Cálculo C - Lista 9

Equações de variáveis separáveis

Resolva as equações

1. $y' = \cancel{kx}$ $(k = \text{cte})$ $K \cdot y$
2. $y' = y \cot x$
3. $y' = 3x^2 y$
4. $y' + ay + b = 0 \quad (a \neq 0)$
5. $y' = e^x y^3$
6. $y' = \sqrt{1 - y^2}$
7. $(x \ln x) y' = y$
8. $y' \sin 2x = y \cos 2x$
9. $(1 + y^2)\, dx + xy\, dy = 0$
10. $(y^2 + xy^2) y' + x^2 - yx^2 = 0$
11. $(1 + y^2)\, dx = x dy$
12. $x\sqrt{1 + y^2} + yy'\sqrt{1 + x^2} = 0$
13. $x\sqrt{1 - y^2}\, dx + y\sqrt{1 - x^2}\, dy = 0$
    $y(0) = 1$
14. $e^{-y}(1 + y') = 1$
15. $y \ln y\, dx + x\, dy = 0$
    $y(1) = 1$
16. $y' = a^{x+y} \quad a > 0,\ a \neq 1$
17. $e^y(1 + x^2) dy - 2x(1 + e^y)\, dx$ $= 0$
18. $(1 + e^x) yy' = e^y$
    $y(0) = 0$
19. $(1 + y^2)(e^{2x}\, dx - e^y\, dy) - (1 + y)\, dy = 0$
20. $(xy^2 - y^2 + x - 1)\, dx + (x^2 y - 2xy + x^2 + 2y - 2x + 2)\, dy = 0$
21. $y' = \sin(x - y)$
22. $y' = ax + by + c \quad (a, b, c$ são constantes$)$
23. $(x + y)^2 y' = a^2$
24. $(1 - y) e^y y' + \frac{y^2}{x \ln x} = 0$
25. $(1 + y^2)\, dx = (y - \sqrt{1 + y^2})(1 + x^2)^{3/2}\, dy$
26. $xy^2(xy' + y) = a^2$ (Sugestão : use $v = yx$)
27. $(x^2 y^2 + 1)\, dx + 2x^2\, dy = 0$
    [Use a substituição: $xy = u$]
28. $(1 + x^2 y^2) y + (xy - 1)^2 x y' = 0$
    [Use a substituição: $xy = u$]
29. $(x^2 y^3 + y + x - 2)\, dx + (x^3 y^2 + x)\, dy = 0$
    [Use a substituição: $xy = u$]
30. $(x^6 - 2x^5 + 2x^4 - y^3 + 4x^2 y)\, dx + (xy^2 - 4x^3)\, dy = 0$
    [Use a substituição: $y = ux$]
31. $$y' + 1 = \frac{(x + y)^m}{(x + y)^n + (x + y)^p}$$
32. $(\ln x + y^3)\, dx - 3xy^2\, dy = 0$
33. $(xy + 2xy(\ln y)^2 + y \ln y)\, dx + (2x^2 \ln y + x)\, dy = 0$
    [Use a substituição: $x \ln y = t$]
34. $y - xy' = a(1 + x^2 y')$
35. $(a^2 + y^2)\, dx + 2x\sqrt{ax - x^2}\, dy = 0$
    $y(a) = 0$
36. $y' + \sin \frac{x+y}{2} = \sin \frac{x-y}{2}$

Lista

1. $y' = k y$

$\frac{dy}{dx} = ky$

$\int \frac{dy}{y} = \int k\,dx + c$

$\ln|y| = kx + c$

$|y| = e^{kx} e^{c}$

$\boxed{y = C_1 e^{kx}}$

2. $y' = y \cot x$

$\frac{dy}{y} = \cot g x\, dx$

$\int \frac{dy}{y} = \int \cot g x\, dx + c$

$\ln|y| = \ln|\sin x| + c$

$\ln|y| - \ln|\sin x| = c$

$\ln \left|\frac{y}{\sin x}\right| = c$

$\left|\frac{y}{\sin x}\right| = e^{c}$ cte

$\boxed{y = C_1 \sin x}$

3. $y' = 3x^2 y$

$\frac{y'}{y} = 3x^2$

$\frac{dy}{y} = 3x^2 dx$

$\int \frac{dy}{y} = 3\int x^2 dx + c$

$\ln|y| = \cancel{3}\frac{x^3}{\cancel{3}} + c$

$|y| = e^{x^3} e^{c}$

$\boxed{y = C_1 e^{x^3}}$

4. $y' + ay + b = 0 \quad (a \neq 0)$

$y' = -(ay+b)$

$\frac{y'}{ay+b} = -1$

$\frac{dy}{ay+b} = -dx$

$\frac{1}{a} \ln|ay+b| = -x + c$

$\ln|ay+b| = -ax + C_1$

$|ay+b| = e^{-ax} e^{C_1}$

$ay + b = C_2 e^{-ax}$

$y = \frac{C_2 e^{-ax} - b}{a}$

5. $y' = e^x y^3$

$$\frac{dy}{y^3} = e^x dx$$

$$\int \frac{dy}{y^3} = \int e^x dx + C$$

$$\frac{y^{-2}}{-2} = e^x + C$$

$$y^{-2} = -2e^x - 2C$$

$$y^{-2} = -2e^x + C_1$$

$$y^2 = \frac{1}{-2e^x + C_1}$$

$$y = \pm \frac{1}{\sqrt{C_1 - 2e^x}}$$

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6. $y' = \sqrt{1-y^2}$

$$\frac{dy}{\sqrt{1-y^2}} = dx$$

$$\underbrace{\int \frac{dy}{\sqrt{1-y^2}}} = \int dx + C$$

$$\arcsin y = x + C$$

$$y = \sin(x+C)$$

7. $(x \ln x) y' = y$

$$\frac{y'}{y} = \frac{1}{x \ln x}$$

$$\frac{dy}{y} = \frac{dx}{x \ln x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x \ln x} + C$$

$$\ln|y| = \ln|\ln x| + C$$

$$\int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln|u|$$

$$u = \ln x, \quad du = \frac{1}{x} dx$$

$$= \ln|\ln x|$$

$$|y| = e^{\ln|\ln x|} e^C$$

$$y = C_1 e^{\ln|\ln x|}$$

$$\boxed{y = C_1 |\ln x|}$$

8. $y' \sin 2x = y \cos 2x$

$$\frac{y'}{y} = \cot g\, 2x$$

$$\frac{dy}{y} = \cot g\, 2x \, dx$$

$$\ln|y| = \frac{1}{2} \ln|\sin 2x| + c$$

$$|y| = e^{\frac{1}{2}\ln|\sin 2x|} e^{c}$$

$$y = C_1 e^{\ln|\sin 2x|^{1/2}}$$

$$= C_1 |\sin 2x|^{1/2}$$

$$y = C_1 \sqrt{|\sin 2x|}$$

9.

9. $(1+y^2)dx + xy\,dy = 0$

$xy\,dy = -(1+y^2)dx$

$\frac{y}{1+y^2}dy = -\frac{dx}{x}$

$\int \frac{y}{1+y^2}dy = -\int \frac{dx}{x} + C$

$\frac{1}{2}\ln|1+y^2| = -\ln|x| + C$

$\ln(1+y^2)^{1/2} + \ln|x| = C$

$\ln(1+y^2)^{1/2}|x| = C$

$|x|(1+y^2)^{1/2} = e^C$

$\boxed{x^2(1+y^2) = C_1}$

. $(y^2+xy^2)y' + x^2 - yx^2 = 0$

$y^2(1+x)y' + x^2(1-y) = 0$

$y^2(1+x)y' = -x^2(1-y)$

$\frac{y^2}{(1-y)}y' = \frac{-x^2}{1+x}$

$\frac{y^2}{1-y}dy = \frac{-x^2}{1+x}dx$

$\int \frac{y^2}{1-y}dy = -\int \frac{x^2}{1+x}dx + C$ (*)

Mas

$\int \frac{y^2}{1-y}dy = \int \frac{(1-u)^2}{u}(-du)$

$u = 1-y$
$du = -dy$

$= \int \frac{-1+2u-u^2}{u}du$

$= \int -\frac{1}{u}du + \int 2du - \int u\,du$

$= -\ln|u| + 2u - \frac{u^2}{2}$

$= -\ln|1-y| + 2(1-y) - \frac{(1-y)^2}{2}$

$\int \frac{x^2}{1+x}dx = \int \frac{(v-1)^2}{v}dv$

$v = 1+x$, $dv = dx$

$= \int \frac{v^2-2v+1}{v}dv$

$= \int \left(v - 2 + \frac{1}{v}\right)dv$

$= \frac{v^2}{2} - 2v + \ln|v|$

$= \frac{(1+x)^2}{2} - 2(1+x) + \ln|1+x|$

(*) ⇒

$-\ln|1-y| + 2(1-y) - \frac{(1-y)^2}{2} =$

$= -\frac{(1+x)^2}{2} + 2(1+x) - \ln|1+x|$
$+ C$

$\therefore$

$-\ln|1-y| + \ln|1+x| + 2\cancel{(1}-y) +$
$- 2\cancel{(1}+x) - \frac{(1-y)^2}{2} + \frac{(1+x)^2}{2} = C$

$\therefore$

$\ln\left|\frac{1+x}{1-y}\right| + -2y - 2x +$
$+ -\frac{(1-y)^2}{2} + \frac{(1+x)^2}{2} = C$

$\ln\left|\frac{1+x}{1-y}\right| - 2(x+y)$
$- \cancel{\frac{1}{2}} + y - \frac{y^2}{2} + \cancel{\frac{1}{2}} + x + \frac{x^2}{2} = C$

$\ln\left|\frac{1+x}{1-y}\right| - 2(x+y) + (x+y)$
$+ \frac{1}{2}(x-y)(x+y) = C$

$\ln\left|\frac{1+x}{1-y}\right| - (x+y) + \frac{1}{2}(x+y)(x-y) = C$

$\ln\left|\frac{1+x}{1-y}\right| + (x+y)\left(\frac{1}{2}(x-y) - 1\right) = C$

$\boxed{\ln\left|\frac{1+x}{1-y}\right| + \frac{(x+y)}{2}(x-y-2) = C}$

11 $(1+y^2)\,dx = x\,dy$

$\frac{dy}{1+y^2} = \frac{dx}{x}$

$\int \frac{dy}{1+y^2} = \int \frac{dx}{x} + C$

$\operatorname{arctg} y = \ln|x| + C$

$\boxed{y = \operatorname{tg}(\ln|x| + C)}$

(?)

12.

$$x\sqrt{1+y^2} + yy'\sqrt{1+x^2} = 0$$

$$yy'\sqrt{1+x^2} = -x\sqrt{1+y^2}$$

$$\frac{yy'}{\sqrt{1+y^2}} = \frac{-x}{\sqrt{1+x^2}}$$

$$\frac{y\,dy}{\sqrt{1+y^2}} = -\frac{x}{\sqrt{1+x^2}}\,dx$$

$$\int\frac{y\,dy}{\sqrt{1+y^2}} = -\int\frac{x}{\sqrt{1+x^2}}\,dx + C$$

$$\sqrt{1+y^2} = -\sqrt{1+x^2} + C$$

$$\boxed{\sqrt{1+y^2} + \sqrt{1+x^2} = C}$$

3.

$$x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$$

$$y(0) = 1$$

$$\frac{y\,dy}{\sqrt{1-y^2}} = -\frac{x}{\sqrt{1-x^2}}\,dx$$

$$-\sqrt{1-y^2} = \sqrt{1-x^2} + C$$

$$\sqrt{1-x^2} + \sqrt{1-y^2} = C_1$$

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$y(0) = 1$:

$$\sqrt{1-0^2} + \sqrt{1-1} = C_1$$

$$C_1 = 1$$

$\therefore$

$$\boxed{\sqrt{1-x^2} + \sqrt{1-y^2} = 1}$$

$y(0) = 1$

14. $e^{-y}(1+y') = 1$

$$1 + y' = e^y$$

$$1 + \frac{dy}{dx} = e^y$$

$$dx + dy = dx\,e^y$$

$$dy = dx\,(e^y - 1)$$

$$\frac{dy}{e^y-1} = dx$$

$$\int\frac{dy}{e^y-1} = \int dx + C \quad (*)$$

Hos

$$\int\frac{dy}{e^y-1} = \int\frac{du}{(u+1)u}$$

$u = e^y - 1$

$du = e^y\,dy \Rightarrow dy = \frac{du}{\ } = \frac{du}{\ }$

$$\int \frac{dy}{e^y - 1} = \int \frac{du}{u(u+1)}$$

$$= \int \frac{du}{u} - \int \frac{du}{u+1}$$

$$= \ln|u| - \ln|u+1|$$

$$= \ln\left|\frac{u}{u+1}\right|$$

$$= \ln\left|\frac{e^y - 1}{e^y}\right|$$

$\therefore$

$(*)$:

$$\ln\left|\frac{e^y-1}{e^y}\right| = x + C$$

$$\left|\frac{e^y-1}{e^y}\right| = e^{x+C} = e^x e^C$$

$$\frac{e^y-1}{e^y} = C_1 e^x$$

$$1 - e^{-y} = C_1 e^x$$

$$\boxed{e^x = C_2(1 - e^{-y})}$$

$$\begin{cases} y \ln y \, dx + x\, dy = 0 \\ y(1) = 1 \qquad (y > 0) \end{cases}$$

$$\frac{dy}{y \ln y} = -\frac{dx}{x}$$

$$\int \frac{dy}{y\ln y} = -\int \frac{dx}{x} + C \qquad (*)$$

Mas

$$u = \ln y \;,\; du = \frac{1}{y}\, dy$$

$$\int \frac{dy}{y \ln y} = \int \frac{du}{u}$$

$$= \ln|u|$$

$$= \ln|\ln y|$$

$(*)$:

$$\ln|\ln y| = -\ln|x| + C$$

$$\ln|x \ln y| = C$$

$$|x \ln y| = e^C$$

$$x \ln y = C_1$$

$$\ln y = \frac{C_1}{x}$$

$$y = e^{\frac{C_1}{x}} \;;\; y(1) = 1$$

$$\Rightarrow 1 = e^{C_1} \Rightarrow C_1 = 0 \therefore \boxed{y = 1}$$

16. $\begin{cases} y' = a^{x+y} \\ a > 0,\ a \neq 1 \end{cases}$

$$\frac{dy}{dx} = a^x a^y$$

$$a^{-y} dy = a^x dx$$

$$\int a^{-y} dy = \int a^x dx + C$$

$$-\frac{a^{-y}}{\ln a} = \frac{a^x}{\ln a} + C$$

$$\left[\frac{da^x}{dx} = \ln a \; a^x\right]$$

$$\frac{a^x}{\ln a} + \frac{a^{-y}}{\ln a} = C$$

$$\boxed{a^x + a^{-y} = C_1}$$

17.

$$e^y(1+x^2)\,dy - 2x(1+e^y)\,dx = 0$$

$$\frac{e^y}{1+e^y} dy = \frac{2x}{1+x^2} dx$$

$$\int \frac{e^y}{1+e^y} dy = \int \frac{2x}{1+x^2} dx + C$$

$$\ln|1+e^y| = \ln|1+x^2| + C$$

$$\ln \frac{(1+e^y)}{1+x^2} = C$$

$$\frac{1+e^y}{1+x^2} = e^C$$

$$1+e^y = C_1(1+x^2)$$

$$e^y = C_1(1+x^2) - 1$$

$$\boxed{y = \ln\left[C_1(1+x^2) - 1\right]}$$

18. $\begin{cases} (1+e^x)\, y\, y' = e^y \\ y(0)=0 \end{cases}$

$$\frac{y\,dy}{e^y} = \frac{dx}{1+e^x}$$

$$\int \frac{y}{e^y}\,dy = \int \frac{dx}{1+e^x} + C \quad (*)$$

Mas

$$\int \frac{y}{e^y}\,dy = \int y e^{-y}\,dy$$

$u = y \rightarrow du = dy$

$dv = e^{-y}dy \rightarrow v = -e^{-y}$

$$= -ye^{-y} + \int e^{-y}\,dy$$

$$= -ye^{-y} - e^{-y}$$

$$\int \frac{dx}{1+e^x} = \int \frac{du}{(u-1)u}$$

$u = 1+e^x$

$du = e^x dx$

$dx = \frac{du}{u-1}$

$$= \int \frac{du}{u-1} - \int \frac{du}{u}$$

$$= \ln|u-1| - \ln|u|$$

$$= \ln e^x - \ln|e^x+1|$$

$(*)$:

$$-ye^{-y} - e^{-y} = \ln e^x - \ln e^x+1 + C$$

$$-(y+1)e^{-y} = \ln \frac{e^x}{e^x+1} + C$$

$$-(y+1)e^{-y} = \ln e^x - \ln(e^x+1)$$

$$-(y+1)e^{-y} = x - \ln(e^x+1) +$$

$$(y+1)e^{-y} = \ln(e^x+1) - x + C_1$$

$\therefore \; y(0)=0$

$$1 = \ln 2 + C_1$$

$$// \; C_1 = 1 - \ln 2 \; //$$

$\therefore$

$$(y+1)e^{-y} = \ln(e^x+1) - x + 1 - \ln 2$$

$$\boxed{(y+1)e^{-y} = \ln\left(\frac{e^x+1}{2}\right) - x}$$

19. $(1+y^2)(e^{2x}dx - e^y dy) - (1+y)dy = 0$

$(1+y^2)e^{2x}dx - (1+y^2)e^y dy - (1+y)dy = 0$

$(1+y^2)e^{2x}dx = \left[(1+y^2)e^y + (1+y)\right] dy$

$e^{2x}dx = \dfrac{(1+y^2)e^y + (1+y)}{1+y^2} dy$

$+\int e^{2x}dx = \int \left(e^y + \dfrac{1+y}{1+y^2}\right) dy$

$C + \dfrac{e^{2x}}{2} = e^y + \int \dfrac{1+y}{1+y^2} dy$; $u = 1+y^2$, $du = 2y\,dy$

Mos $= e^y + \int \dfrac{dy}{1+y^2} + \int \dfrac{y}{1+y^2} dy$

$C + \dfrac{e^{2x}}{2} = e^y + \operatorname{arctg} y + \dfrac{1}{2}\ln(1+y^2)$

$$\boxed{\frac{1}{2}e^{2x} - e^y - \operatorname{arctg} y - \ln\sqrt{1+y^2} = C}$$

20. $(xy^2-y^2+x-1)\,dx+(x^2y-2xy+x^2+2y-2x+2)\,dy=0$

$$\left[y^2(x-1)+(x-1)\right]dx+\left[x^2(y+1)-2x(y+1)+2(y+1)\right]dy=0$$

$$(x-1)(y^2+1)\,dx+(y+1)(x^2-2x+2)\,dy=0$$

$$\frac{y+1}{y^2+1}\,dy=-\frac{x-1}{x^2-2x+2}\,dx$$

$$\int\frac{y+1}{y^2+1}\,dy=-\int\frac{x-1}{x^2-2x+2}\,dx+C$$

$$\int\frac{y}{y^2+1}\,dy+\int\frac{dy}{y^2+1}=-\frac{1}{2}\ln|x^2-2x+2|+C$$

$$\frac{1}{2}\ln(y^2+1)+\arctan y=-\frac{1}{2}\ln|\underbrace{x^2-2x+2}_{\geq 0}|+C$$

$$\frac{1}{2}\ln\left[(y^2+1)(x^2-2x+2)\right]+\arctan y=C$$

$$\ln\left[(y^2+1)(x^2-2x+2)\right]+2\arctan y=C_1$$

$$\ln\left[(y^2+1)(x^2-2x+2)\right]=C_1-2\arctan y$$

$$(y^2+1)(x^2-2x+2)=e^{C_1}e^{-2\arctan y}$$

$$\boxed{(y^2+1)(x^2-2x+2)\,e^{2\arctan y}=C_2}$$

21. $y' = \sin(x-y)$ (*)

Seja $u = x - y \therefore y = x - u$

$y' = 1 - u'$ (**)

$\therefore$

(**) $\to$ (*) : $1 - u' = \sin u$

$u' = 1 - \sin u$

$$\frac{du}{1 - \sin u} = dx$$

$$\therefore \int \frac{du}{1-\sin u} = \int dx + C = x + C \quad (***)$$

Mas $\displaystyle\int \frac{du}{1-\sin u} = \int \frac{1+\sin u}{1-\sin^2 u}\, du = \int \frac{1+\sin u}{\cos^2 u}\, du$

$$= \underbrace{\int \sec^2 u\, du}_{} + \underbrace{\int \operatorname{tg} u \sec u\, du}_{}$$

$$= \operatorname{tg} u + \sec u$$

(***) :

$\operatorname{tg} u + \sec u = x + C$

$$\boxed{\operatorname{tg}(x-y) + \sec(x-y) = x + C}$$

Obs - Verificando a solução

$$tg(x-y) + sec(x-y) = x + C$$

$$\therefore \frac{d}{dx} tg(x-y) + \frac{d}{dx} sec(x-y) = 1$$

$$sec^2(x-y)(1-y') + sec(x-y)\,tg(x-y)\,(1-y') = 1$$

$$(1-y')\left[sec^2(x-y) + sec(x-y)\,tg(x-y)\right] = 1$$

$$(1-y')\left[\frac{1}{cos^2(x-y)} + \frac{sin(x-y)}{cos^2(x-y)}\right] = 1$$

$$(1-y')\,\frac{1+sin(x-y)}{cos^2(x-y)} = 1$$

$$(1-y')(1+sin(x-y)) = cos^2(x-y)$$

$$1-y' = \frac{cos^2(x-y)}{1+sin(x-y)}$$

$$y' = 1 - \frac{cos^2(x-y)}{1+sin(x-y)} = \frac{1 + sin(x-y) - cos^2(x-y)}{1+sin(x-y)}$$

$$= \frac{sin^2(x-y) + sin(x-y)}{1+sin(x-y)}$$

$$= \frac{sin(x-y)\,\cancel{(1+sin(x-y))}}{\cancel{(1+sin(x-y))}}$$

$$y' = sin(x-y) \quad \text{ok!}$$

22. $y' = ax + by + c$ (*)

Seja $u = ax + by + c$

$$\frac{du}{dx} = a + by' \implies y' = \frac{u' - a}{b} \quad (**)$$

(**) $\rightarrow$ (*) : $\frac{u' - a}{b} = u$

$$u' - a = bu$$

$$u' = a + bu$$

$$\frac{u'}{a + bu} = 1$$

$$\frac{du}{a + bu} = dx$$

$$\therefore \frac{1}{b} \ln |a + bu| = x + C$$

$$|a + bu| = C_1 e^{bx} \quad ; \quad C_1 > 0$$

$$\therefore a + bu = \pm C_1 e^{bx}$$

$$a + bu = C_2 e^{bx} \qquad (C_2 = \pm C_1)$$

$$a + b(ax + by + c) = C_2 e^{bx}$$

$$\boxed{b(ax + by + c) + a = C_2 e^{bx}}$$

23. $(x+y)^2 y' = a^2$ $\quad (*)$

$u = x+y \implies y' = u'-1 \quad (**)$

$\therefore (**) \to (*)$: $\quad u^2(u'-1) = a^2$

$$u^2u' - u^2 = a^2$$

$$u^2u' = u^2+a^2$$

$$\frac{u^2}{u^2+a^2}\,du = dx$$

$$\underbrace{\int \frac{u^2}{u^2+a^2}\,du}\; = x + C \quad (***)$$

Mas $\displaystyle\int \frac{u^2}{u^2+a^2}\,du = \int du - \int \frac{a^2\,du}{a^2+u^2}$

$$= u - a^2\frac{1}{a}\operatorname{arctg}\frac{u}{a}$$

$$= u - a\operatorname{arctg}\frac{u}{a}$$

$(***)$:

$$u - a\operatorname{arctg}\frac{u}{a} = x + C$$

$$\cancel{x}+y - a\operatorname{arctg}\left(\frac{x+y}{a}\right) = \cancel{x} + C$$

$$y - C = a\operatorname{arctg}\left(\frac{x+y}{a}\right)$$

$\therefore \operatorname{tg}\left(\frac{y-C}{a}\right) = \frac{x+y}{a} \Rightarrow \boxed{x+y = a\operatorname{tg}\left(\frac{y}{a}+C_1\right)}$

24. $(1-y)e^y y' + \frac{y^2}{x\ln x} = 0$

$$\frac{(1-y)e^y\,dy}{y^2} = -\frac{1}{x\ln x}\,dx$$

$$\underbrace{\int \frac{(1-y)e^y}{y^2}\,dy}_{(*)} = \underbrace{-\int \frac{1}{x\ln x}\,dx}_{(**)} + C$$

$$(*) = \int \frac{(1-y)e^y}{y^2}\,dy = \int -\,du = -u = -\frac{e^y}{y}$$

$$u = \frac{e^y}{y} \rightarrow du = \left(\frac{e^y}{y} - \frac{e^y}{y^2}\right)dy = \frac{ye^y - e^y}{y^2}\,dy = \frac{e^y(y-1)}{y^2}\,dy$$

$$(**) = -\int \frac{1}{x\ln x}\,dx = -\ln|\ln x|$$

$\therefore$

$$-\frac{e^y}{y} = -\ln|\ln x| + C$$

$$\boxed{\frac{e^y}{y} = \ln|\ln x| + C_1}$$

25. $(1+y^2)\,dx = (y-\sqrt{1+y^2})(1+x^2)^{3/2}\,dy$

$$\Leftrightarrow$$

$$\frac{dx}{(1+x^2)^{3/2}} = \frac{y-\sqrt{1+y^2}}{1+y^2}\,dy$$

$$\int \frac{y-\sqrt{1+y^2}}{1+y^2}\,dy = \underbrace{\int \frac{dx}{(1+x^2)^{3/2}}}\ + C$$

$$\int \frac{y}{1+y^2}\,dy - \int \frac{1}{\sqrt{1+y^2}}\,dy = \underbrace{\frac{x}{\sqrt{1+x^2}}}\ + C$$

Obs

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}} + C$$

$$\int \frac{y}{1+y^2}\,dy = \frac{1}{2}\ln(1+y^2)$$

$$\int \frac{1}{\sqrt{1+y^2}}\,dy = \ln|y+\sqrt{y^2+1}|$$

$$\frac{1}{2}\ln(1+y^2) - \ln|y+\sqrt{y^2+1}| = \frac{x}{\sqrt{1+x^2}} + C$$

$$\boxed{\ln \frac{\sqrt{1+y^2}}{\left| y+\sqrt{y^2+1} \right|} = \frac{x}{\sqrt{1+x^2}} + C}$$

26. $xy^2(xy'+y) = a^2$

Seja $u = xy \Rightarrow y = \frac{u}{x}$

$u' = xy' + y$

$$\cancel{x} \frac{u^2}{x^{\cancel{2}}} u' = a^2$$

$$u^2 u' = a^2 x$$

$$u^2 du = a^2 x dx$$

$$\int u^2 du = a^2 \frac{x^2}{2} + C$$

$$\frac{u^3}{3} = \frac{a^2}{2} x^2 + C$$

$$\boxed{y^3 x^3 = \frac{3a^2}{2} x^2 + C_1}$$

27. $(x^2y^2+1)\,dx + 2x^2dy = 0$

$u = xy \quad \therefore \quad y = \frac{u}{x}$

$dy = \frac{1}{x}du - \frac{u}{x^2}dx$

$(1+u^2)\,dx + 2x^2\left(\frac{1}{x}du - \frac{u}{x^2}dx\right) = 0$

$(1+u^2)\,dx + 2x\,du - 2u\,dx = 0$

$(1+u^2-2u)\,dx + 2x\,du = 0$

$\frac{dx}{2x} + \frac{du}{u^2-2u+1} = 0$

$\frac{du}{(u-1)^2} = -\frac{dx}{2x}$

$\int \frac{du}{(u-1)^2} = -\frac{1}{2}\ln|x| + C$

$\frac{-1}{(u-1)} = -\frac{1}{2}\ln|x| + C$

$\frac{-1}{xy-1} = -\frac{1}{2}\ln|x| + C$

$\boxed{\frac{1}{1-xy} + \frac{1}{2}\ln|x| = C}$

28. $(1+x^2y^2)y + (xy-1)^2 x y' = 0$

$xy = u \quad \rightarrow \quad y = \frac{u}{x}$

$dy = \frac{1}{x}du - \frac{1}{x^2}u\,dx$

$$(1+x^2y^2)y\,dx + (xy-1)^2 x\,dy = 0$$

$$(1+x^2y^2)y\,dx + (xy-1)^2 x\left(\frac{1}{x}du - \frac{1}{x^2}u\,dx\right) = 0$$

$$(1+x^2y^2)y\,dx + (xy-1)^2\,du - \frac{(xy-1)^2}{x}u\,dx = 0$$

$$\left(1+\cancel{x^2}\frac{u^2}{\cancel{x^2}}\right)\frac{u}{x}\,dx + (u-1)^2\,du - \frac{(u-1)^2}{x}u\,dx = 0$$

$$\left[(1+u^2)u\frac{1}{x} - \frac{u(u-1)^2}{x}\right]dx + (u-1)^2\,du = 0$$

$$\frac{1}{x}\left(u(1+u^2) - \underbrace{u(u-1)^2}_{-u(u^2-2u+1)}\right)dx + (u-1)^2\,du = 0$$

$$\frac{1}{x}\left(\cancel{u} + \cancel{u^3} - \cancel{u^3} + 2u^2 - \cancel{u}\right)dx + (u-1)^2\,du = 0$$

$$\frac{2u^2}{x}dx + (u-1)^2\,du = 0$$

$$\frac{2}{x}dx + \frac{(u-1)^2}{u^2}du = 0$$

$$\int \frac{(u-1)^2}{u^2}du = -2\int\frac{dx}{x} + c$$

$$\int\left(1-\frac{2}{u}+\frac{1}{u^2}\right)du = -2\ln|x| + c$$

$$u - 2\ln|u| - \frac{1}{u} = +\ln|x|^{-2} + c$$

$$\frac{u^2-1}{u} - 2\ln|u| = \ln|x|^{-2} + C$$

$$\frac{u^2-1}{u} = \ln|u|^2 + \ln|x|^{-2} + C$$

$$= \ln\left(\frac{u}{x}\right)^2 + C$$

$$\ln\left(\frac{u}{x}\right)^2 = \frac{u^2-1}{u} - C$$

$$\left(\frac{u}{x}\right)^2 = e^{\frac{u^2-1}{u} - C} = e^{u-\frac{1}{u}} e^{-C}$$

$u = xy$: $\quad y^2 = e^{xy - \frac{1}{xy}} C_1$

$$\therefore \quad \boxed{C_2 y^2 = e^{xy - \frac{1}{xy}}}$$

2a.

$$(x^2y^3 + y + x - 2)\,dx + (x^3y^2 + x)\,dy = 0$$

$$xy = u \quad \rightarrow \quad dy = \frac{du}{x} - \frac{u}{x^2}\,dx$$

$$\left(\cancel{x^2}\frac{u^3}{\cancel{x^3}} + \frac{u}{x} + x - 2\right)dx + \left(\cancel{x^3}\frac{u^2}{\cancel{x^2}} + x\right)\left(\frac{du}{x} - \frac{u}{x^2}\,dx\right) = 0$$

$$\left(\frac{u^3}{x} + \frac{u}{x} + x - 2\right)dx + (u^2x + x)\left(\frac{du}{x} - \frac{u}{x^2}\,dx\right) = 0$$

$$\left[\frac{u^3}{x} + \frac{u}{x} + x - 2 - \frac{u}{x^2}(u^2x + x)\right]dx + \frac{(u^2x + x)}{x}\,du = 0$$

$$\left[\cancel{\frac{u^3}{x}} + \cancel{\frac{u}{x}} + x - 2 - \cancel{\frac{u^3}{x}} - \cancel{\frac{u}{x}}\right]dx + (u^2 + 1)\,du = 0$$

$$(x-2)\,dx + (u^2+1)\,du$$

$$\int (x-2)\,dx + \int (u^2+1)\,du = C$$

$$\frac{x^2}{2} - 2x + \frac{u^3}{3} + u = C$$

$(\times 6)$ $$\frac{x^2}{2} - 2x + \frac{x^3y^3}{3} + xy = C$$

$$\boxed{3x^2 - 12x + 2x^3y^3 + 6xy = C_1}$$

30. $(x^6-2x^5+2x^4-y^3+4x^2y)\,dx+(xy^2-4x^3)\,dy=0$

$y=ux \longrightarrow dy=du\,x+u\,dx$

$(x^6-2x^5+2x^4-u^3x^3+4x^2ux)\,dx+(xu^2x^2-4x^3)\cdot$
$\cdot(du\,x+u\,dx)=0$

$(x^6-2x^5+2x^4-u^3x^3+4x^3u)\,dx+(u^2x^3-4x^3)\cdot$
$\cdot(du\,x+u\,dx)=0$

$\left[x^6-2x^5+2x^4-u^3x^3+4x^3u+(u^2x^3-4x^3)u\right]dx$
$+(u^2x^3-4x^3)x\,du=0$

$\therefore \left[x^6-2x^5+2x^4-\cancel{u^3x^3}+\cancel{4x^3u}+\cancel{u^3x^3}-\cancel{4x^3u}\right]dx$
$+(u^2-4)x^4\,du$

$\therefore \dfrac{[x^6-2x^5+2x^4]\,dx}{x^4}+(u^2-4)\,du=0$

$(x^2-2x+2)\,dx+(u^2-4)\,du=0$

$\int(x^2-2x+2)\,dx+\int(u^2-4)\,du=C$

$\frac{x^3}{3}-x^2+2x+\frac{u^3}{3}-4u=C$

$\boxed{\frac{x^3}{3}-x^2+2x+\frac{y^3}{3x^3}-4\frac{y}{x}=C}$

31. $$y' + 1 = \frac{(x+y)^m}{(x+y)^n + (x+y)^p}$$

$$u = x + y \quad , \quad du = dx + dy$$

$$\underbrace{dy} + dx = \frac{(x+y)^m}{(x+y)^n + (x+y)^p}\, dx$$

$$\underbrace{du - \cancel{dx}} + \cancel{dx} = \frac{u^m}{u^n + u^p}\, dx$$

$$\frac{u^n + u^p}{u^m}\, du = dx$$

$$\left(u^{n-m} + u^{p-m}\right) du = dx$$

$$\int u^{n-m}\, du + \int u^{p-m}\, du = x + C$$

$$\frac{u^{n-m+1}}{n-m+1} + \frac{u^{p-m+1}}{p-m+1} = x + C$$

Con
$n - m \neq -1$
$p - m \neq -1$

$$\boxed{x = \frac{(x+y)^{n-m+1}}{n-m+1} + \frac{(x+y)^{p-m+1}}{p-m+1} + C_1 \qquad (n-m \neq -1 \ , \ p-m \neq -1)}$$

52. $(\ln x + y^3)\,dx - 3xy^2\,dy = 0$

Seja $u = \ln x + y^3$

$\therefore \quad du = \frac{1}{x}dx + 3y^2 dy$

$\therefore \quad 3y^2 dy = du - \frac{1}{x}dx$

$\therefore$

$$u\,dx - x\left(du - \frac{1}{x}dx\right) = 0$$

$$u\,dx - x\,du + dx = 0$$

$$(u+1)\,dx - x\,du = 0$$

$$\frac{1}{x}dx - \frac{1}{1+u}du = 0$$

$$\ln|x| - \ln|1+u| = c$$

$$\ln\left|\frac{x}{1+u}\right| = c$$

$$\left|\frac{x}{1+u}\right| = e^{c}$$

$$\frac{x}{1+u} = C_1$$

$$1+u = C_2 x$$

$$1 + \ln x + y^3 = C_2 x$$

$$\boxed{y^3 = Cx - \ln x - 1}$$

33. $(xy + 2xy(\ln y)^2 + y\ln y)\,dx + (2x^2\ln y + x)\,dy = 0$

$$x\ln y = t \implies dx\ln y + \frac{x}{y}\,dy = dt$$

$$dy = \frac{y}{x}(dt - \ln y\,dx)$$

$\therefore$

$$0 = (xy + 2xy(\ln y)^2 + y\ln y)\,dx + (2x^2\ln y + x)\frac{y}{x}(dt - \ln y\,dx)$$

$$0 = \left[xy + 2xy(\ln y)^2 + y\ln y - (2x^2\ln y + x)\frac{y}{x}\ln y\right]dx$$

$$+ \frac{y}{x}(2x^2\ln y + x)\,dt$$

$$0 = \left[xy + \cancel{2xy(\ln y)^2} + \cancel{y\ln y} - \cancel{2xy(\ln y)^2} - \cancel{y\ln y}\right]dx$$

$$+ (2xy\ln y + y)\,dt$$

$$0 = xy\,dx + (2xy\ln y + y)\,dt$$

$$= xe^{\frac{t}{x}}\,dx + \left(2e^{\frac{t}{x}}t + e^{\frac{t}{x}}\right)dt$$

$$= e^{\frac{t}{x}}x\,dx + e^{\frac{t}{x}}(2t+1)\,dt$$

$x\ln y = t$

$\ln y = \frac{t}{x}$

$y = e^{\frac{t}{x}}$

$$0 = x\,dx + (2t+1)\,dt$$

$$C = \int x\,dx + \int (2t+1)\,dt$$

$$C = \frac{x^2}{2} + t^2 + t$$

$$C = x^2 + x^2\ln^2 y + x\ln y$$

$$C = \frac{x^2}{2} + (x\ln y)^2 + x\ln y$$

$$C = \frac{x^2}{2} + \left((x\ln y) + \frac{1}{2}\right)^2 - \frac{1}{4}$$

$$C = \frac{x^2}{2} + \frac{(2x\ln y) + 1)^2}{4} - \frac{1}{4}$$

$$C_1 = 2x^2 + (2x\ln y + 1)^2 - 1$$

$$2x^2 + (2x\ln y + 1)^2 = \underbrace{C_1 + 1}$$

$$\boxed{2x^2 + (2x\ln y + 1)^2 = C}$$

34.

$$y - xy' = a(1 + x^2 y')$$

$$y - xy' - a - ax^2 y' = 0$$

$$(y - a) - (x + ax^2)y' = 0$$

$$(y-a)\,dx - (x + ax^2)\,dy = 0$$

$$\frac{dx}{x + ax^2} = \frac{dy}{y - a}$$

$$\frac{dx}{a(x^2 + \frac{x}{a})} = \frac{dy}{y - a}$$

$$\frac{dx}{a\left[\left(x+\frac{1}{2a}\right)^2-\frac{1}{4a^2}\right]}=\frac{dy}{y-a}$$

$$\int\frac{dx}{a\left[\left(x+\frac{1}{2a}\right)^2-\frac{1}{4a^2}\right]}=\int\frac{dy}{y-a}+C$$

$u=x+\frac{1}{2a}$

$du=dx$

$$\frac{1}{a}\int\frac{du}{u^2-\frac{1}{(2a)^2}}=\ln|y-a|+C$$

$$\left(\int\frac{1}{x^2-b^2}\,dx=\frac{1}{2b}\ln\left|\frac{x-b}{x+b}\right|\right)$$

$b=\frac{1}{2a}$

$$\frac{1}{\cancel{a}}\cdot\frac{1}{\cancel{2}\frac{1}{\cancel{2a}}}\ln\left|\frac{u-\frac{1}{2a}}{u+\frac{1}{2a}}\right|=\ln|y-a|+C$$

$$\ln\left|\frac{2au-1}{2au+1}\right|=\ln|y-a|+C$$

$$\ln\left|\frac{2au-1}{2au+1}\,\frac{1}{y-a}\right|=C$$

$$\frac{2au-1}{2au+1}\,\frac{1}{y-a}=e^C$$

$$\frac{(2ax+1)-1}{}\quad\frac{1}{}=e^C$$

$$\frac{2ax}{2ax+2} \cdot \frac{1}{y-a} = e^{C} = C_1$$

$$C_1(y-a) = \frac{2ax}{2ax+2}$$

$$y-a = \frac{1}{C_1}\,\frac{\cancel{2}ax}{\cancel{2}(ax+1)}$$

$$\boxed{y = a + \frac{Cx}{ax+1}}$$

35.

$$(a^2+y^2)\,dx + 2x\sqrt{ax-x^2}\,dy = 0 \quad ; \quad y(a) = 0$$

$$\frac{1}{2x\sqrt{ax-x^2}}\,dx + \frac{1}{a^2+y^2}\,dy = 0$$

$$\int \frac{dx}{2x\sqrt{ax-x^2}} + \int \frac{dy}{y^2+a^2} = C$$

Mas

$a=2b$ $$\int \frac{dx}{x\sqrt{2bx-x^2}} = -\frac{\sqrt{2bx-x^2}}{bx}$$

$$\frac{1}{2}\left[-\frac{\sqrt{ax-x^2}}{\frac{a}{2}x}\right] + \frac{1}{a}\operatorname{arctg}\frac{y}{a} = C$$

$$-\frac{\sqrt{ax-x^2}}{ax}+\frac{1}{a}\operatorname{arctg}\frac{y}{a}=C$$

$$\operatorname{arctg}\frac{y}{a}=aC+\frac{\sqrt{ax-x^2}}{x}$$

$$\frac{y}{a}=\operatorname{tg}\left(C_1+\frac{\sqrt{ax-x^2}}{x}\right)$$

$$y=a\operatorname{tg}\left(C_1+\frac{\sqrt{ax-x^2}}{x}\right)$$

$$\rightarrow y=a\operatorname{tg}\left(C_1+\frac{\sqrt{ax-x^2}}{x}\right)$$

$$\therefore y(a)=0 \Rightarrow 0=a\operatorname{tg}(C_1)$$

$$\Rightarrow \underline{C_1=0}$$

$$\therefore \boxed{y=a\operatorname{tg}\frac{\sqrt{ax-x^2}}{x}}$$

---

36. $y'+\sin\frac{x+y}{2}=\sin\frac{x-y}{2}$

$$y'+\sin\frac{x+y}{2}-\sin\frac{x-y}{2}=0$$

$$y'+\cancel{\sin\frac{x}{2}\cos\frac{y}{2}}+\sin\frac{y}{2}\cos\frac{x}{2}-\cancel{\sin\frac{x}{2}\cos\frac{y}{2}}+\sin\frac{y}{2}\cos\frac{x}{2}=0$$

$$y'+2\sin\frac{y}{2}\cos\frac{x}{2}=0$$

$$\frac{dy}{\sin\frac{y}{2}} + 2\cos\frac{x}{2}\,dx = 0$$

$$\therefore \underbrace{\int \operatorname{cosec}\frac{y}{2}\,dy} + 2\int \cos\frac{x}{2}\,dx = C$$

$u = \frac{y}{2}$

$$-2\ln\left|\underbrace{\operatorname{cosec}\frac{y}{2} + \operatorname{cotg}\frac{y}{2}}\right| + 4\sin\frac{x}{2} = C$$

$$-2\ln\left|\frac{1+\cos\frac{y}{2}}{\sin\frac{y}{2}}\right| + 4\sin\frac{x}{2} = C$$

$$-2\ln\left|\frac{\cancel{2}\cos^{\cancel{2}}\frac{y}{4}}{\cancel{2}\sin\frac{y}{4}\cancel{\cos\frac{y}{4}}}\right| + 4\sin\frac{x}{2} = C$$

$$-2\ln\left|\operatorname{cotg}\frac{y}{4}\right| + 4\sin\frac{x}{2} = C$$

$$\underbrace{-\ln\left|\operatorname{cotg}\frac{y}{4}\right|} + 2\sin\frac{x}{2} = C_1$$

$$\ln\left|\operatorname{tg}\frac{y}{4}\right| + 2\sin\frac{x}{2} = C_1$$

$$\boxed{\ln\left|\operatorname{tg}\frac{y}{4}\right| = C_1 - 2\sin\frac{x}{2}}$$

Cálculo C - Lista 9 - Resposta

. $y = Ce^{kx}$

. $y = C\sin x$

. $y = Ce^{x^3}$

. $y = Ce^{-ax} - \frac{b}{a}$

. $y^2 = \frac{1}{-2e^x + C}$

. $y = \sin(x + C)$

$y = C|\ln x|$

. $y = C\sqrt{|\sin 2x|}$

. $x^2(1 + y^2) = C$

. $\ln\left|\frac{1+x}{1-y}\right| + \frac{(x+y)}{2}(x - y - 2) = C$

$y = \operatorname{tg}(\ln|x| + C)$

. $\sqrt{1+y^2} + \sqrt{1+x^2} = C$

$\sqrt{1-x^2} + \sqrt{1-y^2} = 1$

$e^x = C(1 - e^{-y})$

$y = 1$

16. $a^x + a^{-y} = C$

17. $y = \ln[C(1+x^2) - 1]$

18. $(y+1)e^{-y} = \ln\left(\frac{e^x+1}{2}\right) - x + 1$

19. $\frac{e^{2x}}{2} - e^y - \operatorname{arctg} y - \ln\sqrt{1+y^2} = C$

20. $(y^2+1)(x^2 - 2x + 2)e^{2\operatorname{arctg} y} = C$

21. $\operatorname{tg}(x-y) + \sec(x-y) = x + C$

→ 22. $b(ax + by + c) + a = C_1 e^{bx}$

23. $x + y = a\operatorname{tg}\left(\frac{y}{a} + C\right)$

24. $\frac{e^y}{y} = \ln|\ln x| + C$

25. $\ln\frac{\sqrt{1+y^2}}{|y + \sqrt{y^2+1}|} = \frac{x}{\sqrt{1+x^2}} + C$

26. $y^3x^3 = \frac{3a^2}{2}x^2 + C$

27. $\frac{1}{1-xy} + \frac{1}{2}\ln|x| = C$

28. $Cy^2 = e^{xy - \frac{1}{xy}}$

29. $3x^2 - 12x + 2x^3y^3 + 6xy = C$

30. $\frac{x^3}{3} - x^2 + 2x + \frac{y^3}{3x^3} - \frac{4y}{x} = C$

31. $x = \frac{(x+y)^{n-m+1}}{n-m+1} + \frac{(x+y)^{p-m+1}}{p-m+1} + C,$

$(n-m \neq -1\ ;\ p-m \neq -1)$

32. $y^3 = Cx - \ln x - 1$

33. $2x^2 + (2x \ln y + 1)^2 = C$

34. $y = a + \frac{Cx}{ax+1}$

35. $y = a\, tg \frac{\sqrt{ax - x^2}}{x}$

36. $\ln \left| tg \frac{y}{4} \right| = C - 2 \sin \frac{x}{2}$