

Cálculo C - Prova 2

1. ~~Use o teorema de Stokes para calcular $\oint_{\sigma} \vec{F} \cdot d\vec{S}$ onde σ é a superfície do cilindro de equação $x^2 + y^2 = 1$, com $0 \leq z \leq 1$, e orientada com vetor normal apontando para fora do cilindro,~~

~~e $\vec{F} = (-yz, -xz, xy)$.~~

Use o teorema de Gauss para calcular o volume da esfera.

2. Resolva fazendo a substituição $u = 1 + xy$

$$\frac{dy}{dx} = \frac{y(1 + xy)}{x(1 - xy)}$$

3. Resolva

$$(3x + 2y^2)dx + 2xy dy = 0$$

4. Resolva usando o método da variação dos parâmetros

$$y' \sin x - y \cos x = \sin^2 x$$

5. Resolva usando o método dos coeficientes a determinar

$$y'' + y = \cos x + 3x$$

Fórmulas

- $Mdx + Ndy = 0$

Fator integrante dependente de x : $\frac{1}{N(x)} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \equiv f(x) \therefore F(x) = e^{\int f(x) dx}$.

Fator integrante dependente de y : $\frac{1}{M(y)} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \equiv g(y) \therefore F(y) = e^{\int g(y) dy}$.

$$1. \int_S \vec{E} \cdot d\vec{S} = \int_V \text{div } \vec{E} \, dV$$

Seja $\vec{E} = (x, y, z)$

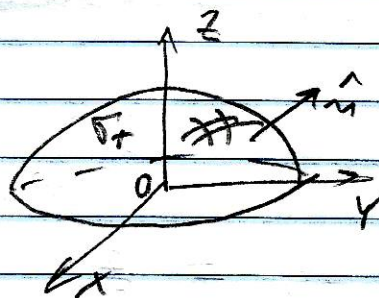
$$\vec{\nabla} \cdot \vec{E} = 3$$

Então

$$V = \frac{1}{3} \int_S \vec{E} \cdot d\vec{S}$$

$$= \frac{1}{3} \int_S (x, y, z) \cdot d\vec{S}$$

$\left. \begin{array}{l} \text{Superior} \\ \text{hemisfério} \end{array} \right\} \begin{array}{l} z = \sqrt{R^2 - x^2 - y^2} \\ x^2 + y^2 \leq R^2 \end{array}$



$$d\vec{S} = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right) dx dy$$

$$= \left(\frac{x}{\sqrt{R^2 - x^2 - y^2}}, \frac{y}{\sqrt{R^2 - x^2 - y^2}}, 1 \right) dx dy$$

$$\text{Volume hemisfério} = \frac{1}{2} V$$

∴

$$V = 2 \underbrace{(\text{Volume do hemisfério})}$$

$$= 2 \cdot \frac{1}{3} \int_{D(x,y)} (x,y,z) \cdot \left(\frac{x}{\sqrt{R^2-x^2-y^2}}, \frac{y}{\sqrt{R^2-x^2-y^2}}, 1 \right) dx dy$$

$$= \frac{2}{3} \int_{D(x,y)} \left(\frac{x^2}{\sqrt{R^2-x^2-y^2}} + \frac{y^2}{\sqrt{R^2-x^2-y^2}} + z(x,y) \right) dx dy$$

$$= \frac{2}{3} \int_{D(x,y)} \left(\frac{x^2}{\sqrt{R^2-x^2-y^2}} + \frac{y^2}{\sqrt{R^2-x^2-y^2}} + \sqrt{R^2-x^2-y^2} \right) dx dy$$

$$= \frac{2}{3} \int_{D(x,y)} \frac{x^2 + y^2 + R^2 - x^2 - y^2}{\sqrt{R^2-x^2-y^2}} dx dy$$

$$\Rightarrow = \frac{2}{3} \int_{D(x,y)} \frac{R^2}{\sqrt{R^2-x^2-y^2}} dx dy$$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

$$D: \begin{cases} 0 \leq r \leq R \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$\Rightarrow = \frac{2}{3} \int_{\theta=0}^{2\pi} \int_{r=0}^R \frac{R^2}{\sqrt{R^2-r^2}} r dr d\theta$$

$$= \frac{2}{3} \int_{\theta=0}^{2\pi} R^2 (-) \sqrt{R^2 - r^2} \bigg|_{r=0}^R d\theta$$

$$= \frac{2}{3} \int_{\theta=0}^{2\pi} R^2 R d\theta$$

$$= \frac{2}{3} \int_{\theta=0}^{2\pi} R^3 d\theta$$

$$= \frac{2}{3} R^3 \theta \bigg|_{\theta=0}^{2\pi}$$

$$= \frac{2}{3} R^3 2\pi$$

$$= \frac{4\pi}{3} R^3$$

$$2. \quad u = 1 + xy \rightarrow y = \frac{u-1}{x}$$

$$u' = x'y + x y' = y + x y'$$

$$\therefore y' = \frac{u' - y}{x}$$

$$= \frac{u' - \frac{u-1}{x}}{x}$$

$$y' = \frac{xu' - u + 1}{x^2}$$

Dañ

$$y' = \frac{y(1+xy)}{x(1-xy)}$$

\therefore

$$\frac{xu' - u + 1}{x^2} = \frac{\frac{u-1}{x} \cdot u}{x(1 - x \frac{u-1}{x})}$$

$$\frac{xu' - u + 1}{x^2} = \frac{(u-1)u}{x^2(2-u)}$$

$$xu' = u-1 + (u-1) \frac{u}{(2-u)}$$

$$xu' = (u-1) \left(1 + \frac{u}{2-u} \right)$$

$$xu' = (u-1) \left(\frac{2-u+u}{2-u} \right)$$

$$2u' = (u-1) \frac{2}{2-u}$$

$$u' \frac{(2-u)}{u-1} = \frac{2}{2-u}$$

$$\frac{2-u}{u-1} du = \frac{2}{2-u} du$$

$$\int \frac{2-u}{u-1} du = \int \frac{2}{2-u} du$$

$$z = u-1 \therefore u = z+1$$

$$dz = du$$

$$\int \frac{2-z-1}{z} dz = 2 \ln|z| + C$$

$$\int \frac{1-z}{z} dz = 2 \ln|z| + C$$

$$\ln|z| - z = 2 \ln|z| + C$$

$$\ln|z| - 2 \ln|z| = z + C$$

$$\ln \frac{|z|}{|z|^2} = z + C$$

$$\frac{|z|}{z^2} = e^{z+C}$$

$$|z| = e^C z^2 e^z$$

$$z = kx^2 e^z$$

↓

$$u-1 = kx^2 e^{u-1}$$

$$u-1 = \underbrace{k e^{-1}} x^2 e^u$$

$$u-1 = k_1 x^2 e^u$$

$$x+y - 1 = k_1 x^2 e^{(1+x+y)}$$

$$xy = \underbrace{k_1 x^2 e} e^{xy}$$

$$// xy = k_2 x^2 e^{xy} //$$

$$y = k_2 x e^{xy}$$

$$3. \quad \underbrace{(3x+2y^2)}_M dx + \underbrace{2xy}_N dy = 0$$

$$\begin{cases} \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(3x+2y^2) = 4y \\ \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y \end{cases} \quad \uparrow \neq$$

$$\frac{1}{N(x)} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{2xy} (4y - 2y) = \frac{\cancel{2y}}{\cancel{2xy}} = \frac{1}{x} = f(x)$$

$$F(x) = e^{\int f(x) dx}$$

$$= e^{\int \frac{1}{x} dx} = e^{\ln(x)} = |x|$$

$$\therefore \quad \underline{F(x) = x}$$

$$(3x+2y^2)dx + 2xy dy = 0$$

$$\nearrow \quad \underbrace{\tilde{M}}_{(3x^2+2xy^2)} dx + \underbrace{\tilde{N}}_{2x^2y} dy = 0 \quad \therefore \text{Eq. exact}$$

$$\underline{\underline{u(x,y)}}$$

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial x} = \tilde{M} \\ \frac{\partial u}{\partial y} = \tilde{N} \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial x} = \tilde{M} \\ \frac{\partial u}{\partial y} = \tilde{N} \end{array} \right.$$

$$\rightarrow \frac{\partial u}{\partial x} = 3x^2 + 2xy^2$$

$$u(x, y) = x^3 + x^2y^2 + h(y)$$

$$\rightarrow \frac{\partial u}{\partial y} = 2x^2y$$

$$\frac{\partial}{\partial y} (x^3 + x^2y^2 + h(y)) = 2x^2y$$

$$\therefore 2x^2y + h'(y) = 2x^2y$$

$$h(y) = c + 1.$$

$$\therefore u(x, y) = x^3 + x^2y^2 + k$$

Solving the eq.

$$u(x, y) = C$$

$$x^3 + x^2y^2 + k = C$$

$$\parallel x^3 + x^2y^2 = k_1 \parallel$$

$$4. \quad y' \sin x - y \cos x = \sin^2 x$$

Eg. homogeneous : $y' \sin x - y \cos x = 0$

$$y' \sin x = y \cos x$$

$$\frac{1}{y} y' = \frac{\cos x}{\sin x}$$

$$\frac{1}{y} y' = \cot x$$

$$\int \frac{1}{y} dy = \int \cot x dx$$

$$\ln |y| = -\ln |\sec x| + K$$

$$\ln |y| + \ln |\sec x| = K$$

$$\ln |y \sec x| = K$$

$$|y \sec x| = e^K$$

$$y \sec x = \pm e^K$$

$$y = C \frac{1}{\sec x}$$

$$y = C \sin x$$

$$y_p = u \sin x$$

$$y'_p = u' \sin x + u \cos x$$

$$y' \sin x - y \cos x = x^2 x$$

$$(u' \sin x + u \cos x) \sin x - u \sin x \cos x = x \sin^2 x$$

$$u' \sin^2 x + \cancel{u \cos x \sin x} - \cancel{u \sin x \cos x} = x \sin^2 x$$

$$u' \sin^2 x = x \sin^2 x$$

$$u' = 1$$

$$\therefore u = x$$

$$// y_p = x \sin x //$$

Ans, $y = y_h + y_p$

$$y = C \sin x + x \sin x$$

$$7. \text{ B. } y'' + y = \cos x + 3x$$

Eg - Homogeneous : $y'' + y = 0$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$$y_H = C_1 \cos x + C_2 \sin x$$

Solução particular : y_p

$$r(x) = \cos x + 3x$$

$$\cos x \rightarrow \underbrace{k_1 \cos x + k_2 \sin x}_{\text{presente na solução homogênea}} \rightarrow k_1 x \cos x + k_2 x \sin x$$

$$3x \rightarrow k_3 x + k_4$$

Daí

$$y_p = k_1 x \cos x + k_2 x \sin x + k_3 x + k_4$$

$$y_p' = k_1 \cos x - k_1 x \sin x + k_2 \sin x + k_2 x \cos x + k_3$$

$$y_p'' = -k_1 \sin x - k_1 \sin x - k_1 x \cos x + k_2 \cos x + k_2 \cos x - k_2 x \sin x$$

$$= -2k_1 \sin x + 2k_2 \cos x - k_1 x \cos x - k_2 x \sin x$$

Dan

$$y'' + y = \cos x + 3x$$

$$\begin{aligned} & -2k_1 \sin x + 2k_2 \cos x - \cancel{k_1 x \cos x} - \cancel{k_2 x \sin x} \\ & + \cancel{k_1 x \cos x} + \cancel{k_2 x \sin x} + k_3 x + k_4 = \cos x + 3x \end{aligned}$$

$$\therefore -2k_1 \sin x + 2k_2 \cos x + k_3 x + k_4 = \cos x + 3x$$

$$\Rightarrow 2k_2 = 1 \Rightarrow k_2 = 1/2$$

$$-2k_1 = 0 \Rightarrow k_1 = 0$$

$$k_3 = 3 \Rightarrow k_3 = 3$$

$$k_4 = 0 \Rightarrow k_4 = 0$$

$$\therefore y_p = \frac{1}{2} x \sin x + 3x$$