Cálculo C - Prova 1

2.0

1. (i) Determine a equação vetorial que representa a curva de interseção das superfícies $z = x^2 + y^2$ e $x^2 + y^2 = 5$ orientada de modo que x aumenta no primeiro octante. 1.5
(ii) Faça um esboço da curva 0.5

1.5

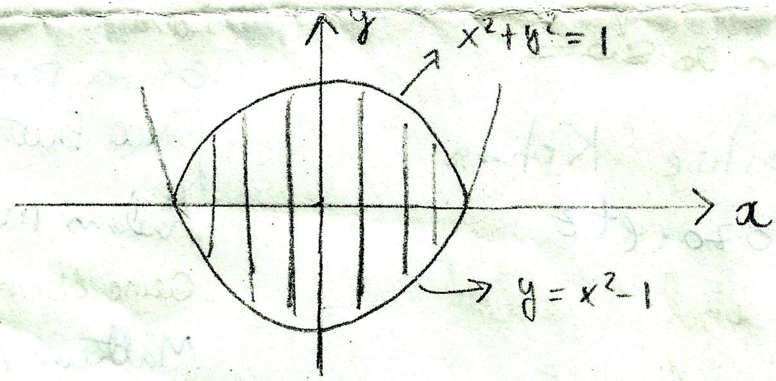
2. Seja $\vec{v} = \vec{\omega} \times \vec{r}$ onde $\vec{\omega}$ é um vetor constante e $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$. Mostre que $\vec{\omega} = \frac{1}{2}(\nabla \times \vec{v})$. 1.5

1.5

3. (i) Mostre que $\vec{F}(x,y,z) = 2xyz\ \hat{\imath} + x^2 z\ \hat{\jmath} + (x^2 y + 1)\ \hat{k}$ é campo conservativo. 1.0
(ii) Usando o teorema fundamental da integral de linha calcule $\int_\gamma \vec{F} \cdot d\vec{r}$ onde γ é a reta de equação vetorial $\vec{r}(t) = (1+t, t, 2t)$, $0 \leq t \leq 1$. 0.5

2.0

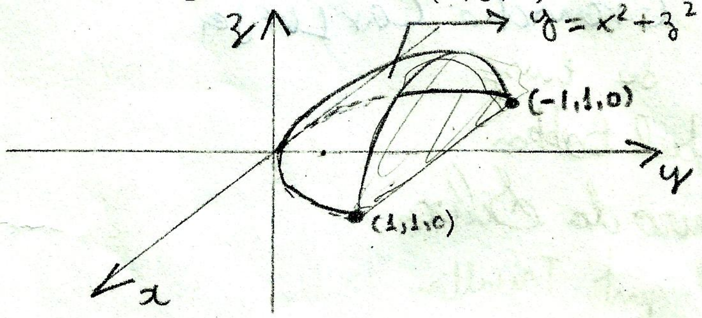
4. Calcule a área da região mostrada na figura usando uma integral de linha 1.0



2.0

5. Calcule a integral de superfície $\int_\Omega \vec{F} \cdot d\vec{S}$ onde Ω é a superfície mostrada na figura, sendo a orientação de Ω determinada por um vetor normal que aponta para fora da superfície e $\vec{F} = (x, y, z)$.

obs. Ω é a superfície fechada limitada por $\begin{cases} y = x^2 + z^2 \\ z = 0 \\ y = 1 \end{cases}$

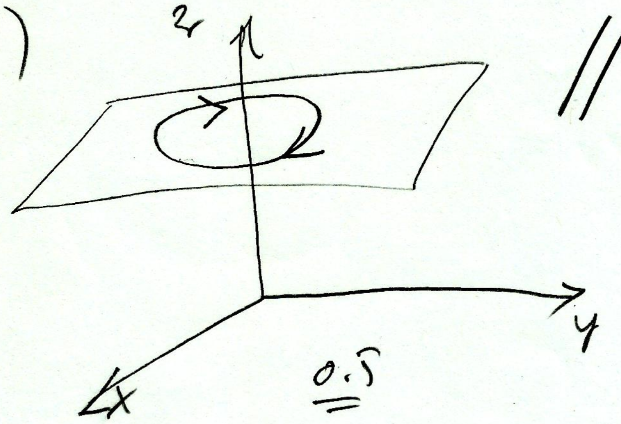


$A = \int_\gamma x\, dy$, $A = -\int_\gamma y dx$, $A = \frac{1}{2}\int_\gamma x dy - y dx$

1.

i) $\begin{cases} z = x^2+y^2 \\ x^2+y^2 = 5 \end{cases} \Longrightarrow \gamma: \begin{cases} z=5 \\ x=\sqrt{5}\sin t \\ y=\sqrt{5}\cos t \end{cases}, \ 0 \le t \le 2\pi$

ii)



$\underline{\underline{0.5}}$

$//\vec{r}(t) = (\sqrt{5}\sin t, \sqrt{5}\cos t, 5)//$

$0 \le t \le 2\pi$

$\underline{\underline{1.5}}$

2. $\vec{v} = \vec{\omega} \times \vec{r}$, $\vec{\omega} = cte.$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{v} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{pmatrix}$$

$$= \hat{i}(\omega_2 z - \omega_3 y) - \hat{j}(\omega_1 z - \omega_3 x) + $$
$$+ \hat{k}(\omega_1 y - \omega_2 x)$$

0.5

Dai

$$\vec{\nabla} \times \vec{v} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \omega_2 z - \omega_3 y & -\omega_1 z + \omega_3 x & \omega_1 y - \omega_2 x \end{pmatrix}$$

$$= \hat{i}\left[\partial_y(\omega_1 y - \omega_2 x) - \partial_z(-\omega_1 z + \omega_3 x)\right]$$
$$- \hat{j}\left[\partial_x(\omega_1 y - \omega_2 x) - \partial_z(\omega_2 z - \omega_3 y)\right]$$
$$+ \hat{k}\left[\partial_x(-\omega_1 z + \omega_3 x) - \partial_y(\omega_2 z - \omega_3 y)\right]$$

$$= \hat{i}\left[\omega_1 + \omega_1\right] - \hat{j}\left[-\omega_2 - \omega_2\right] + \hat{k}\left[\omega_3 + \omega_3\right]$$

$$= 2\omega_1 \hat{i} + 2\omega_2 \hat{j} + 2\omega_3 \hat{k}$$

$$= 2\vec{\omega}$$

$$\therefore \quad \vec{\omega} = \frac{1}{2} \vec{\nabla} \times \vec{v}$$

(1.5)

3

i) Verifique se $\vec{F}(x,y,z) = 2xyz\,\hat{i} + x^2z\,\hat{j} + (x^2y+1)\hat{k}$ é campo gradiente.

$$\vec{F} = \nabla\varphi$$

$$2xyz = \partial_x\varphi \implies \varphi = x^2yz + h(y,z)$$

$$x^2z = \partial_y\varphi \implies x^2z = \partial_y(x^2yz + h(y,z))$$

$$x^2y+1 = \partial_z\varphi$$

$$\cancel{x^2z} = \cancel{x^2z} + \partial_y h(y,z)$$

$$0 = \partial_y h(y,z)$$

$$\therefore \quad h = h(z)$$

$$\varphi(x,y,z) = x^2yz + h(z)$$

$$\Downarrow$$

$$x^2y+1 = \partial_z\varphi$$

$$x^2y+1 = \partial_z(x^2yz + h(z))$$

$$\cancel{x^2y}+1 = \cancel{x^2y} + \frac{dh}{dz}$$

$$1 = \frac{dh}{dz} \implies h(z) = z.$$

$$\therefore \quad \varphi(x,y,z) = x^2yz + z + k$$

ii) Calcule $\int_\gamma \vec{F}\cdot d\vec{r}$ onde γ é a reta de equação $\vec{r}(t) = (1+t, t, 2t)$, $0 \le t \le 1$.

$$\begin{cases} \vec{r}(0) = (1,0,0) \\ \vec{r}(1) = (2,1,2) \end{cases}$$

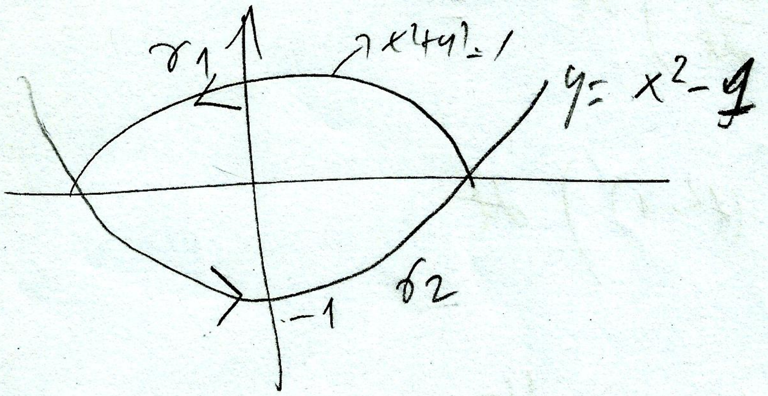
$$\int_\gamma \vec{F}\cdot d\vec{r} = \varphi(\vec{r}(1)) - \varphi(\vec{r}(0))$$

$$= \varphi(2,1,2) - \varphi(1,0,0)$$

$$= 8+2+k-0-k$$

$$= 10$$

4.



$$\gamma_1 : \begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases}, \quad 0 \le t \le \pi \quad \underline{0.5}$$

$$\gamma_2 : \begin{cases} x = t \\ y = t^2-1 \end{cases}, \quad -1 \le t \le 1 \quad \underline{\underline{0.5}}$$

$$A = \frac{1}{2}\int_{\gamma_1 \cup \gamma_2} x\,dy - y\,dx$$

Mas

$$\frac{1}{2}\int_{\gamma_1} x\,dy - y\,dx = \frac{1}{2}\int_0^{\pi} \left(x(t)\frac{dy(t)}{dt} - y(t)\frac{dx(t)}{dt} \right) dt$$

$$= \frac{1}{2}\int_0^{\pi} \left(\cos t \cos t - \sin t\,(-)\sin t \right) dt$$

$$= \frac{1}{2}\int_0^{\pi} (\cos^2 t + \sin^2 t)\,dt$$

$$= \frac{1}{2}\int_0^{\pi} dt = \frac{1}{2}t\Big]_0^{\pi} = \frac{\pi}{2} \quad \underline{0.5}$$

$$\frac{1}{2}\int_{\sigma_2}\left(x(t)\frac{dy}{dt} - y(t)\frac{dx}{dt}\right)dt =$$

$$= \frac{1}{2}\int_{-1}^{1}\left(t\,2t - (t^2-1)\right)dt$$

$$= \frac{1}{2}\int_{-1}^{1}\left(2t^2 - t^2 + 1\right)dt$$

$$= \frac{1}{2}\int_{-1}^{1}(t^2+1)\,dt = \frac{1}{2}\left(\frac{t^3}{3}+t\right)\Big]_{t=-1}^{1}$$

$$= \frac{1}{2}\left(\frac{1}{3}+1\right) - \frac{1}{2}\left(-\frac{1}{3}-1\right)$$

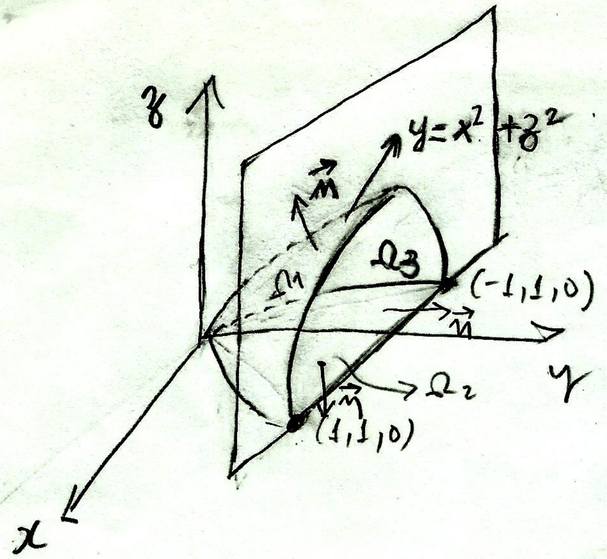
$$= \cancel{\frac{1}{2}}\,\frac{\cancel{4}^{2}}{3} + \cancel{\frac{1}{2}}\,\frac{\cancel{4}^{2}}{3} = \frac{4}{3}$$

O.T

Daí

$$\boxed{A = \frac{\pi}{2} + \frac{4}{3}}$$

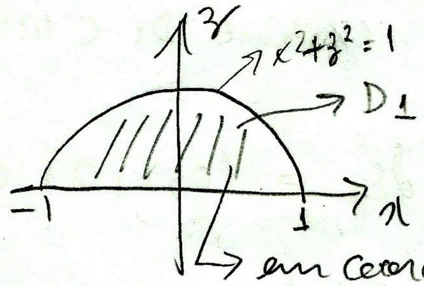
5.



$$\left\{\begin{array}{l} \vec{F}(x,y,z) = (x,y,z) \\ \int_{\Omega} \vec{F}\cdot d\vec{S} \end{array}\right.$$

$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$

$\rightarrow \underline{\underline{\Omega_1}} = \{ (x,y,z) \in \mathbb{R}^3 ; \ y = x^2 + z^2, \ (x,z) \in D_1 \subset \mathbb{R}^2 \}$



$\rightarrow$ em coordenadas polares:

$D_1: 0 \leq r \leq 1, \ 0 \leq \theta \leq \pi$

$$d\vec{S_1} = \left(\frac{\partial y}{\partial x}, -1, \frac{\partial y}{\partial z}\right) dx\,dz$$

$$= (2x, -1, 2z)\, dx\,dz$$

0.5

$$\int_{\Omega_1} \vec{F}\cdot d\vec{S}_1 = \int_{D_1(x,z)} \left(2x^2 - y(x,z) + 2z^2\right) dx\,dz$$

$$= \int_{D_1(x,z)} (2x^2 - x^2 - z^2 + 2z^2)\, dx\,dz$$

$$= \int_{D_1(x,z)} \underbrace{(x^2+z^2)}\, \underbrace{dx\,dz}$$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{1} \underbrace{r^2}\, r\,dr\,d\theta =$$

$$= \int_{\theta=0}^{\pi}\int_{r=0}^{1} r^3 \, dr \, d\theta$$

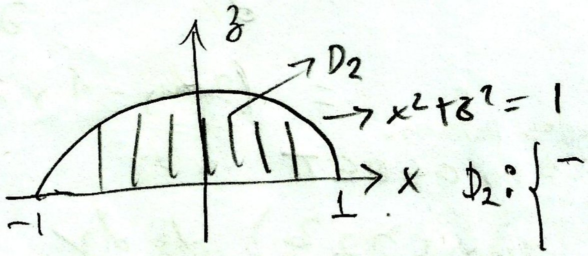
$$= \int_{\theta=0}^{\pi} \left.\frac{1}{4} r^4\right]_0^1 d\theta$$

$$= \int_{\theta=0}^{\pi} \frac{1}{4}\, d\theta = \left.\frac{1}{4}\theta\right]_0^{\pi} = \frac{\pi}{4}$$

O.T

$\rightarrow \underline{\underline{\Omega_2}}$: $\{ (x,y,z) \in \mathbb{R}^3 : y = 1 , (x,z) \in D_2 \subset \mathbb{R}^2 \}$

$$d\vec{S_2} = (0,1,0)\, dx\, dz$$



$D_2 : \begin{cases} -1 \leq x \leq 1 \\ 0 \leq z \leq \sqrt{1-x^2} \end{cases}$

O.T

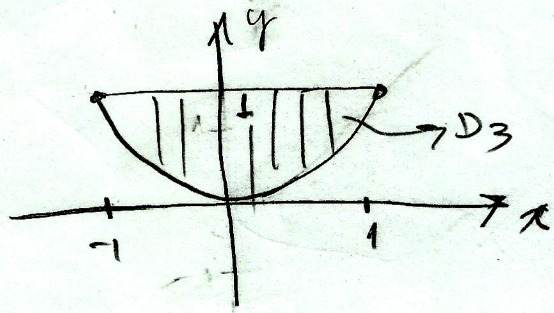
$$\int_{\Omega_2} \vec{F} \cdot d\vec{S_2} = \int_{D_2(x,z)} (x,y,z) \cdot (0,1,0)\, dx\, dz$$

$$= \int_{D_2(x,z)} \underbrace{y(x,z)}_{1}\, dx\, dz$$

$$= \int_{D_2(x,z)} 1 \, dx\, dz = \frac{\pi}{2}$$ (área de D_2)

O.T

$\rightarrow \underline{\Omega_3} = \{(x,y,z) \in \mathbb{R}^3 : z=0, (x,y)\in D_3 \subset \mathbb{R}^2\}$



$d\vec{S_3} = (0,0,-1)\,dx\,dy$

$D_3 : \begin{cases} -1 \le x \le 1 \\ x^2 \le y \le 1 \end{cases}$ 0.5

$$\int_{\Omega_3} \vec{F}\cdot d\vec{S_3} = \int_{D_3(x,y)} (x,y,z)\cdot(0,0,-1)\,dx\,dy$$

$$= \int \underbrace{-z}\,dx\,dy$$

$$= \int \underbrace{0}\,dx\,dy = 0$$ 0.5

Daí:

$$\int_{\Omega} \vec{F}\cdot d\vec{S} = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4} \quad //$$