Cálculo C - Prova 3

Nome:

1. Resolva usando o método dos coeficientes a determinar **[2 pts]**

$$y''' - y'' - 4y' + 4y = 6e^x$$

2. Resolva **[2 pts]**

$$x^2y'' - xy' = 2x^3e^x$$

3. Resolva o problema de valor inicial usando transformada de Laplace **[2 pts]**

$$y'' + y = 2\cos t$$
$$y(0) = 2, \quad y'(0) = 0$$

4. Calcule **[2 pts]**

$$\mathcal{L}^{-1}\left(\frac{e^{-4s}}{s(s^2+3)}\right)$$

Cálculo C - Prova 2

$r(x)$	y_p
$C_nX^n + C_{n-1}x^{n-1} + ... + C_1x + C_0$	$A_nX^n + A_{n-1}x^{n-1} + ... + A_1x + A_0$
Ce^{rx}	Ae^{rx}
$C\cos kx$	$A\cos kx + B\sin kx$
$C\sin kx$	$A\cos kx + B\sin kx$
$(C_nX^n + C_{n-1}x^{n-1} + ... + C_1x + C_0)e^{rx}$	$(A_nX^n + A_{n-1}x^{n-1} + ... + A_1x + A_0)e^{rx}$
$(C_nX^n + C_{n-1}x^{n-1} + ... + C_1x + C_0)\cos kx$	$(A_nX^n + A_{n-1}x^{n-1} + ... + A_1x + A_0)\cos kx +$ $+(B_nX^n + B_{n-1}x^{n-1} + ... + B_1x + B_0)\sin kx$
$(C_nX^n + C_{n-1}x^{n-1} + ... + C_1x + C_0)\sin kx$	$(A_nX^n + A_{n-1}x^{n-1} + ... + A_1x + A_0)\cos kx +$ $+(B_nX^n + B_{n-1}x^{n-1} + ... + B_1x + B_0)\sin kx$
$Ce^{rx}\cos kx$	$Ae^{rx}\cos kx + Be^{rx}\sin kx$
$Ce^{rx}\sin kx$	$Ae^{rx}\cos kx + Be^{rx}\sin kx$
$(C_nX^n + C_{n-1}x^{n-1} + ... + C_1x + C_0)e^{rx}\cos kx$	$(A_nX^n + A_{n-1}x^{n-1} + ... + A_1x + A_0)e^{rx}\cos kx +$ $+(B_nX^n + B_{n-1}x^{n-1} + ... + B_1x + B_0)e^{rx}\sin kx$
$(C_nX^n + C_{n-1}x^{n-1} + ... + C_1x + C_0)e^{rx}\sin kx$	$(A_nX^n + A_{n-1}x^{n-1} + ... + A_1x + A_0)e^{rx}\cos kx +$ $+(B_nX^n + B_{n-1}x^{n-1} + ... + B_1x + B_0)e^{rx}\sin kx$

- $y_h = C_1y_1 + C_2y_2 \qquad y_p = u\,y_1 + v\,y_2$
  $u = -\int \frac{r(x)y_2}{W(y_1,y_2)}\,dx \qquad v = \int \frac{r(x)y_1}{W(y_1,y_2)}\,dx$

- $x^2y'' + axy' + by = 0$

⌟ 2 raízes reais distintas : $m_1, m_2 \longrightarrow y = C_1x^{m_1} + C_2x^{m_2}$

⌟ 1 raiz real repetida : $m \longrightarrow y = x^m(C_1 + C_2\ln x)$

⌟ par de raiz complexa e seu conjugado:

$m_1 = k_1 + ik_2$, $m_2 = k_1 - ik_2 \longrightarrow y = x^{k_1}\Big(C_1\cos(k_2\ln x) + C_2\sin(k_2\ln x)\Big)$

- $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$
  $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0)$
- $\mathcal{L}\big(\int_0^t f(u)du\big) = \frac{1}{s}\mathcal{L}(f)$
- $\mathcal{L}\big(e^{at}f(t)\big) = F(s-a)$ com $F(s) = \mathcal{L}(f)$
- $\mathcal{L}\big(t^nf(t)\big) = (-1)^n\frac{d^n}{ds^n}F(s)$ com $F(s) = \mathcal{L}(f)$
- $\mathcal{L}\big(u_a(t)f(t-a)\big) = e^{-as}F(s)$ $(a \geq 0)$ com $F(s) = \mathcal{L}(f)$
- $\mathcal{L}(f * g) = F(s)G(s)$ com $F(s) = \mathcal{L}(f)$ e $G(s) = \mathcal{L}(g)$
- $f * g := \int_0^t f(t-x)g(x)dx$ ou $= \int_0^t f(x)g(t-x)dx$

$f(t)$	$\mathcal{L}(f(t))$
1	$1/s$, $s>0$
t^n	$\frac{n!}{s^{n+1}}$, $s>0$, $n \in \mathbb{Z}$
e^{at}	$\frac{1}{s-a}$, $s>a$
$\sin at$	$\frac{a}{s^2+a^2}$, $s>0$
$\cos at$	$\frac{s}{s^2+a^2}$, $s>0$
$\sinh at$	$a/(s^2-a^2)$, $s>\lvert a\rvert$
$\cosh at$	$s/(s^2-a^2)$, $s>\lvert a\rvert$

1. $y''' - y'' - 4y' + 4y = 6e^{+x}$

<u>Eq. homogénea</u>

$$y''' - y'' - 4y' + 4y = 0$$

$$\lambda^3 - \lambda^2 - 4\lambda + 4 = 0$$

$\lambda = 1$ é raiz.

$$\begin{array}{l|l} \cancel{\lambda^3} - \cancel{\lambda^2} - 4\lambda + 4 & \lambda - 1 \\ \hline -\lambda^3 + \lambda^2 & \lambda^2 - 4 \\ \hline \quad -4\lambda + 4 & \\ \quad +4\lambda - 4 & \\ \hline \quad\ \ 0 \quad 0 & \end{array}$$

$$\therefore \quad \lambda^3 - \lambda^2 - 4\lambda + 4 = (\lambda - 1)(\lambda^2 - 4)$$
$$= (\lambda - 1)(\lambda - 2)(\lambda + 2)$$

$y_h = C_1 e^x + C_2 e^{2x} + C_3 e^{-2x}$ 1.0

$\rightarrow y_p = A_1 x e^x$ 1.5

$\rightarrow y'_p = A_1 e^x + A_1 x e^x$

$\rightarrow y''_p = A_1 e^x + A_1 e^x + A_1 x e^x$

$= 2A_1 e^x + A_1 x e^x$

$\rightarrow y'''_p = 2A_1 e^x + A_1 e^x + A_1 x e^x = 3A_1 e^x + A_1 x e^x$

$$3A_1e^x + A_1x\cancel{e^x} - 2A_1e^x - A_1\cancel{xe^x}$$

$$-4A_1e^x - 4A_1\cancel{xe^x} + 4A_1\cancel{xe^x} = 6e^x$$

$$\therefore \quad -3A_1e^x = 6e^x$$

$$-3A_1 = 6 \Rightarrow \underline{\underline{A_1 = -2}}$$

2.0

$$y_p = -2xe^x$$

$$\therefore \quad y = y_h + y_p$$

$$\boxed{y = C_1e^x + C_2e^{2x} + C_3e^{-2x} - 2xe^x}$$

2. $x^2y'' - xy' = 2x^3 e^x$

Eq. homogénea : $x^2y'' - xy' = 0$

$$\begin{cases} y = x^m \\ y' = m x^{m-1} , \quad y'' = m(m-1)\, x^{m-2} \end{cases}$$

$$x^2y'' - xy' = 0$$

$$\therefore \quad x^2 m(m-1)\, x^{m-2} - x m\, x^{m-1} = 0$$

$$m(m-1)\, x^m - m x^m = 0$$

$$m^2 - m - m = 0$$

$$m^2 - 2m = 0$$

$$m(m-2) = 0 \Rightarrow \begin{matrix} m = 0 \\ m = 2 \end{matrix}$$

$$y_h = c_1 1 + c_2 x^2$$

0.5

Eq. particular

$$x^2y'' - xy' = 2x^3 e^x$$

$$y'' - \frac{1}{x} y' = \underbrace{2x e^x}_{r(x)}$$

$$y_p = u \cdot \underbrace{1}_{y_1} + v \cdot \underbrace{x^2}_{y_2} \quad , \quad \begin{cases} W(y_1, y_2) = \det \begin{pmatrix} 1 & x^2 \\ 0 & 2x \end{pmatrix} \\ = 2x \end{cases}$$

Temos

$$u(x) = -\int \frac{r(x)\, y_2}{W(y_1, y_2)}\, dx = -\int \frac{\cancel{2x}\, e^x x^2}{\cancel{2x}}\, dx \quad (1.0)$$

$$= -\int e^x x^2\, dx$$

$$\int e^x x^2\, dx = x^2 e^x - 2\int x e^x\, dx \quad (*)$$

$u = x^2 \rightarrow du = 2x\, dx$

$dv = e^x dx \rightarrow v = e^x$

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x \quad (**)$$

$u = x \rightarrow du = dx$

$dv = e^x dx \rightarrow v = e^x$

$(**) \rightarrow (*)$:

$$\int e^x x^2 dx = x^2 e^x - 2\left(x e^x - e^x\right)$$

$$= x^2 e^x - 2x e^x + 2 e^x$$

$$= (x^2 - 2x + 2)\, e^x$$

$$\therefore \quad \boxed{u(x) = -(x^2 - 2x + 2)\, e^x}$$

Temos

$$v(x) = \int \frac{r(x)\, y_1}{W(y_1, y_2)}\, dx = \int \frac{2x e^{x} \cdot 1}{2x}\, dx$$

$$= \int e^{x}\, dx = e^{x}$$

0.5

$$\therefore \quad // \; v(x) = e^{x} \; //$$

$$\therefore \quad y_p = -(\cancel{x^2} - 2x + 2)\, e^{x} + \cancel{x^2} e^{x}$$

$$// \; y_p = 2x e^{x} - 2 e^{x} \; //$$

$$\therefore \quad y = y_h + y_p$$

$$\boxed{y = C_1 + C_2 x^2 + 2(x-1)\, e^{x}}$$

3. $y'' + y = 2\cos t$

$$\begin{cases} y(0) = 2 \\ y'(0) = 0 \end{cases}$$

$\mathcal{L}(y'') + \mathcal{L}(y) = 2\,\mathcal{L}(\cos t)$

$s^2\mathcal{L}(y) - s\underbrace{y(0)}_{2} - \cancel{y'(0)}^{\,0} + \mathcal{L}(y) = 2\,\dfrac{s}{s^2+1}$

$(s^2+1)\,\mathcal{L}(y) - 2s = 2\,\dfrac{s}{s^2+1}$

$(s^2+1)\,\mathcal{L}(y) = 2\left(s + \dfrac{s}{s^2+1}\right)$

$\mathcal{L}(y) = 2\left(\dfrac{s}{s^2+1} + \dfrac{s}{(s^2+1)^2}\right)$

$y = 2\mathcal{L}^{-1}\left(\dfrac{s}{s^2+1}\right) + 2\,\mathcal{L}^{-1}\left(\dfrac{s}{(s^2+1)^2}\right)$

(*) (**) 0.5

Mais $\mathcal{L}^{-1}\left(\dfrac{s}{s^2+1}\right) = \cos t$ 0.5

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right):$$

Temos $\displaystyle \frac{s}{(s^2+1)^2} = -\frac{1}{2}\frac{d}{ds}\frac{1}{(s^2+1)}$ (***)

e $\displaystyle \mathcal{L}\left(t^n f(t)\right) = (-1)^n \frac{d^n}{ds^n} F(s)$

De (***):

$$\frac{2\cdot s}{(s^2+1)^2} = -\frac{d}{ds}\frac{1}{s^2+1}$$

$$= \mathcal{L}\left(t\, f(t)\right)$$

$$= \mathcal{L}\left(t \sin t\right)$$

$$\therefore\ 2\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right) = t\sin t \quad (**)$$

Daí

$$\boxed{y = 2\cos t + t\sin t}$$

1,0

4. $\mathcal{L}^{-1}\left(\dfrac{e^{-4s}}{s(s^2+3)}\right)$

Temos

$$\mathcal{L}\left(u_a(t)\,f(t-a)\right) = e^{-as}F(s)$$

com $F(s) = \mathcal{L}(f(t))$

Daí

$$\frac{e^{-4s}}{s(s^2+3)} = e^{-4s}\cdot \underbrace{\frac{1}{s(s^2+3)}}_{F(s)}$$

$$f(t) \longleftrightarrow F(s)$$

$$F(s) = \frac{1}{s(s^2+3)} = \frac{A}{s} + \frac{Bs+C}{s^2+3}$$

$$\frac{1}{s(s^2+3)} = \frac{As^2+3A+Bs^2+sC}{s(s^2+3)}$$

$$\frac{1}{s(s^2+3)} = \frac{(A+B)s^2+sC+3A}{s(s^2+3)}$$

$\therefore$ $A+B=0 \Rightarrow B=-1/3$

$C=0$

$3A=1 \Rightarrow A=1/3$

$$\therefore \quad \frac{1}{s(s^2+3)} = \frac{1}{3s} - \frac{1s}{3(s^2+3)}$$

0.5

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+3)}\right) = \frac{1}{3}\underbrace{\mathcal{L}^{-1}\left(\frac{1}{s}\right)}_{1} - \frac{1}{3}\mathcal{L}^{-1}\left(\frac{s}{s^2+3}\right)$$

$$= \frac{1}{3} - \frac{1}{3}\cos\sqrt{3}\,t$$

1.5

$\therefore$

$$f(t) = \frac{1}{3} - \frac{1}{3}\cos\sqrt{3}\,t$$

Daí

$$e^{-4s}\frac{1}{s(s^2+3)} = \mathcal{L}\left(u_4(t)\, f(t-4)\right)$$

$$\boxed{\mathcal{L}^{-1}\left(e^{-4s}\frac{1}{s(s^2+3)}\right) = u_4(t)\left[\frac{1}{3} - \frac{1}{3}\cos\sqrt{3}(t-4)\right]}$$

2.0