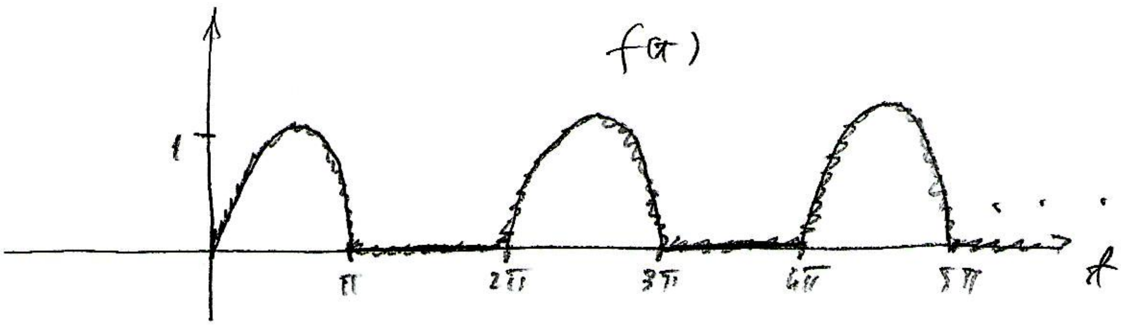
1. $y''' - 2y'' + y' = 2 + e^{x}$ 1,5

Use o método dos coeficientes a determinar

2. $y'' - 2y' + y = t$; $y(0) = 0$, $y'(0) = 0$ 1,5

Resolva usando transformada de Laplace

3. Seja $f(t)$ a função cujo gráfico é mostrado abaixo



i) Escreva $f(t)$ em termos de funções degrau 1,0

ii) Usando a expressão obtida em (i) calcule $\mathcal{L}(f(t))$. 1,0

[obtenha a expressão para $\mathcal{L}(f(t))$ na forma mais simplificada possível]

4. $\mathcal{L}^{-1}\left(\operatorname{arctg} \frac{1}{s}\right)$ 1,0

$\left(2e^{t-2} + 2e^{4-2t}\right) u(t-2) =$

$\mathcal{L}^{-1}\left(\dfrac{e^{-2s}(4s+2)}{(s-1)(s+2)}\right)$

1. $y''' - 2y'' + y' = x + e^x$

Eq. hom.: $y''' - 2y'' + y' = 0$

$$r^3 - 2r^2 + r = 0$$

$$r(r^2 - 2r + 1) = 0$$

$r = 0$, $r = 1$ mult. 2

$$y_h = C_1 + C_2 e^x + C_3 x e^x$$ 0.25

Solução particular

$x = 2 + e^x$

$y_p = K_1 + K_2 e^x \rightarrow$ presente no homogénea

$\hookrightarrow$ presente no homogénea

$= K_1 x + K_2 x^2 e^x$ 0.25

$y'_p = K_1 + 2K_2 x e^x + K_2 x^2 e^x$

$y''_p = 2K_2 e^x + 2K_2 x e^x + 2K_2 x e^x + K_2 x^2 e^x$

$= 2K_2 e^x + 4K_2 x e^x + K_2 x^2 e^x$

$y'''_p = 2K_2 e^x + 4K_2 e^x + 4K_2 x e^x + 2K_2 x e^x + K_2 x^2 e^x$

$= 6K_2 e^x + 6K_2 x e^x + K_2 x^2 e^x$

Daí

$$y''' - 2y'' + y' = 2 + e^x$$

$$\underline{6k_2 e^x} + \cancel{6k_2 x e^x} + \cancel{k_2 x^2 e^x} - \underline{4k_2 e^x} - \cancel{8k_2 x e^x} - \cancel{2k_2 x^2 e^x}$$

$$+ k_1 + \cancel{2k_2 x e^x} + \cancel{k_2 x^2 e^x} = 2 + e^x$$

$$\therefore \quad 2k_2 e^x + k_1 = 2 + e^x$$

$$\therefore \quad k_1 = 2 \,, \quad k_2 = \frac{1}{2}$$

$$y_p = 2 + \frac{1}{2} x^2 e^x$$

$$y = y_h + y_p$$

$$\boxed{y = C_1 + C_2 e^x + C_3 x e^x + 2x + \frac{1}{2} x^2 e^x}$$

1,0

2. $y'' - 2y' + y = t$, $y(0) = y'(0) = 0$

$\mathcal{L}(y'') - 2\mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(t)$

$s^2\mathcal{L}(y) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} - 2(s\mathcal{L}(y) - \cancel{y(0)}^{0}) + \mathcal{L}(y) = \frac{1}{s^2}$

$s^2\mathcal{L}(y) - 2s\mathcal{L}(y) + \mathcal{L}(y) = \frac{1}{s^2}$

$(s^2 - 2s + 1)\mathcal{L}(y) = \frac{1}{s^2}$

$(s-1)^2\mathcal{L}(y) = \frac{1}{s^2}$

$\mathcal{L}(y) = \frac{1}{s^2}\frac{1}{(s-1)^2}$

$y = \mathcal{L}^{-1}\left(\frac{1}{s^2}\frac{1}{(s-1)^2}\right)$ 0,5

Mas

$\mathcal{L}^{-1}(F(s)\,G(s)) = f * g$

$F(s) = \frac{1}{s^2} \rightarrow f(t) = t$

$G(s) = \frac{1}{(s-1)^2} \rightarrow g(t) = e^t t$

Daí

$f * g = \int_0^t f(t-x)\,g(x)\,dx$

$$= \int_0^t (t-x)\, e^x x \, dx$$

$$= t \int_0^t x e^x \, dx - \int_0^t x^2 e^x \, dx$$

Mà

$$\int x e^x \, dx =$$

$$\left(\begin{array}{l} u = x \longrightarrow du = dx \\ dv = e^x dx \rightarrow v = e^x \end{array} \right)$$

$$= x e^x - \int e^x \, dx = x e^x - e^x$$

rõ $\displaystyle\int_0^t x e^x \, dx = \left(x e^x - e^x \right) \Big|_{x=0}^{t}$

$$= t e^t - e^t + 1$$

$$= e^t (t - 1) + 1$$

$$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$$

$$\left(\begin{array}{l} u = x^2 \longrightarrow du = 2x \, dx \\ dv = e^x dx \rightarrow v = e^x \end{array} \right)$$

$$= x^2 e^x - 2 \int x e^x \, dx$$

$$= x^2 e^x - 2 \left(x e^x - e^x \right)$$

$$= x^2e^x - 2xe^x + 2e^x$$

$$\int_0^t x^2 e^x \, dx = e^x(x^2 - 2x + 2)\Big|_{x=0}^{t}$$

$$= e^t(t^2 - 2t + 2) - 2$$

Daí,

$$f * g = t\left(e^t(t-1) + 1\right) - e^t(t^2 - 2t + 2) + 2$$

$$= e^t(t^2 - t) + t - e^t(t^2 - 2t + 2) + 2$$

$$= e^t(\cancel{t^2} - t - \cancel{t^2} + 2t - 2) + t + 2$$

$$= e^t(t - 2) + t + 2$$

$$\therefore \quad y(t) = e^t(t-2) + t + 2$$

1,0

Outra solução

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\,\frac{1}{(s-1)^2}\right)$$

$$\frac{1}{s^2}\,\frac{1}{(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$$

$$= \frac{s\overbrace{(s-1)^2}^{s^2-2s+1}A + B(s-1)^2 + Cs^2(s-1) + Ds^2}{s^2(s-1)^2}$$

$$= \frac{(s^3-2s^2+s)A + B(s^2-2s+1) + C(s^3-s^2) + Ds^2}{s^2(s-1)^2}$$

$$= \frac{As^3 - 2As^2 + As + Bs^2 - 2Bs + B + Cs^3 - Cs^2 + Ds^2}{s^2(s-1)^2}$$

$$\frac{1}{s^2(s-1)^2} = \frac{(A+C)s^3 + (-2A+B-C+D)s^2 + (A-2B)s + B}{s^2(s-1)^2}$$

$$\Rightarrow \quad A+C=0 \longrightarrow \underline{C=-A=-2}$$

$$-2A+B-C+D=0 \longrightarrow -4+1+2+D=0 \quad \therefore \underline{D=1}$$

$$A-2B=0 \longrightarrow \underline{A=2}$$

$$\underline{B=1}$$

$$\therefore \quad \frac{1}{s^2}\,\frac{1}{(s-1)^2} = \frac{2}{s} + \frac{1}{s^2} - \frac{2}{(s-1)} + \frac{1}{(s-1)^2}$$

Dai

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\,\frac{1}{(s-1)^2}\right) = 2\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - 2\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right)$$

$$= 2 + t - 2e^t + e^t t$$

$$\therefore \quad \boxed{y(t) = e^t t - 2e^t + t + 2}$$

outra solução

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\,\frac{1}{(s-1)^2}\right)$$

$$\mathcal{L}\left(\int_0^t f(x)\,dx\right) = \frac{1}{s}\underbrace{\mathcal{L}(f(t))}_{F(s)}$$

$$\Leftrightarrow$$

$$\int_0^t f(x)\,dx = \mathcal{L}^{-1}\left(\frac{1}{s}F(s)\right) \quad (*)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\,\frac{1}{(s-1)^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\,\overbrace{\frac{1}{s}\,\frac{1}{(s-1)^2}}^{F_1(s)}\right)$$

$$\overset{(*)}{=} \int_0^t f_1(x)\,dx \quad (**)$$

onde $f_1(t) = \mathcal{L}^{-1}(F_1(s))$

Mas

$$\underbrace{\mathcal{L}^{-1}(F_1(s))}_{f_1(t)} = \mathcal{L}^{-1}\left(\frac{1}{s}\,\overbrace{\frac{1}{(s-1)^2}}^{F_2(s)}\right)$$

$$f_1(t) \overset{(*)}{=} \int_0^t f_2(x)\,dx \quad (***)$$

onde

$$f_2(t) = \mathcal{L}^{-1}(F_2(s))$$

$$= \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right) =$$

$$= e^t t$$

Substituindo $f_2(t)$ em $(***)$ obtem-se:

$$f_1(t) = \int_0^t e^x x \, dx$$

$$\begin{pmatrix} u = x & \longrightarrow & du = dx \\ dv = e^x dx & \longrightarrow & v = e^x \end{pmatrix}$$

$$= \left(x e^x - \int e^x dx \right) \Big]_{x=0}^{t}$$

$$= \left(x e^x - e^x \right) \Big|_{x=0}^{t}$$

$$= t e^t - e^t + 1$$

Substituindo em $(**)$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2} \frac{1}{(s-1)^2} \right) = \int_0^t (x e^x - e^x + 1) \, dx$$

$$= \left[\underline{\int x e^x dx} - \int e^x dx + \int 1 dx \right]_{x=0}^{t}$$

$$= \left(x e^x - e^x - e^x + x \right) \Big|_{x=0}^{t}$$

$$= \left(x e^x - 2 e^x + x \right) \Big|_{x=0}^{t}$$

$$= t e^t - 2 e^t + t + 2$$

$$\boxed{y(t) = t e^t - 2 e^t + t + 2}$$

$\cdot 3 \cdot$

1.

$0 < t < \pi$; $f(t) = \sin t$

$$[u(t-0) - u(t-\pi)] \sin t$$

$\pi < t < 2\pi$; $f(t) = 0$

$2\pi < t < 3\pi$; $f(t) = \sin t$

$$[u(t-2\pi) - u(t-3\pi)] \sin t$$

$3\pi < t < 4\pi$; $f(t) = 0$

$\vdots$

$2n\pi < t < (2n+1)\pi$; $[u(t-2n\pi) - u(t-(2n+1)\pi)] \sin t$

$(2n+1)\pi < t < (2n+2)\pi$: $f(t) = 0$

$\vdots$

$$f(t) = [u(t-0) - u(t-\pi)] \sin t +$$
$$[u(t-2\pi) - u(t-3\pi)] \sin t +$$
$$+ \dots + [u(t-2n\pi) - u(t-(2n+1)\pi)] \sin t + \dots$$

$$f(t) = \sum_{n=0}^{\infty} \left[u(t-2n\pi) - u(t-(2n+1)\pi) \right] \sin t$$

1.0

ii) $$\mathcal{L}(f(t)) = \sum_{n=0}^{\infty} \Big[\mathcal{L}\big(u(t-2n\pi)\sin t \big) -$$
$$- \mathcal{L}\big(u(t-(2n+1)\pi)\sin t \big) \Big]$$

Mas

$$\mathcal{L}\big(u(t-2n\pi)\sin t \big) =$$
$$= \mathcal{L}\big(u(t-2n\pi)\sin(t-2n\pi) \big)$$
$$= e^{-2n\pi s}\mathcal{L}(\sin t) = e^{-2n\pi s}\frac{1}{s^2+1}$$

$$\mathcal{L}\big(u(t-(2n+1)\pi)\sin t \big)$$

$$\sin(t-(2n+1)\pi)$$
$$= \sin(t-\pi-2n\pi)$$
$$= \sin(t-\pi)$$
$$= \sin t \cos\pi = -\sin t$$

$$= \mathcal{L}\big(u(t-(2n+1)\pi)(-1)\sin(t-(2n+1)\pi) \big)$$
$$= -\mathcal{L}\big(u(t-(2n+1)\pi)\sin(t-(2n+1)\pi) \big)$$
$$= -e^{-(2n+1)\pi s}\frac{1}{s^2+1}$$

$$\therefore \quad \mathcal{L}(f(t)) = \sum_{n=0}^{\infty} \left[e^{-2n\pi s} + e^{-(2n+1)\pi s} \right] \frac{1}{s^2+1}$$

$$= \sum_{n=0}^{\infty} e^{-2n\pi s} \left(1 + e^{-(2n+1)\pi s + 2n\pi s} \right) \frac{1}{s^2+1}$$

$$= \sum_{n=0}^{\infty} e^{-2n\pi s} \frac{\left(1 + e^{-\pi s} \right)}{s^2+1}$$

Mas

$$S_n = \underset{e^{-2\pi s}}{\underbrace{1 + e^{-2\pi s}}} + e^{-4\pi s} + \dots + e^{-2n\pi s}$$

$$e^{-2\pi s} S_n = e^{-2\pi s} + e^{-4\pi s} + \dots + e^{-2n\pi s} + e^{-2(n+1)\pi s}$$

$$\left(1 - e^{-2\pi s}\right) S_n = 1 - e^{-2(n+1)\pi s}$$

$$S_n = \frac{1 - e^{-2(n+1)\pi s}}{1 - e^{-2\pi s}}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1 - \cancelto{0}{e^{-2(n+1)\pi s}}}{1 - e^{-2\pi s}}$$

$$= \frac{1}{1 - e^{-2\pi s}}$$

$$\therefore \boxed{\mathcal{L}(f(t)) = \frac{\left(1 + e^{-\pi s}\right)}{\left(1 - e^{-2\pi s}\right)} \frac{1}{(s^2+1)}}$$

1,0

Mais

$$\frac{(1+e^{-\pi s})}{(1-e^{-2\pi s})}$$

$$= \frac{\left(1+\frac{1}{e^{\pi s}}\right)}{\left(1-\frac{1}{e^{2\pi s}}\right)}$$

$$= \frac{\frac{e^{\pi s}+1}{e^{\pi s}}}{\frac{e^{2\pi s}-1}{e^{2\pi s}}} = \frac{(e^{\pi s}+1)\ e^{\pi s}}{(e^{2\pi s}-1)}$$

$$= \frac{\cancel{(e^{\pi s}+1)}}{(e^{\pi s}-1)\cancel{(e^{\pi s}+1)}}\ e^{\pi s}$$

$$= \frac{e^{\pi s}}{e^{\pi s}-1}$$

Donc

$$\boxed{L(f(x)) = \frac{1}{(s^2+1)}\ \frac{e^{\pi s}}{(e^{\pi s}-1)}}$$

4. $\mathcal{L}^{-1}\left(\operatorname{arctg}\frac{1}{s}\right)$

$$\left\{\begin{array}{l} \mathcal{L}\left(t^n f(t)\right) = (-1)^n \frac{d^n}{ds^n} F(s) \\ \Leftrightarrow \\ t^n f(t) = (-1)^n \mathcal{L}^{-1}\left(\frac{d^n}{ds^n} F(s)\right) \quad \text{onde} \quad F(s) = \mathcal{L}(f(t)) \end{array}\right.$$

$$t^n \mathcal{L}^{-1}(F(s)) = (-1)^n \mathcal{L}^{-1}\left(\frac{d^n}{ds^n} F(s)\right)$$

Seja $F(s) = \operatorname{arctg}\frac{1}{s}$

Então $\frac{d}{ds} F(s) = \frac{1}{1+\frac{1}{s^2}} \left(-\frac{1}{s^2}\right)$

$$= \frac{1}{\frac{s^2+1}{s^2}} \left(-\frac{1}{s^2}\right) = -\frac{1}{(s^2+1)} \quad 0.5$$

Então,

$$t\mathcal{L}^{-1}(F(s)) = -\mathcal{L}^{-1}\left(\frac{d}{ds}\operatorname{arctg}\frac{1}{s}\right)$$

$$= -\mathcal{L}^{-1}\left(-\frac{1}{s^2+1}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \operatorname{sin} t$$

$$\therefore \mathcal{L}^{-1}(F(s)) = \frac{1}{t}\operatorname{sin} t$$

$$// \mathcal{L}^{-1}\left(\operatorname{arctg}\frac{1}{s}\right) = \frac{1}{t}\operatorname{sin} t // \quad 1.0$$