Cálculo C - Lista 6

Teorema de Green

Determine $\int_\gamma M(x,y)\,dx + N(x,y)\,dy$ onde γ é orientada no sentido anti-horário.

1. $\int_\gamma y\,dx$, onde γ a curva no primeiro quadrante formada por parte do círculo $x^2+y^2=4$ e dos intervalos $[0,2]$ nos eixos x e y.

2. $\int_\gamma xy\,dx + (x^{\frac{3}{2}} + y^{\frac{3}{2}})\,dy$, onde γ é borda do quadrado com vértices $(0,0),(1,0),(1,1),(0,1)$.

3. $\int_\gamma (x^2+y^2)^{\frac{3}{2}}\,dx + (x^2+y^2)^{\frac{3}{2}}\,dy$, onde γ é o círculo $x^2+y^2=1$.

Use o teorema de Green para calcular a integral de linha. Assuma que cada curva é orientada no sentido anti-horário.

4. $\int_\gamma e^x \sin y\,dx + e^x \cos y\,dy$ onde γ é composta de parte do gráfico de $\sqrt{x}+\sqrt{y}=5$ e dos intervalos $[0,25]$ nos eixos x e y.

5. $\int_\gamma xy\,dx + (\frac{1}{2}x^2 + xy)\,dy$ onde γ é composta do intervalo $[-1,1]$ sobre o eixo x e a parte de cima da elipse $x^2+4y^2=1$.

6. $\int_\gamma (\cos^3 x + e^x)\,dx + e^y\,dy$ onde γ é o gráfico de $x^6+y^8=1$.

Nos exercícios a seguir use o teorema de Green para calcular as integrais de linha $\int_\gamma \vec{F}\cdot d\vec{r}$, onde γ é orientada no sentido anti-horário.

7. $\vec{F}(x,y) = y\vec{i} + 3x\vec{j}$ onde γ é o círculo $x^2+y^2=4$.

8. $\vec{F}(x,y) = y\sin x\,\vec{i} - \cos x\vec{j}$ onde γ é composta do semi-círculo $x^2+y^2=9$ com $y\geq 0$ e a linha $y=0$ com $-3\leq x\leq 3$.

Nos exercícios a seguir use o teorema de Green para encontrar a área da região dada.

9. A região limitada pelo eixo y, a linha $y=\frac{1}{4}$ e a curva parametrizada por $\vec{r}(t) = \sin \pi t\,\vec{i} + t(1-t)\,\vec{j}$ com $0\leq t\leq \frac{1}{2}$.

10. Sejam

$$M(x,y) = \frac{-y}{x^2+y^2}, \quad N(x,y) = \frac{x}{x^2+y^2}$$

(a) Verifique que

$$\int_\gamma M(x,y)\,dx + N(x,y)\,dy = \int_D dA\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

onde D é a região limitada pelos círculos: $x^2+y^2=4$ orientado no sentido anti-horário e $x^2+y^2=1$ orientado no sentido horário.

(b) Mostre que

$$\int_\gamma M(x,y)\,dx + N(x,y)\,dy \neq \int_D dA\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

onde D é o disco cuja borda é o círculo $x^2+y^2=1$.

(c) O resultado em (b) viola o teorema de Green? Explique.

11. Assuma que D é uma região do plano. Sejam γ_1 e γ_2 curvas suaves fechadas em D ambas orientadas no sentido anti-horário. Suponha que $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ em D. Use o teorema de Green para mostrar que

$$\int_{\gamma_1} M(x,y)\,dx + N(x,y)\,dy = \int_{\gamma_2} M(x,y)\,dx + N(x,y)\,dy$$

12. (a) Seja γ o segmento de reta no plano unindo os pontos (x_1,y_1) e (x_2,y_2). Mostre que

$$\frac{1}{2}\int_\gamma x\,dy - y\,dx = \frac{1}{2}(x_1y_2 - x_2y_1)$$

(b) Considere um polígono orientado no sentido anti-horário cujos vértices são $(x_1,y_1),(x_2,y_2),...,(x_n,y_n)$. Usando a parte (a) mostre que a área do polígono é dada por

$$A = \frac{1}{2}(x_1y_2 - x_2y_1) + \frac{1}{2}(x_2y_3 - x_3y_2) + ... + \\ + \frac{1}{2}(x_{n-1}y_n - x_ny_{n-1}) + \frac{1}{2}(x_ny_1 - x_1y_n)$$

(c) Encontre a área do quadrilátero com vértices $(0,0),(1,0),(2,3),(-1,1)$.

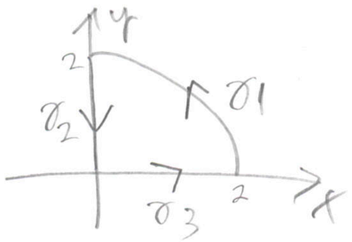
Respostas

1. $-\pi$ 2. $\frac{1}{2}$ 3. 0 4. 0
5. $\frac{1}{6}$ 6. 0 7. 8π 8. 0,
9. $\frac{1}{\pi} - \frac{2}{\pi^2}$ 10. Não 12. (c) 4

Cálculo C - Lista 6

1. $\int_{\gamma} y\,dx$

γ :



Aqui,

$$\vec{F} = F_1\hat{i} + F_2\hat{j} = y\hat{i} + 0\hat{j}$$

$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$$

$$\gamma_1 : \begin{cases} \vec{r}_1 = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} \\ 0 \leq t \leq \frac{\pi}{2} \end{cases}$$

$$\gamma_2 : \begin{cases} \vec{r}_2 = 0\hat{i} - t\,\hat{j} \\ -2 \leq t \leq 0 \end{cases}$$

$$\gamma_3 : \begin{cases} \vec{r}_3 = t\,\hat{i} + 0\hat{j} \\ 0 \leq t \leq 2 \end{cases}$$

$$\oint_{\gamma} y\,dx = \int_{\gamma_1} y\,dx + \int_{\gamma_2} y\,dx + \int_{\gamma_3} y\,dx$$

Mas

$$\int_{\gamma_1} y\,dx = \int_0^{\frac{\pi}{2}} y(t)\,\frac{dx}{dt}\,dt$$

$$= \int_0^{\frac{\pi}{2}} 2\sin t\,(-2\sin t)\,dt$$

$$= -4\int_0^{\frac{\pi}{2}} \sin^2 t\,dt$$

$$= -4\int_0^{\frac{\pi}{2}} \frac{1-\cos 2t}{2}\,dt$$

$$= -2\int_0^{\frac{\pi}{2}} (1-\cos 2t)\,dt$$

$$= -2\left(t - \frac{\sin 2t}{2}\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= -2\left(\frac{\pi}{2}\right) = -\pi$$

$$\int_{\gamma_2} y\,dx = \int_{-2}^{0} y(t)\,\frac{dx}{dt}\,dt$$

$$= \int_{-2}^{0} -t \cdot 0\,dt$$

$$= 0$$

$$\int_{\gamma_3} y\,dx = \int_0^2 \underbrace{y(t)}_{0} \frac{dx}{dt}\,dt = \underline{\underline{0}}$$

$\therefore$

$$\boxed{\oint_\gamma y\,dx = -\pi}$$

<u>Obs.</u>: Usando o teorema de Green:

$$\int_\gamma y\,dx + 0\,dy = \int_R \left(\frac{\partial \cancel{0}^{\,0}}{\partial x} - \frac{\partial y}{\partial y}\right) dA$$

$$= \int_R -dA$$

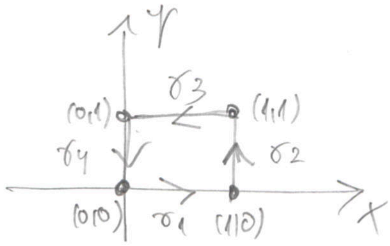
$$= -\underbrace{\int_R dA}_{\frac{1}{4}\pi(2)^2}$$

1/4 da área do círculo de raio 2

$$= -\frac{1}{\cancel{4}}\pi \cancel{4}$$

$$= \underline{\underline{-\pi}} \quad \underline{\text{Ok!}}$$

2. $$\int_\gamma xy\,dx + (x^{3/2} + y^{3/2})\,dy$$



$$\gamma_1 : \begin{cases} \vec{r}_1(t) = t\,\hat{i} + 0\,\hat{j} \\ 0 \le t \le 1 \end{cases}$$

$$\gamma_2 : \begin{cases} \vec{r}_2(t) = 1\,\hat{i} + t\,\hat{j} \\ 0 \le t \le 1 \end{cases}$$

$$\gamma_3 : \begin{cases} \vec{r}_3(t) = t\,\hat{i} + 1\,\hat{j} \\ 0 \le t \le 1 \end{cases}$$

$\downarrow$ inverte a orientação

$$\begin{cases} \vec{r}_3(1-t) = (1-t)\,\hat{i} + 1\,\hat{j} \\ 0 \le t \le 1 \end{cases}$$

$$\gamma_4 : \begin{cases} \vec{r}_4(t) = 0\,\hat{i} + (1-t)\,\hat{j} \\ 0 \le t \le 1 \end{cases}$$

2. cont.

$$\int_\gamma xy\,dx + (x^{3/2}+y^{3/2})\,dy =$$

$$= \int_{\gamma_1}(\cdots) + \int_{\gamma_2}(\cdots) + \int_{\gamma_3}(\cdots) + \int_{\gamma_4}(\cdots)$$

$$\int_{\gamma_1} xy\,dx + (x^{3/2}+y^{3/2})\,dy =$$

$$= \int_{\gamma_1}\left(x(t)\,\cancelto{0}{y(t)}\,\frac{dx}{dt} + (x(t)^{3/2}+y^{3/2}(t))\,\cancelto{0}{\frac{dy}{dt}}\right)dt$$

$$= 0$$

$$\int_{\gamma_2} xy\,dx + (x^{3/2}+y^{3/2})\,dy =$$

$$= \int_{\gamma_2}\left(x(t)y(t)\,\cancelto{0}{\frac{dx}{dt}} + (x(t)^{3/2}+y^{3/2}(t))\frac{dy}{dt}\right)dt$$

$$= \int_0^1 (1+t^{3/2})\,dt$$

$$= t + \frac{2}{5}t^{5/2}\Big]_0^1$$

$$= 1+\frac{2}{5} = \frac{7}{5}$$

$$\int_{\gamma_3} xy\,dx + (x^{3/2}+y^{3/2})\,dy =$$

$$= \int_0^1\left(x(t)y(t)\frac{dx}{dt} + x(t)^{3/2}+y^{3/2}(t)\,\cancelto{0}{\frac{dy}{dt}}\right)dt$$

$$= \int_0^1 (1-t)(-)\,dt = \frac{(1-t)^2}{2}\Big]_0^1 = -\frac{1}{2}$$

$$\int_{\gamma_4} xy\,dx + (x^{3/2}+y^{3/2})\,dy =$$

$$= \int_0^1\left(x(t)y(t)\,\cancelto{0}{\frac{dx}{dt}} + (x(t)^{3/2}+y(t)^{3/2})\frac{dy}{dt}\right)dt$$

$$= \int_0^1 (1-t)^{3/2}(-1)\,dt$$

$$= \frac{2}{5}(1-t)^{5/2}\Big]_0^1$$

$$= -\frac{2}{5}$$

$\therefore$

$$\boxed{\int_\gamma xy\,dx + (x^{3/2}+y^{3/2})\,dy =}$$

$$= 0 + \frac{7}{5} - \frac{1}{2} - \frac{2}{5} = 0 + 1 - \frac{1}{2}$$

$$= \boxed{\frac{1}{2}}$$

Obs.: Usando o teorema de Green:

$$\int_\gamma \underbrace{xy}_{M}\,dx + \underbrace{(x^{3/2}+y^{3/2})}_{N}\,dy =$$

$$= \int_R \left[\frac{\partial(x^{3/2}+y^{3/2})}{\partial x} - \frac{\partial(xy)}{\partial y}\right] dA$$

$$= \int_R \left(\frac{3}{2}x^{1/2} - x\right) dA$$

$$= \int_{x=0}^{1} dx \int_{y=0}^{1} dy \left(\frac{3}{2}x^{1/2} - x\right)$$

$$= \int_{x=0}^{1} dx \left(\frac{3}{2}x^{1/2} - x\right) y \Big]_0^1$$

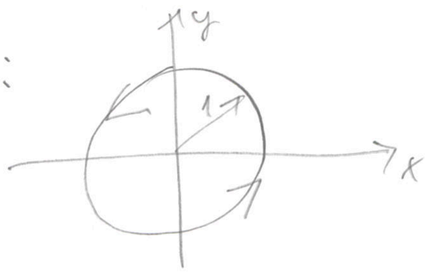
$$= \int_{x=0}^{1} dx \left(\frac{3}{2}x^{1/2} - x\right)$$

$$= x^{3/2} - \frac{x^2}{2}\Big]_0^1$$

$$= 1 - \frac{1}{2} = \frac{1}{2} \quad \text{ok!}$$

3. $$\int_\gamma (x^2+y^2)^{\frac{3}{2}}dx + (x^2+y^2)^{\frac{3}{2}}dy$$

γ :



$$\gamma : \begin{cases} \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} \\ 0 \leq t \leq 2\pi \end{cases}$$

$$\int_\gamma (x^2+y^2)^{3/2}dx + (x^2+y^2)^{3/2}dy =$$

$$= \int_0^{2\pi} \left(\underbrace{(x(t)^2+y(t)^2)}_{1}{}^{3/2}\frac{dx}{dt} + (x^2(t)+y^2(t))^{3/2}\frac{dy}{dt}\right)$$

$$= \int_0^{2\pi} (-\sin t + \cos t)\, dt$$

$$= \cos t + \sin t \Big]_0^{2\pi}$$

$$= \cos 2\pi + \cancel{\sin 2\pi}^{\,0} - \cos 0 - \cancel{\sin}$$

$$= 1 - 1 = 0$$

Obs.: Usando o teorema de Green

$$\int_\gamma \underbrace{(x^2+y^2)^{3/2}}_{M}\, dx + \underbrace{(x^2+y^2)^{3/2}}_{N}\, dy =$$

$$= \int_R \left(\frac{\partial (x^2+y^2)^{3/2}}{\partial x} - \frac{\partial}{\partial y}(x^2+y^2)^{3/2}\right) dA$$

$$= \int_R \left(\frac{3}{2}(x^2+y^2)^{\frac{1}{2}}\, 2x - \frac{3}{2}(x^2+y^2)^{\frac{1}{2}}\, 2y\right) dA$$

$$= \int_R \left(3x(x^2+y^2)^{\frac{1}{2}} - 3y(x^2+y^2)^{\frac{1}{2}}\right) dA$$

$$= \int_R 3(x-y)(x^2+y^2)^{1/2}\, dA$$

Seja $x = r\cos\theta$, $y = r\sin\theta$

$$= \int_{r=0}^{1} dr \int_{\theta=0}^{2\pi} d\theta\; 3(r\cos\theta - r\sin\theta)\, r\, \underbrace{r}_{\text{Jacobiano}}$$

$$= \int_{r=0}^{1} dr \int_{\theta=0}^{2\pi} d\theta \left(3r^3\cos\theta - 3r^3\sin\theta\right)$$

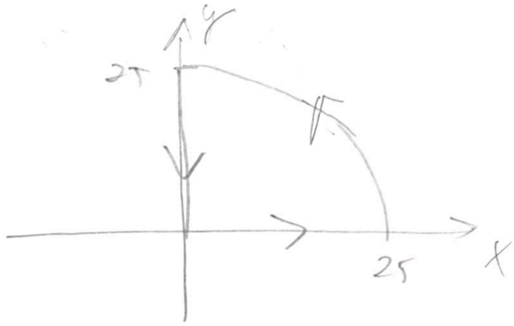
$$= \int_{r=0}^{1} dr \left(3r^3 \cancelto{0}{\sin\theta}\Big]_0^{2\pi} - \right.$$
$$\left. + 3r^3 \cancelto{0}{\cos\theta}\Big]_0^{2\pi} \right)$$

$= 0$ //

4. $$\int_\gamma \underbrace{e^x \sin y}_{M}\, dx + \underbrace{e^x \cos y}_{N}\, dy$$

$\sqrt{x} + \sqrt{y} = 5$

∴



∴ $R := \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 25,\ 0 \le y \le (5-\sqrt{x})^2\}$

∴

$$\int_\gamma e^x \sin y\, dx + e^x \cos y\, dy =$$

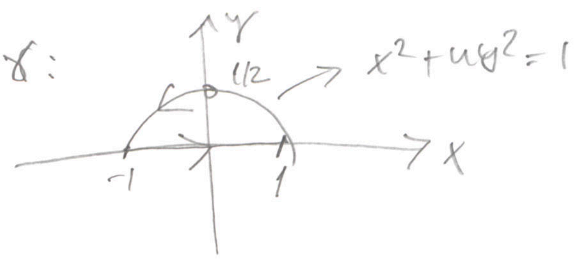
$$= \int_R \left[\frac{\partial}{\partial x}(e^x\cos y) - \frac{\partial}{\partial y}(e^x \sin y)\right] dA$$

$$= \int_R \left(\cancel{e^x\cos y} - \cancel{e^x \cos y}\right) dA$$

$= 0$ //

7. $\int_\gamma \underbrace{xy}_{M}\,dx + \underbrace{\left(\frac{1}{2}x^2 + xy\right)}_{N} dy$

γ:



$$\int_\gamma xy\,dx + \left(\frac{1}{2}x^2 + xy\right) dy =$$

$$= \int_R \left(\frac{\partial}{\partial x}\left(\frac{1}{2}x^2+xy\right) - \frac{\partial}{\partial y}(xy)\right) dA$$

$$= \int_R \left[(x+y) - x\right] dA$$

$$= \int_R y\,dA$$

$$R = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; -1 \le x \le 1,\ 0 \le y \le \frac{1}{2}\sqrt{1-x^2} \right\}$$

$$= \int_{-1}^{1} dx \int_0^{\frac{1}{2}\sqrt{1-x^2}} dy\, y$$

$$= \int_{-1}^{1} dx\, \frac{y^2}{2}\Bigg]_0^{\frac{1}{2}\sqrt{1-x^2}}$$

$$= \int_{-1}^{1} dx\, \frac{1}{2}\,\frac{1}{4}(1-x^2)$$

$$= \frac{1}{8}\int_{-1}^{1} dx\,(1-x^2)$$

$$= \frac{1}{8}\left(x - \frac{x^3}{3}\right)\Bigg]_{-1}^{1}$$

$$= \frac{1}{8}\left(1 - \frac{1}{3} - \left(-1 - \frac{(-1)^3}{3}\right)\right)$$

$$= \frac{1}{8}\left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right)$$

$$= \frac{1}{8}\left(2 - \frac{2}{3}\right)$$

$$= \frac{1}{\cancel{8}_{\,2}}\left(\frac{\cancel{4}}{3}\right)$$

$$= \frac{1}{6}$$

6.

$$\int_\gamma (\cos^3 x + e^x)\,dx + e^y\,dy$$

$$\gamma : \left\{ x^6 + y^8 = 1 \right.$$

$(\pm 1, 0), (0, \pm 1) \in \gamma.$

Qualquer que seja a curva γ podemos escolher para a região R (interior de γ) a seguinte parametrização:

$$R = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; -1 \leq x \leq 1, \; -\sqrt[8]{1-x^6} \leq y \leq \sqrt[8]{1-x^6} \right\}$$

2.º

$$\int_\gamma \underbrace{(\cos^3 x + e^x)}_{M}\,dx + \underbrace{e^y}_{N}\,dy =$$

$$= \int_\gamma \left(\cancel{\frac{\partial e^y}{\partial x}}^{\,0} - \cancel{\frac{\partial (\cos^3 x + e^x)}{\partial y}}^{\,0} \right) dA$$

$= 0$

7.

$$\int_\gamma \vec{F} \cdot d\vec{r}$$

$$\vec{F}(x,y) = y\,\hat{i} + 3x\,\hat{j}$$

$$\gamma : x^2 + y^2 = 4$$

$$\int_\gamma \vec{F} \cdot d\vec{r} = \int F_1\,dx + F_2\,dy$$

$$= \int_\gamma \underbrace{y}_{M}\,dx + \underbrace{3x}_{N}\,dy$$

$$= \int_R \left(\frac{\partial (3x)}{\partial x} - \frac{\partial y}{\partial y} \right) dA$$

$$= \int_R (3 - 1)\,dA$$

$$= 2 \underbrace{\int_R dA}_{\text{área do círculo de raio } 2}$$

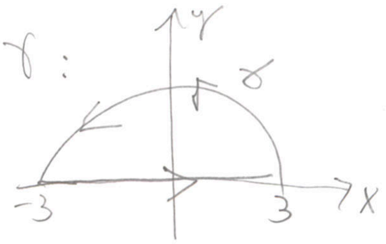
$$= 2\pi (2)^2$$

$$= 8\pi$$

8. $\int_\gamma \vec{F}\cdot d\vec{r}$

$\vec{F} = y\sin x\,\hat{i} - \cos x\,\hat{j}$

γ :



$$\int_\gamma \vec{F}\cdot d\vec{r} = \int_\gamma F_1\,dx + F_2\,dy$$

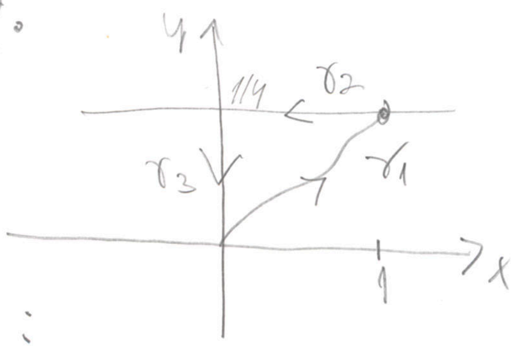
$$= \int_\gamma \underbrace{y\sin x}_{M}\,dx - \underbrace{\cos x}_{N}\,dy$$

$$= \int_R \left[\frac{\partial}{\partial x}(-\cos x) - \frac{\partial}{\partial y}(y\sin x)\right] dA$$

$$= \int_R (\cancel{\sin x} - \cancel{\sin x})\,dA$$

$$= 0 //$$

9.



γ_1 :

$\vec{r}(t) = \sin \pi t\,\hat{i} + t(1-t)\,\hat{j}$

$0 \le t \le \frac{1}{2}$

$\vec{r}(0) = (0,0)$, $\vec{r}(\frac{1}{2}) = (1, \frac{1}{4})$

Há 3 possibilidades para o cálculo da área de uma região via o teorema de Green:

$$A = \int_\gamma x\,dy = -\int_\gamma y\,dx =$$

$$= \frac{1}{2}\int_\gamma (x\,dy - y\,dx)$$

Usaremos a forma

$$A = \int_\gamma x\,dy = \int_{\gamma_1} x\,dy + \int_{\gamma_2} x\,dy + \int_{\gamma_3} x\,dy$$

$\therefore$

$$\int_{\gamma_1} x\,dy = \int_0^{1/2} x(t)\,\frac{dy}{dt}\,dt$$

$$= \int_0^{1/2} \sin \pi t\,(1-2t)\,dt$$

$$= \underbrace{\int_0^{\frac{1}{2}} \sin \pi t\,dt}_{(*)} - \underbrace{\int_0^{1/2} 2t\sin \pi t\,dt}_{(**)}$$

$$(*) = \int_0^{\frac{1}{2}} \sin \pi t\,dt = -\frac{\cos \pi t}{\pi}\Bigg]_0^{1/2}$$

$$= \frac{1}{\pi}$$

9. cont.

Mas $\int 2t \sin \pi t \, dt$

$u = 2t \rightarrow du = 2\, dt$

$dv = \sin \pi t \, dt \rightarrow v = -\dfrac{\cos \pi t}{\pi}$

$$= -\frac{2t}{\pi}\cos \pi t + \int \frac{2}{\pi}\cos \pi t \, dt$$

$$= -\frac{2t\cos \pi t}{\pi} + \frac{2}{\pi^2}\sin \pi t$$

$\therefore$

$$\int_0^{\frac{1}{2}} 2t \sin \pi t \, dt =$$

$$= -\frac{2t}{\pi}\cos \pi t + \frac{2}{\pi^2}\sin \pi t \Big|_0^{1/2}$$

$$= -\frac{1}{\pi}\cancelto{0}{\cos \frac{\pi}{2}} + \frac{2}{\pi^2}\sin \frac{\pi}{2}$$

$$+ \frac{2.0\cos 0}{\pi} - \frac{2}{\pi^2}\cancelto{0}{\sin 0}$$

$$= \frac{2}{\pi^2}$$

$\therefore$

$$\int_{\gamma_1} x\, dy = (*) - (**)$$

$$= \frac{1}{\pi} - \frac{2}{\pi^2}$$

$$= \frac{1}{\pi} - \frac{2}{\pi^2}$$

$\gamma_2 : \begin{cases} \vec{r}(t) = (1-t)\,\hat{i} + \frac{1}{4}\,\hat{j} \\ 0 \le t \le 1 \end{cases}$

$$\int_{\gamma_2} x\, dy = \int_0^1 x(t)\, \cancelto{0}{\frac{dy}{dt}}\, dt = 0$$

$\gamma_3 : \begin{cases} \vec{r}(t) = 0\,\hat{i} + \left(\frac{1}{4} - t\right)\hat{j} \\ 0 \le t \le \frac{1}{4} \end{cases}$

$$\int_{\gamma_3} x\, dy = \int 0 \,\frac{dy}{dt}\, dt = 0$$

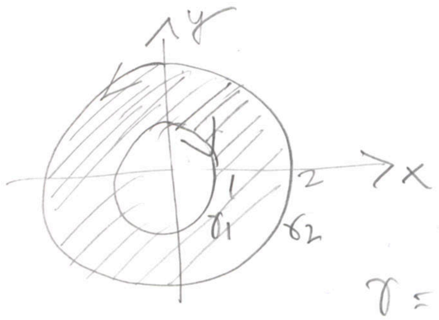
$\therefore$

$$A = \oint_{\gamma} x\, dy = \int_{\gamma_1} x\, dy = \boxed{\frac{1}{\pi} - \frac{2}{\pi^2}}$$

10.

$$\begin{cases} M(x,y) = \dfrac{-y}{x^2+y^2} \\[2ex] N(x,y) = \dfrac{x}{x^2+y^2} \end{cases}$$

a)



$\gamma = \gamma_1 \cup \gamma_2$

Temos

$$\int_\gamma M(x,y)\,dx + N(x,y)\,dy =$$

$$= \int_\gamma \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy =$$

$$= \oint_{\gamma_1} (\quad) + \oint_{\gamma_2} (\quad)$$

Mas

$$\gamma_1 : \begin{cases} \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} \\ 0 \le t \le 2\pi \end{cases}$$

$\downarrow$ reverte a orientação

$$\vec{r}(t) = \cos(2\pi - t)\,\hat{i} + \sin(2\pi - t)\,\hat{j}$$

$$= \cos t\,\hat{i} - \sin t\,\hat{j}, \; 0 \le t \le 2\pi$$

$\therefore$

$$\oint_{\gamma_1} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy =$$

$$= \int_0^{2\pi} \left[\frac{-(-\sin t)}{1}(-\sin t) + \frac{\cos t}{1}(-\cos t) \right] dt$$

$$= \int_0^{2\pi} (-\sin^2 t - \cos^2 t)\,dt$$

$$= \underline{-2\pi}$$

$$\gamma_2 : \begin{cases} \vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} \\ 0 \le t \le 2\pi \end{cases}$$

$$\oint_{\gamma_2} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$= \int_0^{2\pi} \left(\frac{-2\sin t}{4}(-2\sin t) + \frac{2\cos t}{4} 2\cos t \right) dt$$

$$= \int_0^{2\pi} \left(\frac{4\sin^2 t}{4} + \frac{4\cos^2 t}{4} \right) dt$$

$$= \underline{2\pi}$$

10. Cont.

$$\int_\gamma \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy =$$

$$= -2\pi + 2\pi = \underline{\underline{0}}$$

Agora:

$$\int_D dA \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) =$$

$$= \int_D dA \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right)$$

$$= \int_D dA \left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right)$$

$$= \int_D dA \left(\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right)$$

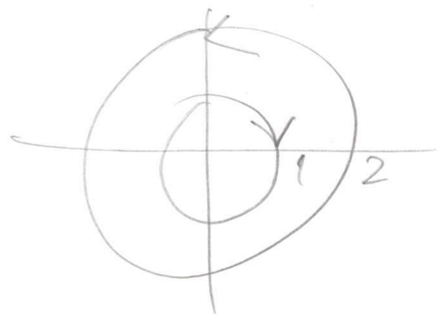
$$= \int_D dA \, \frac{\cancel{2x^2}+\cancel{2y^2}-\cancel{2x^2}-\cancel{2y^2}}{(x^2+y^2)^2}$$

$$= 0$$

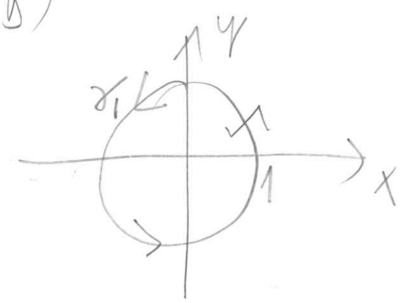
i.e., teorema

$$\boxed{\int_\gamma M(x,y)\,dx + N(x,y)\,dy = \int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA}$$

com D a região



b)



$$\oint_{\gamma_1} M(x,y)\,dx + N(x,y)\,dy$$

$$\gamma_1 : \begin{cases} \vec{r}_1(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} \\ 0 \le t \le 2\pi \end{cases}$$

$\therefore$

$$\oint_{\gamma_1} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$= \int_0^{2\pi} \left(-\sin t(-\sin t) + \cos t \cos t\right) dt$$

$$\iint_D \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{=0} dA = 0$$

Temos então que para a região abaixo

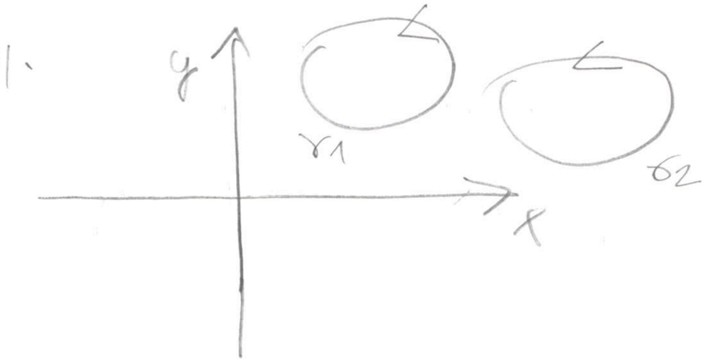
o teorema de Green não se aplica:

$$\oint_\gamma M(x,y)dx + N(x,y)dy \neq \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

c) O resultado (b) não viola o teorema de Green pois as derivadas parciais

$\frac{\partial N}{\partial x}, \frac{\partial M}{\partial y}$ não são contínuas em D, condições necessárias para que se tenha o teorema de Green.

II.



$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \text{ em } D$$

$$\oint_{\gamma_1} M(x,y)dx + N(x,y)dy =$$

$$= \iint_{D_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= 0 \quad \text{pois } \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

também

$$\oint_{\gamma_2} M(x,y)dx + N(x,y)dy =$$

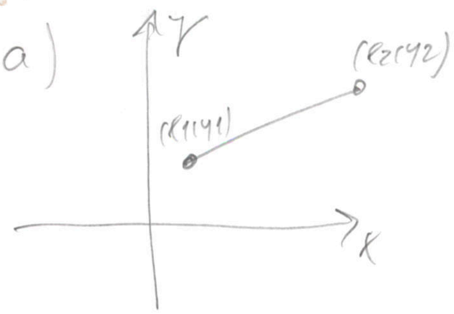
$$= \iint_{D_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= 0$$

$$\therefore \boxed{\oint_{\gamma_1} M dx + N dy = \oint_{\gamma_2} M dx + N dy}$$

12.

a)



$$\frac{1}{2}\int_\gamma x\,dy - y\,dx =$$

γ: $x - x_1 = t(x_2 - x_1)$

$\quad y - y_1 = t(y_2 - y_1)$

$$\begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1) \\ 0 \leq t \leq 1 \end{cases}$$

$$\frac{1}{2}\int_\gamma x\,dy - y\,dx =$$

$$= \frac{1}{2}\int_0^1 \left(x(t)\frac{dy}{dt} - y\frac{dx}{dt} \right) dt$$

$$= \frac{1}{2}\int_0^1 \Big[\big(x_1 + t(x_2 - x_1)\big)(y_2 - y_1) - \big(y_1 + t(y_2 - y_1)\big)(x_2 - x_1) \Big] dt$$

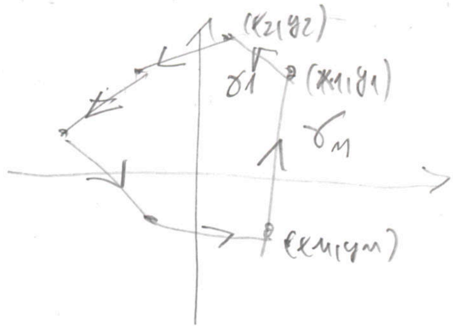
$$= \frac{1}{2}\int_0^1 \Big[x_1(y_2 - y_1) + \cancel{t(x_2 - x_1)(y_2 - y_1)} - y_1(x_2 - x_1) - \cancel{t(y_2 - y_1)(x_2 - x_1)} \Big] dt$$

$$= \frac{1}{2}\big(x_1(y_2 - y_1) - y_1(x_2 - x_1) \big)$$

$$= \frac{1}{2}\left(x_1 y_2 - \cancel{x_1 y_1} - y_1 x_2 + \cancel{y_1 x_1} \right)$$

$$= \frac{1}{2}\left(x_1 y_2 - x_2 y_1 \right)$$

b)



Aqui, usando o teorema de Green podemos calcular a área do polígono como

$$A = \frac{1}{2}\oint_\gamma x\,dy - y\,dx$$

$$= \frac{1}{2}\int_{\gamma_1} x\,dy - y\,dx + \frac{1}{2}\int_{\gamma_2} x\,dy - y\,dx + \dots + \frac{1}{2}\int_{\gamma_n} x\,dy - y\,dx$$

$$= \frac{1}{2}(x_1 y_2 - x_2 y_1) + \frac{1}{2}(x_2 y_3 - x_3 y_2) + \dots + \frac{1}{2}(x_n y_1 - x_1 y_n)$$

$$
\begin{aligned}
A = {} & \tfrac{1}{2}(x_1 y_2 - x_2 y_1) + \\
& + \tfrac{1}{2}(x_2 y_3 - x_3 y_2) \\
& + \dots + \tfrac{1}{2}(x_{n-1} y_n - x_n y_{n-1}) \\
& + \tfrac{1}{2}(x_n y_1 - x_1 y_n)
\end{aligned}
$$

c) $\overset{x_1\,y_1}{(0|0)}$, $\overset{x_2\,y_2}{(1|0)}$, $\overset{x_3\,y_3}{(2|3)}$, $\overset{x_4\,y_4}{(-1|1)}$

$$
\begin{aligned}
A = {} & \tfrac{1}{2}(\cancel{0 \cdot 0}^{\,0} - \cancel{1 \cdot 0}^{\,0}) + \\
& + \tfrac{1}{2}(1 \cdot 3 - \cancel{2 \cdot 0}^{\,0}) \\
& + \tfrac{1}{2}(2 \cdot 1 - (-1) \cdot 3) \\
& + \tfrac{1}{2}(-1 \cdot 0 - 0 \cdot 1) \\
= {} & \tfrac{3}{2} + \tfrac{1}{2}(2 + 3) \\
= {} & \tfrac{3}{2} + \tfrac{5}{2} = 4
\end{aligned}
$$