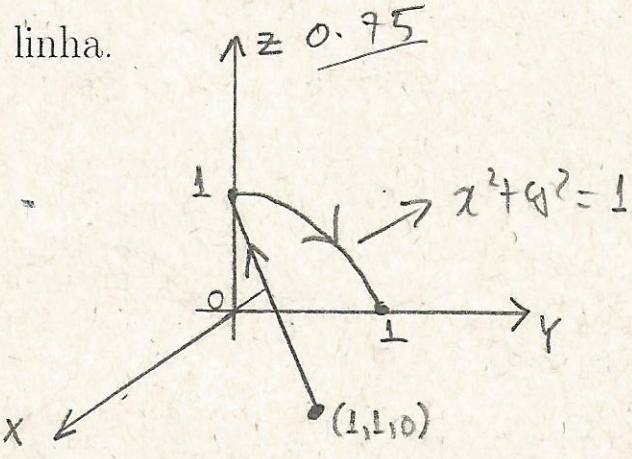
Cálculo C - Prova 1

1.25 1. Mostre que $\vec{\nabla} \times (\varphi \vec{F}) = \varphi \vec{\nabla} \times \vec{F} + (\nabla \varphi) \times \vec{F}$

2.75 2. i) Calcule a integral de linha $\int_\gamma \vec{F} \cdot d\vec{r}$ onde γ é a curva mostrada na figura e $\vec{F} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$. 2.0

ii) Calcule a mesma integral de linha usando o teorema fundamental da integral de linha. 0.75



1.0 3. Use o teorema de Green e calcule a integral de linha $\int_\gamma y^3 dx - x^3 dy$, onde γ é o círculo de equação $x^2 + y^2 = 4$. 1.0
e orientada no sentido anti-horário

2.0 4. Calcule $\int_\sigma \vec{F} \cdot d\vec{r}$ onde $\vec{F} = x\hat{\imath} - z\hat{k}$ e σ é a superfície formada por parte do parabolóide $x = y^2 + z^2$ com $0 \leq x \leq 1$, e pelo disco $y^2 + z^2 \leq 1, x = 1$. Assuma σ orientada com vetor normal apontando p/ fora.

2.0

1. Seja

$$\vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\imath} + F_2(x,y,z)\,\hat{\jmath} + F_3(x,y,z)\,\hat{k}$$

$\varphi(x,y,z)$ (campo escalar)

Temos

$$\vec{\nabla} \times (\varphi \vec{F}) = \det \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \varphi F_1 & \varphi F_2 & \varphi F_3 \end{pmatrix}$$

$$= \left[\partial_y(\varphi F_3) - \partial_z(\varphi F_2)\right]\hat{\imath} + \left[\partial_z(\varphi F_1) - \partial_x(\varphi F_3)\right]\hat{\jmath} + \left[\partial_x(\varphi F_2) - \partial_y(\varphi F_1)\right]\hat{k}$$

$$= \left[\partial_y\varphi\, F_3 + \varphi\,\partial_y F_3 - \partial_z\varphi\, F_2 - \varphi\,\partial_z F_2\right]\hat{\imath} + \left[\partial_z\varphi\, F_1 + \varphi\,\partial_z F_1 - \partial_x\varphi\, F_3 - \varphi\,\partial_x F_3\right]\hat{\jmath} + \left[\partial_x\varphi\, F_2 + \varphi\,\partial_x F_2 - \partial_y\varphi\, F_1 - \varphi\,\partial_y F_1\right]\hat{k}$$

$$= \left(\underline{\partial_y\varphi}\, F_3 - \partial_z\varphi\, F_2\right)\hat{\imath} + \varphi\left(\partial_y F_3 - \partial_z F_2\right)\hat{\imath} + \left(\partial_z\varphi\, F_1 - \underline{\partial_x\varphi}\, F_3\right)\hat{\jmath} + \varphi\left(\partial_z F_1 - \partial_x F_3\right)\hat{\jmath} + \left(\underline{\partial_x\varphi}\, F_2 - \underline{\partial_y\varphi}\, F_1\right)\hat{k} + \varphi\left(\partial_x F_2 - \partial_y F_1\right)\hat{k}$$

$$
\begin{aligned}
&= \partial_x \varphi \left(F_2 \hat{k} - F_3 \hat{j} \right) + \partial_y \varphi \left(F_3 \hat{i} - F_1 \hat{k} \right) \\
&\quad + \partial_z \varphi \left(F_1 \hat{j} - F_2 \hat{i} \right) + \\
&\quad + \varphi \left\{ \left(\partial_y F_3 - \partial_z F_2 \right) \hat{i} + \left(\partial_z F_1 - \partial_x F_3 \right) \hat{j} \right. \\
&\qquad \left. + \left(\partial_x F_2 - \partial_y F_1 \right) \hat{k} \right\}
\end{aligned}
\qquad (*)
$$

0.75

Mas

$$
(\vec{\nabla}\varphi) \times \vec{F} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x \varphi & \partial_y \varphi & \partial_z \varphi \\ F_1 & F_2 & F_3 \end{pmatrix}
$$

$$
\begin{aligned}
&= \hat{i} \left(\partial_y \varphi\, F_3 - \partial_z \varphi\, F_2 \right) \\
&\quad + \hat{j} \left(\partial_z \varphi\, F_1 - \partial_x \varphi\, F_3 \right) \\
&\quad + \hat{k} \left(\partial_x \varphi\, F_2 - \partial_y \varphi\, F_1 \right) \\
&= \partial_x \varphi \left(F_2 \hat{k} - F_3 \hat{j} \right) + \partial_y \varphi \left(F_3 \hat{i} - F_1 \hat{k} \right) \\
&\quad + \partial_z \varphi \left(F_1 \hat{j} - F_2 \hat{i} \right)
\end{aligned}
\qquad (**)
$$

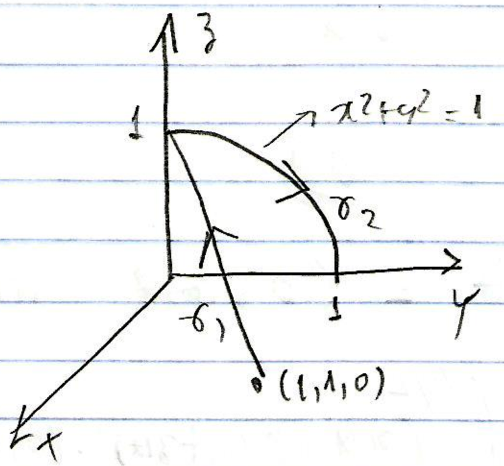
0.5

Daí, usando $(**)$ em $(*)$ temos:

$$
\boxed{\vec{\nabla} \times (\varphi \vec{F}) = (\vec{\nabla}\varphi) \times \vec{F} + \varphi\, \vec{\nabla} \times \vec{F}}
$$

2.

i)



$\gamma = \gamma_1 \cup \gamma_2$

$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$

$\gamma_1 : \begin{cases} x(t) = 1-t \\ y(t) = 1-t \\ z(t) = t \\ 0 \le t \le 1 \end{cases}$ $\underline{0.5}$

$d\vec{r}_1 = \vec{r}_1'(t)\,dt = (-1,-1,1)\,dt$

$\int_{\gamma_1} \vec{F} \cdot d\vec{r}_1 = \int_0^1 (x(t), y(t), z(t)) \cdot (-1,-1,1)\,dt$

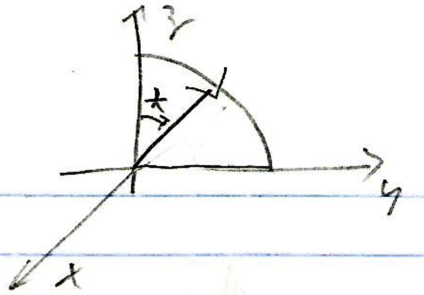
$= \int_0^1 (-x(t) - y(t) + z(t))\,dt$

$= \int_0^1 (-1+t-1+t+t)\,dt$

$= \int_0^1 (-2+3t)\,dt$

$= -2t + 3\frac{t^2}{2} \Big]_{t=0}^{1} = -2 + \frac{3}{2} = -\frac{1}{2}$

$\underline{0.5}$



$$\gamma_2 : \begin{cases} x(t) = 0 \\ y(t) = \sin t \\ z(t) = \cos t \\ 0 \le t \le \frac{\pi}{2} \end{cases}$$

0.1

$$d\vec{r}_2 = \vec{r}_2'(t)\,dt = (0, \cos t, -\sin t)\,dt$$

$$\int_{\gamma_2} \vec{F}\cdot d\vec{r}_2 = \int_0^{\pi/2} (x(t), y(t), z(t))\cdot(0, \cos t, -\sin t)\,dt$$

$$= \int_0^{\pi/2} \left(y(t)\cos t - z(t)\sin t\right) dt$$

$$= \int_0^{\pi/2} \left(\sin t\cos t - \cos t\sin t\right) dt$$

$$= \int_0^{\pi/2} 0\,dt$$

$$= 0$$

0.1

$\therefore$

$$\int_{\gamma} \vec{F}\cdot d\vec{r} = \int_{\gamma_1} \vec{F}\cdot d\vec{r}_1 + \int_{\gamma_2} \vec{F}\cdot d\vec{r}_2$$

$$= -\frac{1}{2} + 0$$

$$= -\frac{1}{2}$$

(1)

$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{F} = \nabla\varphi$$

$\therefore$
$$\begin{cases} \partial_x \varphi = x & \longrightarrow \quad \varphi = \frac{x^2}{2} + h(y,z) \\ \partial_y \varphi = y & \longrightarrow \quad \partial_y\left(\frac{x^2}{2} + h(y,z)\right) = y \\ \partial_z \varphi = +z & \quad\quad \partial_y h(y,z) = y \end{cases}$$

$$h(y,z) = \frac{1}{2}y^2 + k(z)$$

$\therefore$

$$\varphi = \frac{x^2}{2} + \frac{y^2}{2} + k(z)$$

0.5

$$\partial_z \varphi = + z$$

$$\partial_z\left(\frac{1}{2}x^2 + \frac{1}{2}y^2 + k(z)\right) = +z$$

$$\partial_z k = +z$$

$\therefore$

$$k(z) = +\frac{1}{2}z^2 + C$$

$$\therefore \quad \varphi(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 + C$$

Temos

$$\int_\gamma \vec{F}\cdot d\vec{r} = \varphi(0,1,0) - \varphi(1,1,0)$$

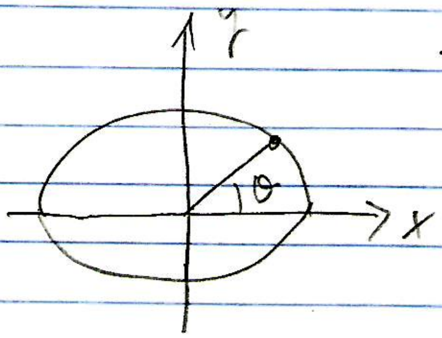
$$= \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = -\frac{1}{2}$$

0.25

3. $\int_\gamma y^3 dx - x^3 dy = \int_D \left(\frac{\partial(-x^3)}{\partial x} - \frac{\partial}{\partial y} y^3 \right) dA$

$$(*) = \int_D (-3x^2 - 3y^2)\, dA$$

0.25



$x = r\cos\theta$
$y = r\sin\theta$

$\begin{cases} 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{cases}$

0.25

$$(*) = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} -3r^2 \; r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\int_{r=0}^{2} -3r^3\, dr \right) d\theta$$

$$= \int_{\theta=0}^{2\pi} -\frac{3r^4}{4} \Big]_{r=0}^{2} d\theta$$

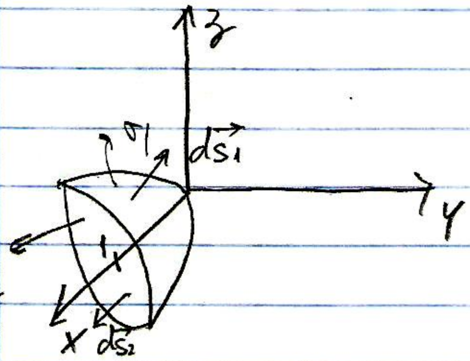
$$= \int_{\theta=0}^{2\pi} -3 \cdot \frac{\cancel{16}^{\,4}}{\cancel{4}}\, d\theta = \int_{\theta=0}^{2\pi} -12\, d\theta$$

$$= -12\theta \Big]_{\theta=0}^{2\pi}$$

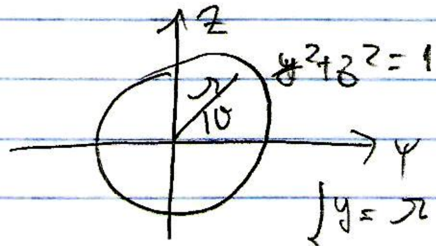
$$= -24\pi$$

0.5

4.



$$\vec{F} = x\hat{\imath} - z\hat{k}$$



$$\begin{cases} y = r\cos\theta \\ z = r\sin\theta \end{cases}$$

$$\sigma_1 : \begin{cases} y = y \\ z = z \\ x = y^2 + z^2 \\ (y,z) \in D_1 : \quad y^2 + z^2 \leq 1 \end{cases} \qquad \underline{0.25}$$

$$d\vec{S_1} = \left(\pm 1, \mp \frac{\partial x}{\partial y}, \mp \frac{\partial x}{\partial z} \right) dy\,dz$$

$$= (\pm 1, \mp 2y, \mp 2z)\, dy\,dz$$

$$\therefore \; d\vec{S_1} = (-1, 2y, 2z)\, dy\,dz \qquad \underline{0.25}$$

$$\int_{\sigma_1} \vec{F} \cdot d\vec{S_1} = \int_{D_1} (x(y,z), 0, -z) \cdot (-1, 2y, 2z)\, dy\,dz$$

$$= \int_{D_1} (-x(y,z) - 2z^2)\, dy\,dz$$

$$= \int_{D_1} (-y^2 - z^2 - 2z^2)\, dy\,dz$$

$$= \int_{D_1} (-y^2 - 3z^2)\, dy\,dz$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (-r^2\cos^2\theta - \underbrace{3r^2\sin^2\theta}) r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (\underbrace{-r^2\cos^2\theta - r^2\sin^2\theta} - 2r^2\sin^2\theta) r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (-r^2 - 2r^2\sin^2\theta) r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (-r^3 - 2r^3\sin^2\theta)\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} -\frac{r^4}{4} - \cancel{2}\frac{r^4}{\cancel{4}}\sin^2\theta \Big]_{r=0}^{1} d\theta$$

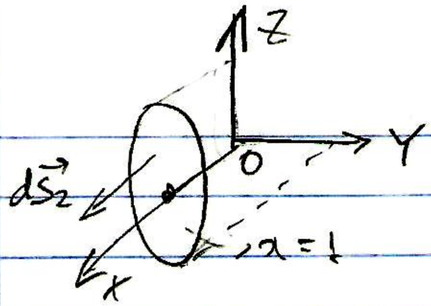
$$= \int_{\theta=0}^{2\pi} \left(-\frac{1}{4} - \frac{1}{2}\sin^2\theta\right) d\theta$$

$$= -\frac{1}{4}\theta\Big]_0^{2\pi} - \frac{1}{2}\int_{\theta=0}^{2\pi} \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= -\frac{1}{4}2\pi - \frac{1}{4}\left(\theta - \frac{1}{2}\cancel{\sin 2\theta}^{\,0}\right)\Big|_{\theta=0}^{2\pi}$$

$$= -\frac{\pi}{2} - \frac{1}{4}(2\pi) = -\frac{\pi}{2} - \frac{\pi}{2} = -\pi$$

0.5



$$\sigma_2 : \begin{cases} y = y \\ z = z \\ x = f(y,z) = 1 \end{cases}$$

$(y,z) \in D_2 : \; y^2 + z^2 \leq 1$ 0.25

$$d\vec{S_2} = \left(+1, -\frac{\partial x}{\partial y}, -\frac{\partial x}{\partial z} \right) dydz$$

$$\therefore \; d\vec{S_2} = (1, 0, 0)\, dydz \qquad 0.25$$

$$\int_{\sigma_2} \vec{F} \cdot d\vec{S_2} = \int_{D_2} (x(y,z), 0, -z) \cdot (1, 0, 0)\, dydz$$

$$= \int_{D_2} x(y,z)\, dydz$$

$$= \int_{D_2} 1\, dydz = \text{Área do } D_2$$

$$= \pi \qquad 0.5$$

Daí

$$\int_{\sigma} \vec{F} \cdot d\vec{S} = \int_{\sigma_1} \vec{F} \cdot d\vec{S_1} + \int_{\sigma_2} \vec{F} \cdot d\vec{S_2}$$

$$= -\pi + \pi$$

$$= 0$$