$\int_{D_2} \vec{\nabla}\times\vec{F} \cdot d\vec{S_2} = \int_{D_2} (\hat{i}+\hat{j}+\hat{k}) \cdot (0,-1,0)\, dS_2$$

$$= -\int_{D_2} dS_2 = -\int_{D_2} \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\, dx\,dz$$



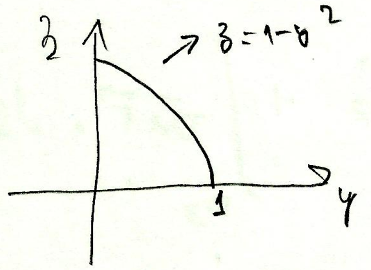
$$= -\int_{D_2} dx\,dz$$

$$= -\int_{x=0}^{1} dx \int_{z=0}^{1-x^2} dz$$

$$\Omega_2 = \{(x,y,z)\in\mathbb{R}^3 : y=0;\ 0\le x\le 1,\ 0\le z\le 1-x^2\}$$

$$= -\int_{x=0}^{1} dx\ z\Big]_0^{1-x^2}$$

$$= -\int_{x=0}^{1} dx\,(1-x^2)$$



$$= -x+\frac{x^3}{3}\Big]_0^1$$

$$= -1+\frac{1}{3} = -\frac{2}{3} \quad // \quad (**)$$

$$\int_{D_3} \vec{\nabla}\times\vec{F} \cdot d\vec{S_3} = \int_{D_3} (\hat{i}+\hat{j}+\hat{k}) \cdot (-1,0,0)\, dS_3$$

$$= -\int_{D_3} dS_3 = -\int_{D_3} \sqrt{1+\left(\frac{\partial x}{\partial z}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\, dy\,dz$$

$$\Omega_3 = \{(x,y,z)\in\mathbb{R}^3 : x=0,\ 0\le y\le 1,\ 0\le z\le 1-y^2\}$$

$$= -\int_{D_3} dy\,dz = -\frac{2}{3} \quad // \quad (***)$$

cálculo análogo

$$\therefore \oint_\gamma \vec{F}\cdot d\vec{r} = * + ** + *** = -\frac{\pi}{4} - \frac{4}{3} \quad //$$

ok!