7 - Começo

Cálculo C - Lista 5

O teorema fundamental de integrais de linha

Nos exercícios a seguir mostre que as integrais de linha são independentes do caminho, e calcule as integrais.

1. $\int_\gamma (e^x + y)dx + (x + 2y)dy$ onde γ é qualquer curva suave por partes no plano xy de $(0, 1)$ a $(2, 3)$.

2. $\int_\gamma ydx + (x + z)dy + ydz$ onde γ é a curva parametrizada por $\vec{r}(t) = \frac{t^2+1}{t^2-1}\vec{i} + \cos \pi t\,\vec{j} + 2t \sin \pi t\,\vec{k}$, $0 \leq t \leq \frac{1}{2}$.

3. $\int_\gamma (3x^2 + y)dx + xdy$ onde γ é a reta de $(2, 1, 5)$ a $(-3, 2, 4)$.

4. $\int_\gamma 3x^2yzdx + x^3zdy + (x^3y - 4z)dz$ onde γ é curva obtida pela interseção de $x^2+y^2+z^2 = 3$ e $y = x$, e indo de $(-1, -1, 1)$ até $(1, 1, -1)$.

5.
$$\int_\gamma \frac{x}{1 + x^2 + y^2 + z^2}\,dx + \frac{y}{1 + x^2 + y^2 + z^2}\,dy + \\ +\frac{z}{1 + x^2 + y^2 + z^2}\,dz$$
onde γ é a curva parametrizada por $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^4\vec{k}$, $0 \leq t \leq 1$.

6. $\int_\gamma e^{-x} \ln y\, dx - \frac{e^{-x}}{y}dy + zdz$ onde γ é a curva parametrizada por $\vec{r}(t) = (t-1)\vec{i} + e^{t^4}\vec{j} + (t^2 + 1)\vec{k}$, $0 \leq t \leq 1$.

Com derivadas parciais [illegible]

7. Suponha que f, g, h são funções contínuas. Prove que $\int_\gamma f(x)dx + g(y)dy + h(z)dz$ é independente do caminho escolhido.

8. Seja $\vec{F}(x, y) = \frac{y}{x^2+y^2}\vec{i} - \frac{x}{x^2+y^2}\vec{j}$.
(a) Mostre que $\nabla \times \vec{F} = 0$.
(b) Seja D uma região no plano que contem o círculo $x^2 + y^2 = 1$ mas não contêm a origem. Mostre que $\int_\gamma \vec{F} \cdot d\vec{r}$ não é independente do caminho em D. [SUGESTÃO: Calcule primeiro $\int_{\gamma_1} \vec{F} \cdot d\vec{r}$ e depois $\int_{\gamma_2} \vec{F} \cdot d\vec{r}$ onde γ_1 é o semicírculo parametrizado por $\vec{r}_1(t) = \cos t\vec{i} + \sin t\vec{j}$, $0 \leq t \leq \pi$, e γ_2 é o semicírculo parametrizado por $\vec{r}_2(t) = \cos t\vec{i} - \sin t\vec{j}$, $0 \leq t \leq \pi$.]

9. (a) Verifique se a integral de linha
$$\int_\gamma \frac{x}{\sqrt{x^2 + y^2}}\,dx + \frac{y}{\sqrt{x^2 + y^2}}\,dy$$
é independente do caminho no domínio $D = R^2 - \{(0, 0)\}$.
(b) Verifique se a integral de linha
$$\int_\gamma \frac{x}{\sqrt{x^2 + y^2 + z^2}}\,dx + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\,dy + \\ +\frac{z}{\sqrt{x^2 + y^2 + z^2}}\,dz$$
é independente do caminho no domínio $\Omega = R^3 - \{(0, 0, 0)\}$.

10. Em que domínio é a integral de linha
$$\int_\gamma \frac{ydx - xdy}{x^2 + y^2}$$
independente do caminho:
(a) $x > 0$ (b) $x < 0$ (c) $y > 0$ (d) $y < 0$ (e) $x^2 + y^2 > 0$

11. A segunda lei da termodinâmica afirma que a integral de linha
$$I := \int \frac{1}{T}(dU + PdV)$$
é independente do caminho no plano UV.
(a) As equações de estado de um gás ideal são $PV = nRT$, $U = f(T)$ onde n e R são constantes e $f(T)$ é uma dada função da temperatura. Dessa forma é mais conveniente tomar T e V como variáveis independentes e expressar P e U em termos de ~~de~~ T e V. Se isto for feito mostre que
$$I = \int \frac{k}{T}\,dT + \frac{nR}{V}\,dV, \text{ onde } k = \frac{dU}{dT}$$
(b) Uma vez que a integral de linha é independente do caminho existe uma função $S(T, V)$, dita a *entropia*, tal que
$$\int_\gamma \frac{k}{T}\,dT + \frac{nR}{V}\,dV = S(B) - S(A)$$
onde γ é a curva ligando os pontos A e B. Mostre que se k é constante tem-se
$$S = k \ln T + nR \ln V + S_0$$
onde S_0 é uma constante.

12. Seja g uma função contínua de uma variável e considere

$$\vec{F}(x,y,z) = g(x^2+y^2+z^2)(x\vec{i}+y\vec{j}+z\vec{k})$$

(a) Mostre que $\vec{F}$ é conservativo.
[SUGESTÃO: Mostre que $\vec{F} = \nabla f$ onde $f(x,y,z) = \frac{1}{2}h(x^2+y^2+z^2)$ e $h(u) = \int g(u)du$]
(b) Mostre que $\vec{F}$ é irrotacional.

Respostas:

1. $e^2 + 13$
2. 1
3. -43
4. -2
5. $\ln 2$
6. $\frac{1}{2}$

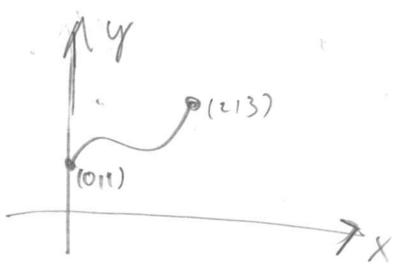
10. a) Independe
b) Independe
c) Independe
d) Independe
e) ~~Ind~~ depende

Cálculo C – Lista 5

1.

$$\int_\gamma (e^x+y)\,dx + (x+2y)\,dy$$

γ :



$$\vec{F} = (e^x+y)\,\hat{i} + (x+2y)\,\hat{j}$$

$$\vec{F} = \nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j}$$

$\therefore$

$$\rightarrow \frac{\partial\varphi}{\partial x} = e^x+y$$

$$\varphi = e^x + yx + C(y) \quad (*)$$

$$\rightarrow \frac{\partial\varphi}{\partial y} = x + \frac{dC}{dy} = (x+2y)$$

$\therefore$ $$\frac{dC}{dy} = 2y$$

$$C(y) = y^2 + K \quad (**)$$

$(**) \rightarrow (*)$:

$\therefore$ $$\| \varphi(x,y) = e^x + yx + y^2 + K \|$$

cte

$$\vec{F} = \nabla\varphi = \nabla\left(e^x + yx + y^2 + K\right)$$

$\therefore$

$$\int_\gamma (e^x+y)\,dx + (x+2y)\,dy =$$

$$= \varphi(2,3) - \varphi(0,1)$$

$$= \left(e^2 + 3\cdot 2 + 3^2 + \cancel{K}\right) - $$
$$- \left(1 + 0\cdot 1 + 1^2 + \cancel{K}\right)$$

$$= e^2 + 6 + 9 - 2$$

$$= \boxed{e^2 + 13}$$

2.

$$\int_\gamma y\,dx + (x+z)\,dy + y\,dz$$

$$\vec{r}(t) = \frac{t^2+1}{t^2-1}\,\hat{i} + \cos\pi t\,\hat{j} + $$
$$+ 2t\sin\pi t\,\hat{k}$$

$$0 \le t \le \frac{1}{2}$$

$$\vec{F} = \nabla\varphi$$

$$\frac{\partial\varphi}{\partial x} = y \Rightarrow \varphi = yx + C(y,z)$$

$$\frac{\partial\varphi}{\partial y} = \cancel{x} + \frac{dC(y,z)}{dy} = \cancel{x} + z$$

$$\therefore \frac{dC}{dy}(y,z) = z$$

$$\therefore C(y,z) = zy + K(z)$$

$$\varphi(x,y,z) = yx + zy + K(z)$$

$$\frac{\partial\varphi}{\partial z} = \cancel{y} + \frac{dK}{dz} = \cancel{y}$$

$$\therefore K = cte.$$

$\therefore$

$$// \ \varphi(x,y,z) = xy + yz + K \ //$$

$\therefore$

$$\vec{r}(0) = 1\hat{i} + \hat{j} + 0\hat{k} = (1,1,0)$$

$$\vec{r}\left(\tfrac{1}{2}\right) = -\frac{5}{3}\hat{i} + 0\hat{j} + \hat{k} = \left(-\tfrac{5}{3},0,1\right)$$

$\therefore$

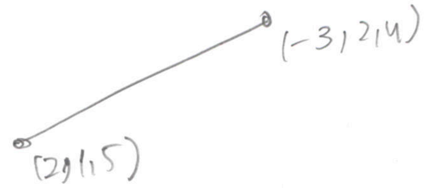
$$\int_\gamma y\,dx + (x+z)\,dy + y\,dz =$$

$$= \underbrace{\varphi\left(-\tfrac{5}{3},0,1\right)} - \varphi(1,1,0)$$

$$= \cancel{K} - (-1 + \cancel{K}) = 1 \ //$$

3. $$\int_\gamma (3x^2+y)\,dx + x\,dy$$

γ :



$$\vec{F} = \nabla\varphi$$

$$\frac{\partial\varphi}{\partial x} = 3x^2+y$$

$$\therefore \varphi = x^3 + yx + K(y,z)$$

$$\frac{\partial\varphi}{\partial y} = \cancel{x} + \frac{dK}{dy} = \cancel{x}$$

$$\frac{dK(y,z)}{dy} = 0 \Rightarrow K = K(z)$$

$\therefore$

$$\varphi(x,y,z) = x^3 + yx + K(z)$$

$$\therefore \frac{\partial\varphi}{\partial z} = \frac{dK}{dz} = 0 \Rightarrow K = cte$$

$$\therefore // \ \varphi(x,y,z) = x^3 + yx + K \ //$$

$$\therefore \int_\gamma (3x^2+y)\,dx + x\,dy =$$

$$= \varphi(-3,2,4) - \varphi(2,1,5)$$

$$= (-27-6+K) - (8+2+K)$$

$$= -43 \ //$$

4.
$$\int_\gamma 3x^2yz\,dx + x^3z\,dy + (x^3y - 4z)\,dz$$

$$\begin{cases} x^2+y^2+z^2 = 3 \\ y = x \end{cases}$$

$A = (-1,-1,1)$

$B = (1,1,-1)$

$\vec{F} = \nabla \varphi(x,y,z)$

$$\frac{\partial \varphi}{\partial x} = 3x^2yz$$

$\therefore\ \varphi(x,y,z) = x^3yz + k(y,z)$

$$\frac{\partial \varphi}{\partial y} = \cancel{x^3z} + \frac{\partial k}{\partial y}(y,z) = \cancel{x^3z}$$

$$\therefore\ \frac{\partial k}{\partial y}(y,z) = 0$$

$$\therefore\ k = k(z)$$

$\varphi(x,y,z) = x^3yz + k(z)$

$$\frac{\partial \varphi}{\partial z} = \cancel{x^3y} + \frac{dk}{dz}(z) = \cancel{x^3y} - 4z$$

$$\frac{dk}{dz} = -4z$$

$$k(z) = -2z^2 + \underset{\text{cte}}{k}$$

$\therefore$

$\varphi(x,y,z) = x^3yz - 2z^2 + k$

$$\int_\gamma 3x^2yz\,dx + x^3z\,dy + (x^3y - 4z)\,dz =$$

$= \varphi(1,1,-1) - \varphi(-1,-1,1)$

$= (-1 - 2(-1)^2 + k) -$

$-(-1(-1)1 - 2 + k)$

$= (-1 - 2 \cancel{+k}) - (1 - 2 \cancel{+k})$

$= -3 - (-1)$

$= -3 + 1$

$= -2$

5.

$\vec{F} = \nabla \varphi$ com

$\varphi(x,y,z) = \frac{1}{2} \ln (1+x^2+y^2+z^2)$

$\vec{r}(0) = (0,0,0)$; $\vec{r}(1) = (1,1,1)$

$$\int_\gamma \frac{x}{1+x^2+y^2+z^2} dx + \frac{y}{1+x^2+y^2+z^2} dy + \frac{z}{1+x^2+y^2+z^2} dz =$$

$$= \varphi(1,1,1) - \varphi(0,0,0)$$

$$= \frac{1}{2} \ln(4) - \frac{1}{2} \cancelto{0}{\ln 1}$$

$$= \frac{1}{2} \ln 2^2 = \cancel{\frac{2}{2}} \ln 2$$

$$= \ln 2$$

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6. $\vec{F} = \nabla \varphi$

$$\frac{\partial \varphi}{\partial x} = e^{-x} \ln y$$

$$\therefore \varphi = -e^{-x} \ln y + k(y,z)$$

$$\frac{\partial \varphi}{\partial y} = \cancel{-\frac{e^{-x}}{y}} + \frac{\partial k}{\partial y} = \cancel{-\frac{e^{-x}}{y}}$$

$$\therefore \frac{\partial k}{\partial y}(y,z) = 0$$

$k = k(z)$

$\therefore$

$$\varphi(x,y,z) = -e^{-x} \ln y + k(z)$$

$$\frac{\partial \varphi}{\partial z} = \frac{dk}{dz} = z$$

$$\therefore k = \frac{z^2}{2} + cte$$

$\therefore$

$$\varphi(x,y,z) = -e^{-x} \ln y + \frac{z^2}{2} +$$

$\vec{r}(0) = (-1,1,1)$; $\vec{r}(1) = (0,e,2)$

$$\int_\gamma e^{-x} \ln y \, dx - \frac{e^{-x}}{y} dy + z \, dz$$

$$= \varphi(0,e,2) - \varphi(-1,1,1)$$

$$= -1 + 2 + k - (\cancelto{0}{-e \cdot 0} + \frac{1}{2} + k$$

$$= 1 + k - \frac{1}{2} - k$$

$$= \frac{1}{2}$$

7. f, g, h : funções contínuas

$$\int_\gamma f(x)dx + g(y)dy + h(z)dz$$

$$\vec{F} = f(x)\,\hat{i} + g(y)\,\hat{j} + h(z)\,\hat{k}$$

Aqui, tentemos analisar a condição,

$$\vec{F} = \nabla\varphi$$

$$f(x) = \frac{\partial\varphi(x,y,z)}{\partial x}$$

$$g(y) = \frac{\partial\varphi(x,y,z)}{\partial y}$$

$$h(z) = \frac{\partial\varphi(x,y,z)}{\partial z}$$

Se $f(x)$, $g(y)$, $h(z)$ são contínuas então existe antiderivadas $\int f(x)dx$, $\int g(y)dy$, $\int h(z)dz$.

Daí,

$$f(x) = \frac{\partial\varphi}{\partial x}$$

$$\therefore \varphi(x,y,z) = \int_0^x f(t)\,dt + K_1(y,$$

$$\frac{\partial\varphi}{\partial y} = \frac{\partial K_1(y,z)}{\partial y} = g(y)$$

$$\therefore K_1(y,z) = \int_0^y g(t)\,dt + K_2(z)$$

$$\therefore \varphi(x,y,z) = \int_0^x f(t)\,dt + \int_0^y g(t)\,dt + K_2(z)$$

$$\therefore \frac{\partial\varphi(x,y,z)}{\partial z} = \frac{dK_2(z)}{dz} = h(z)$$

$$\therefore K_2(z) = \int_0^z h(t)\,dt + \underbrace{K}_{\text{cte}}$$

$$\therefore \left\{ \varphi(x,y,z) = \int_0^x f(t)\,dt + \int_0^y g(t)\,dt + \int_0^z h(t)\,dt + K \right.$$

Isto é $\vec{F} = \nabla\varphi$ e então

$\int_\gamma f(x)dx + g(y)dy + h(z)dz$

independe do caminho

Portanto

Em particular temos

Se
$$\gamma : \vec{r}(t), \quad a \le t \le b$$

e
$$\begin{cases} \vec{r}(a) = (x_a, y_a, z_a) \\ \vec{r}(b) = (x_b, y_b, z_b) \end{cases}$$

Então

$$\boxed{\int_\gamma f(x)\,dx + g(y)\,dy + h(z)\,dz =}$$

$$= \varphi(x_b, y_b, z_b) - \varphi(x_a, y_a, z_a)$$

$$= \int_0^{x_b} f(t)\,dt + \int_0^{y_b} g(t)\,dt + \int_0^{z_b} h(t)\,dt + \cancel{K}$$

$$- \int_0^{x_a} f(t)\,dt - \int_0^{y_a} g(t)\,dt - \int_0^{z_a} h(t)\,dt - \cancel{K}$$

$$\boxed{= \int_{x_a}^{x_b} f(t)\,dt + \int_{x_b}^{y_b} g(t)\,dt + \int_{z_a}^{z_b} h(t)\,dt}$$

8.

$$\vec{F}(x,y) = \underbrace{\frac{y}{x^2+y^2}}_{F_1}\hat{i} - \underbrace{\frac{x}{x^2+y^2}}_{F_2}\hat{j} \qquad F_3 = 0$$

a) Aqui

$$\vec{\nabla}\times\vec{F} = \left(\cancelto{0}{\frac{\partial F_3}{\partial y}} - \cancelto{0}{\frac{\partial F_2}{\partial z}}\right)\hat{i} + \left(\cancelto{0}{\frac{\partial F_1}{\partial z}} - \cancelto{0}{\frac{\partial F_3}{\partial x}}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right)$$

$$= -\frac{1}{x^2+y^2} + \frac{x\,2x}{(x^2+y^2)^2}$$

$$= \frac{-(x^2+y^2)+2x^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$-\frac{\partial F_1}{\partial y} = -\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

Daí,

8. Cont.

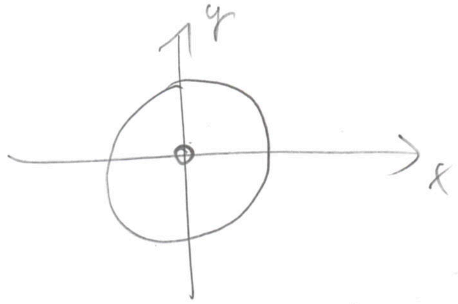
$$\vec{\nabla}\times\vec{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$$= \left(\frac{x^2-y^2}{(x^2+y^2)^2} + \frac{y^2-x^2}{(x^2+y^2)^2}\right)\hat{k}$$

$$= \vec{0}$$

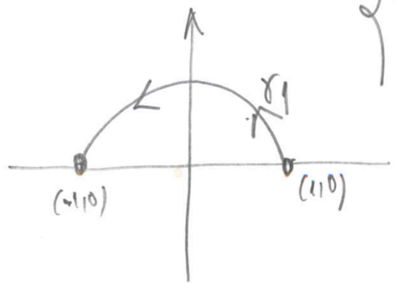
$$\therefore \boxed{\vec{\nabla}\times\vec{F} = \vec{0}}$$

b)



$$\gamma_1 : \begin{cases} \vec{r}_1(t) = \cos t\ \hat{i} + \sin t\ \hat{j} \\ 0 \le t \le \pi \end{cases}$$

$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases}$$



$$\int_{\gamma_1} \vec{F}\cdot d\vec{r} = \int_0^{\pi} \vec{F}(t)\cdot \frac{d\vec{r}_1}{dt}\,dt$$

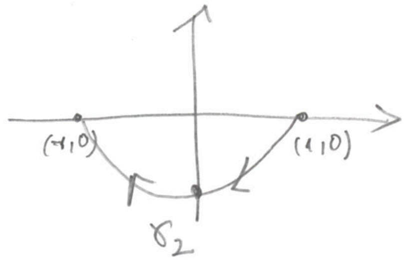
$$= \int_0^{\pi} (\sin t\ \hat{i} - \cos t\ \hat{j})\cdot(-\sin t\ \hat{i} + \cos t\ \hat{j})\,dt$$

$$= \int_0^{\pi} (-\sin^2 t - \cos^2 t)\,dt$$

$$= \int_0^{\pi} -1\,dt = \underline{-\pi}$$

$$\gamma_2 : \begin{cases} \vec{r}_2(t) = \cos t\ \hat{i} - \sin t\ \hat{j} \\ 0 \le t \le \pi \end{cases}$$

$$x(t) = \cos t, \quad y(t) = -\sin t$$



$$\int_{\gamma_2} \vec{F}\cdot d\vec{r} = \int_0^{\pi} \vec{F}(t)\cdot \frac{d\vec{r}_2}{dt}\,dt$$

$$= \int_0^{\pi} (-\sin t\ \hat{i} - \cos t\ \hat{j})\cdot(-\sin t\ \hat{i} - \cos t\ \hat{j})\,dt$$

$$= \int_0^{\pi} (\sin^2 t + \cos^2 t)\,dt$$

$$= \int_0^{\pi} dt = \underline{\pi}$$

Isto é

$$\boxed{\nabla\times\vec{F} = 0 \ \ \underline{\text{mas}} \ \ \int_{\gamma_1}\vec{F}\cdot d\vec{r}_1 \neq \int_{\gamma_2}\vec{F}\cdot d\vec{r}}$$

9.

a) $\int_\gamma \frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy$

Aqui,

$$\vec{F} = \frac{x}{\sqrt{x^2+y^2}}\,\hat{\imath} + \frac{y}{\sqrt{x^2+y^2}}\,\hat{\jmath}$$

$$= \nabla \varphi(x,y)$$

onde

$$\varphi(x,y) = \sqrt{x^2+y^2}$$

Logo

$$\boxed{\int_\gamma \frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy \quad \text{independe do caminho}}$$

b) $$\vec{F} = \frac{x}{\sqrt{x^2+y^2+z^2}}\,\hat{\imath} + \frac{y}{\sqrt{x^2+y^2+z^2}}\,\hat{\jmath} + \frac{z}{\sqrt{x^2+y^2+z^2}}\,\hat{k}$$

$\therefore$ $\vec{F} = \nabla\varphi$ com

$$\varphi = \sqrt{x^2+y^2+z^2}$$

Logo

$$\boxed{\int_\gamma \frac{x}{\sqrt{x^2+y^2+z^2}}\,dx + \frac{y}{\sqrt{x^2+y^2+z^2}}\,dy + \frac{z}{\sqrt{x^2+y^2+z^2}}\,dz \quad \text{independe do caminho}}$$

10. $\int_\gamma \frac{y}{x^2+y^2}\,dx - \frac{x}{x^2+y^2}\,dy$

$$\vec{F} = \frac{y}{x^2+y^2}\,\hat{\imath} - \frac{x}{x^2+y^2}\,\hat{\jmath}$$

Já vimos que

$$\vec{\nabla}\times\vec{F} = 0.$$

Se tivermos as duas condições

$$\begin{cases} \vec{\nabla}\times\vec{F} = 0 \\ \text{Dom } \vec{F} \text{ simplesmente conexo} \end{cases}$$

então $\int_\gamma \vec{F}\cdot d\vec{r}$ independe do caminho

Daí basta analisar se cada uma das regiões é simplesmente conexa

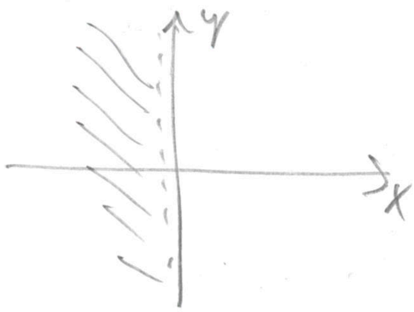
a) $x > 0$

$D = \{(x,y) \mid x > 0\}$ é simplesmente conexa

$\therefore$

$\int_\gamma \vec{F} \cdot d\vec{r}$ <u>independe do caminho</u>

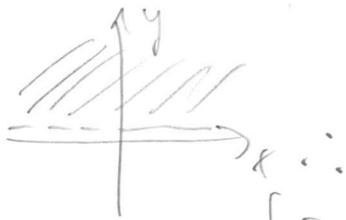
b) $x < 0$



$D = \{(x,y) \mid x < 0\}$ é simplesmente conexa

$\therefore$ $\int_\gamma \vec{F} \cdot d\vec{r}$ ind. caminho

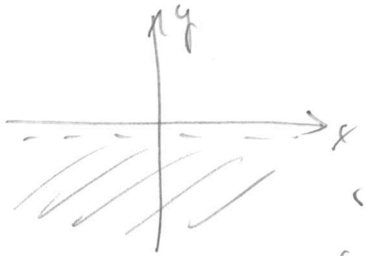
c) $y > 0$ $\rightarrow$ $D = \{(x,y) \mid y > 0\}$ simplesmente conexa



$\therefore$

$\int_\gamma \vec{F} \cdot d\vec{r}$ <u>ind. caminho</u>

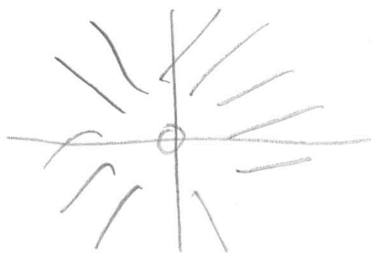
d) $y < 0$ $D = \{(x,y) \mid y < 0\}$ simplesmente conexa



$\therefore$

$\int_\gamma \vec{F} \cdot d\vec{r}$ <u>ind. caminho</u>

e) $x^2 + y^2 > 0$



$D = \{(x,y) \mid x^2 + y^2 > 0\}$ não é simplesmente conexa.

O exercício 8 mostra que $\int_\gamma \vec{F} \cdot d\vec{r}$ <u>depende do caminho</u>

11.

$$I = \int \frac{1}{T} dU + \frac{P}{T} dV$$

independe do caminho no plano UV.

a) $\begin{cases} PV = nRT \Rightarrow P = nR\frac{T}{V} \\ U = f(T) \end{cases}$

$$I = \int \frac{1}{T} dU + \frac{P}{T} dV$$

$$= \int \frac{1}{T} dU + nR\frac{1}{V} dV$$

$$dU = \frac{dU}{dT} dT$$

$$= \int \frac{1}{T}\frac{dU}{dT} dT + \frac{nR}{V} dV$$

$$\boxed{I = \int \frac{K}{T} dT + \frac{nR}{V} dV \qquad K \equiv \frac{dU(T)}{dT}}$$

b) I independe do caminho, i.e., existe $S(T,V)$ t.q.

$$I = S(B) - S(A)$$

$\therefore$

$$\int \frac{K}{T} dT + \frac{nR}{V} dV = S(B) - S(A)$$

Seja $K = \frac{dU}{dT} = \text{cte}$.

Então

$$S(B) - S(A) = \int \frac{K}{T} dT + \frac{nR}{V} dV$$

$$= \int_{T_A}^{T_B} \frac{K}{T} dT + \int_{V_A}^{V_B} \frac{nR}{V} dV$$

$$\left(\int_\gamma f(x)dx + g(y)dy + h(z)dz = \int_{x_a}^{x_b} f(x)dx + \int_{y_a}^{y_b} g(y)dy + \int_{z_a}^{z_b} h(z)dz \right)$$

$$S(B) - S(A) = K\ln T_B - K\ln T_A + nR\ln V_B - nR\ln V_A$$

12. cont

$$\boxed{\vec{F} = \nabla\varphi \quad \text{com} \quad \varphi = \frac{1}{2}\int_0^{x^2+y^2+z^2} g(t)\,dt + \underset{\text{cte}}{\underbrace{k}}}$$

$\therefore$ $\vec{F}$ é conservativa

b) Se $\vec{F} = \nabla\varphi$ segue-se que

$$\vec{\nabla}\times\vec{F} = \vec{\nabla}\times\nabla\varphi = 0$$

Explicitamente:

$$\vec{\nabla}\times\vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$$\frac{\partial F_3}{\partial y} = \frac{\partial}{\partial y}\left(g(x^2+y^2+z^2)\,z\right) = g'(x^2+y^2+z^2)\,2yz$$

$$\frac{\partial F_2}{\partial z} = \frac{\partial}{\partial z}\left(g(x^2+y^2+z^2)\,y\right) = g'(x^2+y^2+z^2)\,2yz$$

$\therefore$

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = 0$$

Analogamente se verifica que

$$\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = 0$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

i.e.

$$\vec{\nabla}\times\vec{F} = \vec{0}$$

[redacted] [redacted] Isto é [redacted]

$$\boxed{S = k\ln T + nR\ln V + \underbrace{S_0}_{\text{cte}}}$$

12.

$\vec{F}(x,y,z) = g(x^2+y^2+z^2)(x\hat{i} + y\hat{j} + z\hat{k})$

a) $\vec{F}$ é conservativo se

$\exists\ \varphi(x,y,z)$ tq.

$\vec{F} = \nabla\varphi$.

∴

$$\frac{\partial\varphi}{\partial x} = g(x^2+y^2+z^2)\,x$$

Aqui uma antiderivada de $\frac{\partial\varphi}{\partial x}$ é:

$$\int g(x^2+y^2+z^2)\,x\,dx =$$

$$t = x^2+y^2+z^2 \to dt = 2x\,dx$$

$$\frac{1}{2}dt = x\,dx$$

$$= \int g(t)\,\frac{1}{2}\,dt$$

∴

$$\| h(t) = \frac{1}{2}\int g(t)\,dt \|$$

$$\frac{\partial\varphi}{\partial x} = g(x^2+y^2+z^2)\,x$$

∴

$$\varphi(x,y,z) = \frac{1}{2}\int_0^{x^2+y^2+z^2} g(t)\,dt + K_1(y,z)$$

$$\frac{\partial\varphi}{\partial y} = \cancel{\frac{1}{2}}\, g(x^2+y^2+z^2)\,\cancel{2y} + \frac{dK}{dy}$$

[Obs.: $F(t) \doteq G(t^2) = \int_0^{t^2} g(u)\,du$

$$\frac{dF}{dt} = \underbrace{\frac{dG}{dt^2}}\,2t = g(t^2)\,2t$$]

$$\frac{\partial\varphi}{\partial y} = F_2 = g(x^2+y^2+z^2)\,y$$

$$\Rightarrow \frac{dK}{dy} = 0 \Rightarrow K = K(z)$$

∴

$$\varphi(x,y,z) = \frac{1}{2}\int_0^{x^2+y^2+z^2} g(t)\,dt + K(z)$$

$$\frac{\partial\varphi}{\partial z} = \cancel{g(x^2+y^2+z^2)\,z} + \frac{dK}{dz} = \cancel{g(x^2+y^2+z^2)\,z}$$

$$\therefore\ \frac{dK}{dz} = 0 \Rightarrow K = \text{cte}$$

∴

$$\varphi(x,y,z) = \frac{1}{2}\int_0^{x^2+y^2+z^2} g(t)\,dt + K$$