#22

#21 :

Cálculo C - Lista 10

Equações exatas

Mostre que as equações a seguir são exatas e resolva as equações

1. $ydx + xdy = 0$

2. $y^2dx + 2xy\,dy = 0$

3. $[(x+1)e^x - e^y]dx - xe^y dy = 0$

4. $\sinh x\,\cos y\,dx = \cosh x\,\sin y\,dy$

5. $e^{-\theta}dr - re^{-\theta}d\theta = 0$

6. $x(2x^2+y^2) + y(x^2+2y^2)y' = 0$

7. $(3x^2+6xy^2)dx + (6x^2y+4y^3)dy = 0$

8. $$\left(\frac{x}{\sqrt{x^2+y^2}} + \frac{1}{x} + \frac{1}{y}\right)dx + \\ +\left(\frac{y}{\sqrt{x^2+y^2}} + \frac{1}{y} - \frac{x}{y^2}\right)dy = 0$$

9. $$\left(3x^2\tan y - \frac{2y^3}{x^3}\right)dx + \\ \left(x^3\sec^2 y + 4y^3 + \frac{3y^2}{x^2}\right)dy = 0$$

10. $$\left(2x + \frac{x^2+y^2}{x^2y}\right)dx = \frac{x^2+y^2}{xy^2}\,dy$$

11. $$\left(\frac{\sin 2x}{y} + x\right)dx + \left(y - \frac{\sin^2 x}{y^2}\right)dy = 0$$

12. $(3x^2 - 2x - y)dx + (2y - x + 3y^2)dy = 0$

13. $$\left(\frac{xy}{\sqrt{1+x^2}} + 2xy - \frac{y}{x}\right)dx + \\ +(\sqrt{1+x^2} + x^2 - \ln x)dy = 0$$

14. $$\frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} + \frac{x\,dy - y\,dx}{x^2} = 0$$

15. $$\left(\sin y + y\sin x + \frac{1}{x}\right)dx + \left(x\cos y - \cos x + \frac{1}{y}\right)dy = 0$$

16. $$\frac{y + \sin x\,\cos^2 xy}{\cos^2 xy}\,dx + \left(\frac{x}{\cos^2 xy} + \sin y\right)dy = 0$$

17. $$\frac{2x}{y^3}\,dx + \frac{(y^2-3x^2)}{y^4}\,dy = 0, \quad y(1) = 1$$

18. $$[n\cos(nx+my) - m\sin(mx+ny)]dx + \\ [m\cos(nx+my) - n\sin(mx+ny)]dy = 0$$

19. $$\frac{x\,dx + y\,dy}{\sqrt{(x^2+y^2)(1-x^2-y^2)}} + \\ \left(\frac{1}{y\sqrt{y^2-x^2}} + \frac{e^{x/y}}{y^2}\right)(y\,dx - x\,dy) = 0$$

20. Mostre que uma equação separável é exata. É uma equação exata separável?

21. Sob que condições tem-se
$$(ax+by)dx + (cx+dy)dy = 0$$
exata? (a,b,c,d são constantes). Resolva a equação exata.

22. Sob que condição tem-se
$$(f(x) + g(y))dx + (h(x) + p(y))dy = 0$$
exata?

23. Sob que condição é $f(x,y)dx + g(x)h(y)dy = 0$ exata?

24. Uma mesma equação diferencial pode ser resolvida por vários métodos. Resolva as equações a seguir usando o seguinte procedimento (i) tornando-a exata, e (ii) pelo método de separação de variáveis.
   (a) $3x^{-4}ydx = x^{-3}dy$
   (b) $2x\,dx + x^{-2}(x\,dy - y\,dx) = 0$

Lista 10

1. $y\,dx + x\,dy = 0$

$$\left.\begin{array}{l} M(x,y) = y \longrightarrow \dfrac{\partial M}{\partial y} = 1 \\ N(x,y) = x \longrightarrow \dfrac{\partial N}{\partial x} = 1 \end{array}\right\}$$

eq. é exata pois: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$\underbrace{y\,dx + x\,dy}_{d(yx)} = 0$$

$$d(yx) = 0 \quad \therefore \quad \boxed{yx = c}$$

2. $y^2dx + 2xy\,dy = 0$

$$\left.\begin{array}{l} N(x,y) = 2xy \longrightarrow \dfrac{\partial N}{\partial x} = 2y \\ M(x,y) = y^2 \longrightarrow \dfrac{\partial M}{\partial y} = 2y \end{array}\right\}$$

$\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$

$\therefore$ eq. é exata

$$\therefore \quad y^2dx + 2xy\,dy = 0$$

$$\therefore \quad d(y^2x) = 0 \Rightarrow \boxed{y^2x = c}$$

3. $\left((x+1)e^x - e^y\right)dx - xe^y\,dy = 0$

$$\begin{cases} N(x,y) = -xe^y \longrightarrow \dfrac{\partial N}{\partial x} = -e^y \\ M(x,y) = (x+1)e^x - e^y \longrightarrow \dfrac{\partial M}{\partial y} = -e^y \end{cases}$$

$\therefore \quad \dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y} \rightarrow$ Eq. é exata.

$\left((x+1)e^x - e^y\right)dx - xe^y\,dy = 0$

$$\frac{\partial \mu}{\partial x} = (x+1)e^x - e^y$$

$$\frac{\partial \mu}{\partial y} = -xe^y \Longrightarrow \mu = \int -xe^y\,dy + k(x)$$

$$= -xe^y + k(x) \quad (*)$$

$(*)$: $\underbrace{\dfrac{\partial \mu}{\partial x}} = -e^y + \dfrac{dk(x)}{dx}$

$$(x+1)e^x - e^y = -e^y + \frac{dk(x)}{dx}$$

$$(x+1)e^x \cancel{-e^y} = \cancel{-e^y} + \frac{dk}{dx}$$

$$\frac{dk}{dx} = (x+1)e^x$$

$(**)$ $\quad k(x) = \int (x+1)e^x + C$

∴ Más

$$\int (x+1)e^x dx = \int xe^x dx + \int e^x dx$$

$$= \int xe^x dx + e^x$$

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x$$

$u = x \rightarrow du = dx$

$dv = e^x dx \rightarrow v = e^x$

$$= xe^x - e^x + e^x$$

$$K(x) = xe^x \quad (**)$$

(**) → (*):

$$u(x,y) = -xe^y + xe^x + C = cte$$

$$\therefore \boxed{x(e^x - e^y) = C_1}$$

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4. $\sinh x \cos y \, dx = \cosh x \sin y \, dy$

$$\cosh x \sin y \, dy - \sinh x \cos y \, dx = 0$$

$N(x,y) = \cosh x \sin y \rightarrow \frac{\partial N}{\partial x} = \sinh x \sin y$

$M(x,y) = -\sinh x \cos y \rightarrow \frac{\partial M}{\partial y} = \sinh x \sin y$

∴ é exata →

$\therefore \quad \cosh x \sin y \, dy - \sinh x \cos y \, dx = 0$

$\therefore \quad d(-\cosh x \cos y) = 0$

$\sinh x = \dfrac{e^x - e^{-x}}{2}$

$\cosh x = \dfrac{e^x + e^{-x}}{2}$

$\therefore \quad -\cosh x \cos y = c$

$$\boxed{\cosh x \cos y = c_1}$$

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5. $e^{-\theta} dr - r e^{-\theta} d\theta = 0$

$M(r,\theta) = -r e^{-\theta} \longrightarrow \dfrac{\partial M}{\partial r} = -e^{-\theta}$

$N(r,\theta) = e^{-\theta} \longrightarrow \dfrac{\partial N}{\partial \theta} = -e^{-\theta}$

$=$

Eq. exata.

$e^{-\theta} dr - r e^{-\theta} d\theta = 0$

$\therefore \quad d(r e^{-\theta}) = 0 \Longrightarrow \boxed{r e^{-\theta} = c}$

5. $x(2x^2+y^2) + y(x^2+2y^2)\,y' = 0$

$y(x^2+2y^2)\,dy + x(2x^2+y^2)\,dx = 0$

$N(x,y) = y(x^2+2y^2) \longrightarrow \dfrac{\partial N}{\partial x} = 2xy$

exata

$M(x,y) = x(2x^2+y^2) \longrightarrow \dfrac{\partial M}{\partial y} = 2xy$

$(yx^2+2y^3)\,dy + (2x^3+xy^2)\,dx = 0$

$u(x,y) = C$

$\dfrac{\partial u}{\partial y} = yx^2+2y^3 \Longrightarrow u = \int (yx^2+2y^3)\,dy + k(x)$

$= \dfrac{y^2x^2}{2} + \dfrac{y^4}{2} + k(x)$ (*)

$\dfrac{\partial u}{\partial x} = 2x^3+xy^2$

$\cancel{y^2x} + \dfrac{dk}{dx} = 2x^3 + \cancel{xy^2}$

$k = \int 2x^3\,dx + c$

$= \dfrac{2x^4}{4} + c = \dfrac{x^4}{2} + c$ (**)

$\therefore$ (**) $\to$ (*) : $u(x,y) = \dfrac{x^2y^2}{2} + \dfrac{y^4}{2} + \dfrac{x^4}{2} + C = cte$

$$\therefore \quad \boxed{x^4 + x^2y^2 + y^4 = C}$$

7. $(3x^2 + 6xy^2)dx + (6x^2y + 4y^3)dy = 0$

$M(x,y) = 3x^2 + 6xy^2 \longrightarrow \dfrac{\partial M}{\partial y} = 12xy$

$N(x,y) = 6x^2y + 4y^3 \longrightarrow \dfrac{\partial N}{\partial x} = 12xy$

Exata

$u(x,y) = cte$

$$\frac{\partial u}{\partial x} = 3x^2 + 6xy^2 \Rightarrow \begin{cases} u = \int (3x^2 + 6xy^2)\,dx + k(y) \\ \quad = x^3 + 3x^2y^2 + k(y) \quad (*) \end{cases}$$

$$\frac{\partial u}{\partial y} = 6x^2y + 4y^3$$

$$(*) \quad \cancel{6x^2y} + \frac{dk}{dy} = \cancel{6x^2y} + 4y^3 \Longrightarrow k = y^4 + C \quad (**)$$

$(**) \rightarrow (*)$; $\quad u = x^3 + 3x^2y^2 + y^4 + C = cte$

$$\boxed{y^4 + 3x^2y^2 + x^3 = C_1}$$

8. $\left(\frac{x}{\sqrt{x^2+y^2}} + \frac{1}{x} + \frac{1}{y}\right) dx + \left(\frac{y}{\sqrt{x^2+y^2}} + \frac{1}{y} - \frac{x}{y^2}\right) dy = 0$

$$M(x,y) = \frac{x}{\sqrt{x^2+y^2}} + \frac{1}{x} + \frac{1}{y}$$

$$\frac{\partial M}{\partial y} = -\frac{xy}{(x^2+y^2)^{3/2}} - \frac{1}{y^2}$$

$=$ Eq. exata

$$N(x,y) = \frac{y}{\sqrt{x^2+y^2}} + \frac{1}{y} - \frac{x}{y^2}$$

$$\frac{\partial N}{\partial x} = \frac{-xy}{(x^2+y^2)^{3/2}} - \frac{1}{y^2}$$

$$\rightarrow \frac{\partial M}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} + \frac{1}{x} + \frac{1}{y}$$

$$\therefore M = \int \frac{x}{\sqrt{x^2+y^2}}\, dx + \int \frac{1}{x}\, dx + \frac{x}{y} + K(x)$$

$$M(x,y) = \sqrt{x^2+y^2} + \ln|x| + \frac{x}{y} + K(x) \quad (*)$$

$$\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} + \frac{1}{y} - \frac{x}{y^2}$$

$(*)$

$$\cancel{\frac{y}{\sqrt{x^2+y^2}}} \cancel{- \frac{x}{y^2}} + \frac{d\sigma}{dy} = \cancel{\frac{y}{\sqrt{x^2+y^2}}} + \frac{1}{y} \cancel{- \frac{x}{y^2}}$$

$$\frac{d\sigma}{dy} = \frac{1}{y}$$

$$\therefore \quad \sigma = \ln|y| + C \quad (**)$$

$(**) \to (*)$:

$$u(x,y) = \sqrt{x^2+y^2} + \ln|x| + \frac{x}{y} + \ln|y| + C = cte$$

$$\boxed{\sqrt{x^2+y^2} + \frac{x}{y} + \ln|xy| = C_1}$$

1. $\left(3x^2 \operatorname{tg} y - \frac{2y^3}{x^3}\right) dx + \left(x^3 \sec^2 y + 4y^3 + \frac{3y^2}{x^2}\right) dy = 0$

$$M(x,y) = 3x^2 \operatorname{tg} y - \frac{2y^3}{x^3}$$

$$\frac{\partial M}{\partial y} = 3x^2 \sec^2 y - \frac{6y^2}{x^3}$$

= Exata

$$N(x,y) = x^3 \sec^2 y + 4y^3 + \frac{3y^2}{x^2}$$

$$\frac{\partial N}{\partial x} = 3x^2 \sec^2 y - \frac{6y^2}{x^3}$$

$\rightarrow \mu(x,y) = \text{cte}$

$$\frac{\partial \mu}{\partial x} = 3x^2 \operatorname{tg} y - \frac{2y^3}{x^3}$$

$$\mu = \int 3x^2 \operatorname{tg} y \, dx - \int \frac{2y^3}{x^3} dx + k(y)$$

$$= x^3 \operatorname{tg} y - \frac{2y^3 x^{-2}}{-2} + k(y)$$

$$\mu = x^3 \operatorname{tg} y + \frac{y^3}{x^2} + k(y) \quad (*)$$

$$\frac{\partial M}{\partial y} = x^3 \sec^2 y + 4y^3 + \frac{3y^2}{x^2}$$

$$\cancel{x^3 \sec^2 y} + \cancel{\frac{3y^2}{x^2}} + \frac{dk}{dy} = \cancel{x^3 \sec^2 y} + 4y^3 + \cancel{\frac{3y^2}{x^2}} \quad (*)$$

$$\frac{dk}{dy} = 4y^3$$

$$k = y^4 + C \quad (**)$$

$(**) \to (*)$: $M = x^3 \operatorname{tg} y + \frac{y^3}{x^2} + y^4 + C = \text{cte}$

$$\therefore \boxed{x^3 \operatorname{tg} y + \frac{y^3}{x^2} + y^4 = C_1}$$

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10. $$\left(2x + \frac{x^2+y^2}{x^2 y}\right) dx = \frac{x^2+y^2}{x y^2}\, dy$$

$$\therefore \left(2x + \frac{x^2+y^2}{x^2 y}\right) dx - \frac{x^2+y^2}{x y^2}\, dy$$

Veamos

$$N(x,y) = -\frac{x^2+y^2}{x y^2}$$

$$\frac{\partial N}{\partial x} = -\frac{2\cancel{x}}{\cancel{x} y^2} + \frac{x^2+y^2}{x^2 y^2} = -\frac{2}{y^2} + \frac{x^2+y^2}{x^2 y^2} = \frac{y^2 - x^2}{x^2 y^2}$$

10. <u>cont.</u>

$$M(x,y) = 2x + \frac{x^2+y^2}{x^2y}$$

$$\frac{\partial M}{\partial y} = \frac{\cancel{2y}}{x^2\cancel{y}} - \frac{x^2+y^2}{x^2y^2} = \frac{2y^2-x^2-y^2}{x^2y^2}$$

$$= \frac{y^2-x^2}{x^2y^2}$$

Daí $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \Rightarrow$ Eq. é exata.

$\therefore$

$$u(x,y) = cte$$

$$\frac{\partial u}{\partial x} = 2x + \frac{x^2+y^2}{x^2y}$$

$$\therefore \quad u = x^2 + \int \frac{x^2+y^2}{x^2y}\,dx + k(y)$$

$$= x^2 + \int\left(\frac{1}{y} + \frac{y}{x^2}\right)dx + k(y)$$

$$u = x^2 + \frac{x}{y} - \frac{y}{x} + k(y) \quad (*)$$

(8) $$\frac{\partial u}{\partial y} = -\frac{x^2+y^2}{xy^2}$$

$$\cancel{-\frac{x}{y^2}} - \cancel{\frac{1}{x}} + \frac{dk}{dy} = -\frac{x^2+y^2}{xy^2}$$

$$= \cancel{-\frac{x}{y^2}} - \cancel{\frac{1}{x}}$$

$$\therefore \quad \frac{dk}{dy} = 0 \quad \Rightarrow \quad k(y) = \text{cte.}$$ (**)

(**) $\to$ (*) :

$$u(x,y) = x^2 + \frac{x}{y} - \frac{y}{x} + c = \text{cte}$$

$$\therefore \quad \boxed{x^2 + \frac{x}{y} - \frac{y}{x} = C_1}$$

ou

$$\therefore \quad \boxed{x^3y + x^2 - y^2 = C_1 xy}$$

11. $\left(\frac{\sin 2x}{y} + x\right) dx + \left(y - \frac{\sin^2 x}{y^2}\right) dy = 0$

$$M(x,y) = \frac{\sin 2x}{y} + x \longrightarrow \frac{\partial M}{\partial y} = -\frac{\sin 2x}{y^2}$$

$$N(x,y) = y - \frac{\sin^2 x}{y^2} \longrightarrow \frac{\partial N}{\partial x} = -\frac{2\sin x \cos x}{y^2}$$

$$= -\frac{\sin 2x}{y^2}$$

= exata

$$\frac{\partial u}{\partial x} = \frac{\sin 2x}{y} + x$$

$$\therefore \quad u = -\frac{\cos 2x}{2y} + \frac{x^2}{2} + k(y) \quad (*)$$

$$\frac{\partial u}{\partial y} = y - \frac{\sin^2 x}{y^2}$$

$$\frac{\cos 2x}{2y^2} + \frac{dk}{dy} = y - \frac{\sin^2 x}{y^2}$$

$$= y - \frac{1 - \cos 2x}{2y^2}$$

$$\cancel{\frac{\cos 2x}{2y^2}} + \frac{dk}{dy} = y - \frac{1}{2y^2} + \cancel{\frac{\cos 2x}{2y^2}}$$

$$\int \quad k = \frac{y^2}{2} + \frac{1}{2y} + C \quad (**)$$

$(**) \to (*)$:

$$\mu = -\frac{\cos 2x}{2y} + \frac{x^2}{2} + \frac{y^2}{2} + \frac{1}{2y} + C = cte$$

$$\therefore \quad \frac{1-\cos 2x}{2y} + \frac{x^2+y^2}{2} = C$$

$$\boxed{\frac{\sin^2 x}{y} + \frac{x^2+y^2}{2} = C}$$

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12. $(3x^2-2x-y)\,dx + (2y-x+3y^2)\,dy = 0$

$$M(x,y) = 3x^2-2x-y \longrightarrow \frac{\partial M}{\partial y} = -1$$

$$N(x,y) = 2y-x+3y^2 \longrightarrow \frac{\partial N}{\partial x} = -1$$

$=$ Exata

$$\frac{\partial \mu}{\partial x} = 3x^2-2x-y \Longrightarrow \mu = x^3-x^2-yx+K(y) \quad (*)$$

$$\frac{\partial \mu}{\partial y} = 2y-x+3y^2$$

$$\cancel{-x}+\frac{dK}{dy} = 2y \cancel{-x} + 3y^2$$

$$\frac{dK}{dy} = 2y+3y^2 \Longrightarrow K(y) = y^2+y^3+C \quad (**)$$

12. Cont.

$$u = x^3 - x^2 - yx + y^2 + y^3 + C = cte$$

$$\therefore \boxed{x^3 + y^3 - x^2 - xy + y^2 = C}$$

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13. $$\left(\frac{xy}{\sqrt{1+x^2}} + 2xy - \frac{y}{x}\right)dx + \left(\sqrt{1+x^2} + x^2 - \ln x\right)dy = 0$$

$$M(x,y) = \frac{xy}{\sqrt{1+x^2}} + 2xy - \frac{y}{x} \rightarrow \frac{\partial M}{\partial y} = \frac{x}{\sqrt{1+x^2}} + 2x - \frac{1}{x}$$

$$N(x,y) = \sqrt{1+x^2} + x^2 - \ln x \rightarrow \frac{\partial N}{\partial x} = \frac{x}{\sqrt{1+x^2}} + 2x - \frac{1}{x}$$

(Exata)

$$\frac{\partial u}{\partial x} = \frac{xy}{\sqrt{1+x^2}} + 2xy - \frac{y}{x}$$

$$\therefore u = \sqrt{1+x^2}\,y + x^2y - y\ln|x| + k(y)$$ (*)

$$\frac{\partial u}{\partial y} = \sqrt{1+x^2} + x^2 - \ln x$$

$$\cancel{\sqrt{1+x^2}} + \cancel{x^2} - \cancel{\ln|x|} + \frac{dk}{dy} = \cancel{\sqrt{1+x^2}} + \cancel{x^2} - \cancel{\ln x}$$

$$\frac{dk}{dy} = 0 \Rightarrow k(y) = cte$$ (**)

$(**) \to (*)$:

$$u(x,y) = \sqrt{1+x^2}\,y + x^2 y - y\ln|x| + C = cte$$

($x>0$ desde o início)

$$\therefore \boxed{y\sqrt{1+x^2} + x^2 y - y\ln x = C} \quad (x>0)$$

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14. $$\frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} + \frac{x\,dy - y\,dx}{x^2} = 0$$

$$\left(\frac{x}{\sqrt{x^2+y^2}} - \frac{y}{x^2}\right)dx + \left(\frac{y}{\sqrt{x^2+y^2}} + \frac{\cancel{x}}{x^{\cancel{2}}}\right)dy = 0$$

$$\begin{cases} M(x,y) = \dfrac{x}{\sqrt{x^2+y^2}} - \dfrac{y}{x^2} \longrightarrow \dfrac{\partial M}{\partial y} = \dfrac{-xy}{(x^2+y^2)^{3/2}} - \dfrac{1}{x^2} \\[2ex] N(x,y) = \dfrac{y}{\sqrt{x^2+y^2}} + \dfrac{1}{x} \longrightarrow \dfrac{\partial N}{\partial x} = -\dfrac{xy}{(x^2+y^2)^{3/2}} - \dfrac{1}{x^2} \end{cases}$$

= Exata

$$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} - \frac{y}{x^2}$$

$$\therefore u = \sqrt{x^2+y^2} + \frac{y}{x} + K(y) \quad (*)$$

14. Cont.

$$\frac{\partial M}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} + \frac{1}{x}$$

(*)

$$\cancel{\frac{y}{\sqrt{x^2+y^2}}} + \cancel{\frac{1}{x}} + \frac{dk}{dy} = \cancel{\frac{y}{\sqrt{x^2+y^2}}} + \cancel{\frac{1}{x}}$$

$$\frac{dk}{dy} = 0 \Rightarrow k = cte \quad (**)$$

$(**) \rightarrow (*)$:

$$M(x,y) = \sqrt{x^2+y^2} + \frac{y}{x} + C = cte$$

$$\therefore \boxed{\sqrt{x^2+y^2} + \frac{y}{x} = C_1}$$

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15. $\left(\sin y + y\sin x + \frac{1}{x}\right)dx + \left(x\cos y - \cos x + \frac{1}{y}\right)dy = 0$

$$M(x,y) = \sin y + y\sin x + \frac{1}{x} \rightarrow \frac{\partial M}{\partial y} = \cos y + \sin x$$

$$N(x,y) = x\cos y - \cos x + \frac{1}{y} \rightarrow \frac{\partial N}{\partial x} = \cos y + \sin x$$

$=$

exata

$$\frac{\partial M}{\partial x} = \sin y + y\sin x + \frac{1}{x} \Rightarrow \int M = (\sin y)x - y\cos x + \ln|x| + k(y) \quad (*)$$

$$\frac{\partial u}{\partial y} = x\cos y - \cos x + \frac{1}{y}$$

(*)

$$\cancel{(\cos y)}\, x \cancel{- \cos x} + \frac{dk}{dy} = \cancel{x\cos y} \cancel{- \cos x} + \frac{1}{y}$$

$$\frac{dk}{dy} = \frac{1}{y}$$

$$\therefore \quad k = \ln|y| + C \qquad (**)$$

(**) $\rightarrow$ (*) :

$$u = x\sin y - y\cos x + \ln|x| + \ln|y| + C = cte$$

$$\therefore \quad \boxed{\ln|xy| + x\sin y - y\cos x = C_1}$$

16. $\dfrac{y+\sin x\cos^2 xy}{\cos^2 xy}\,dx+\left(\dfrac{x}{\cos^2 xy}+\sin y\right)dy=0$

$$M(x,y)=\frac{y+\sin x\cos^2 xy}{\cos^2 xy}$$

$$\frac{\partial M}{\partial y}=\frac{1+\sin x\;2\cos xy\,(-)\sin xy\;x}{\cos^2 xy}+$$

$$+\left(y+\sin x\cos^2 xy\right)\frac{(-2)}{\cos^3 xy}(-\sin xy)\,x$$

$$\frac{\partial M}{\partial y}=\frac{1-2x\sin x\cos xy\sin xy}{\cos^2 xy}+$$

$$+\left(y+\sin x\cos^2 xy\right)\frac{2x\sin xy}{\cos^3 xy}$$

$$N(x,y)=\frac{x}{\cos^2 xy}+\sin y$$

$$\frac{\partial N}{\partial x}=\frac{1}{\cos^2 xy}-\frac{2x}{\cos^3 xy}(-\sin xy)\,y$$

$$\frac{\partial N}{\partial x}=\frac{1}{\cos^2 xy}+2xy\frac{\sin xy}{\cos^3 xy}\quad (*)$$

Nos desenvolvendo (*) temos

$$\frac{\partial M}{\partial y} = \frac{1}{\cos^2 xy} - \cancel{\frac{2x \sin x \sin xy}{\cos xy}}$$

$$+ 2xy \frac{\sin xy}{\cos^3 xy} + \cancel{\frac{2x \sin x \sin xy}{\cos xy}}$$

$$\frac{\partial M}{\partial y} = \frac{1}{\cos^2 xy} + 2xy \frac{\sin xy}{\cos^3 xy} \quad (**)$$

De (*) e (**) vemos que $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$

$\therefore$ A eq. é exata.

Daí,

$$\frac{\partial M}{\partial y} = \frac{x}{\cos^2 xy} + \sin y$$

$$\therefore M = \int x \sec^2 xy \, dy + \int \sin y \, dy + k(x)$$

$$M = tg\, xy - \cos y + k(x) \quad (***)$$

$$\frac{\partial M}{\partial x} = \frac{y + \sin x \cos^2 xy}{\cos^2 xy}$$

$***$

$$\cancel{(\sec^2 xy)}\, y + \frac{dk}{dx} = y\, \cancel{\sec^2 xy} + \sin x$$

$$\frac{dk}{dx} = \sin x \Rightarrow k(x) = -\cos x + C \quad ****$$

$****$ → $**$:

$$u = \operatorname{tg} xy - \cos y - \cos x + C = cte$$

$$\therefore \boxed{\operatorname{tg} xy - \cos x - \cos y = C_1}$$

---

17. $$\frac{2x\,dx}{y^3} + \frac{(y^2 - 3x^2)}{y^4}\,dy = 0 \quad ; \quad y(1) = 1$$

$$M(x,y) = \frac{2x}{y^3} \rightarrow \frac{\partial M}{\partial y} = -6\frac{x}{y^4}$$

$$N(x,y) = \frac{y^2 - 3x^2}{y^4} \rightarrow \frac{\partial N}{\partial x} = \frac{-6x}{y^4}$$

= Exata

$$\rightarrow \frac{\partial u}{\partial x} = \frac{2x}{y^3} \Rightarrow u = \frac{x^2}{y^3} + k(y) \quad *$$

$$\frac{\partial M}{\partial y} = \frac{y^2 - 3x^2}{y^4}$$

$$\cancel{-\frac{3x^2}{y^4}} + \frac{dk}{dy} = \frac{y^2-3x^2}{y^4} =$$

$$= \cancel{-\frac{3x^2}{y^4}} + \frac{1}{y^2}$$

$$\therefore \frac{dk}{dy} = +\frac{1}{y^2} \Longrightarrow k(y) = -\frac{1}{y} + C$$

$$(**) \to (*) :$$

$$M = \frac{x^2}{y^3} - \frac{1}{y} + C = c^{te}$$

$$\frac{x^2}{y^3} - \frac{1}{y} = C_1$$

$$y(1) = 1 \Longrightarrow \frac{1}{1} - \frac{1}{1} = C_1 \quad \therefore C_1 = 0$$

$$\therefore \frac{x^2}{y^3} - \frac{1}{y} = 0$$

$$\frac{x^2}{y^3} = \frac{1}{y} \qquad (y \neq 0)$$

$$x^2\cancel{y} = y^{\cancel{3}} \Longrightarrow x^2 = y^2$$

$$\therefore y(x) = \pm x$$

mas $y(1) = 1$ $\Longrightarrow$ $\boxed{y(x) = x}$

18. $\left[n\cos(mx+my) - m\sin(mx+my)\right]dx \; +$

$+\left[m\cos(nx+my) - n\sin(mx+my)\right]dy = 0$

$M(x,y) = n\cos(nx+my) - m\sin(mx+my)$

$$\frac{\partial M}{\partial y} = -nm\sin(mx+my) - mn\cos(mx+my)$$

$=$

$N(x,y) = m\cos(nx+my) - n\sin(mx+my)$

$$\frac{\partial N}{\partial x} = -mn\sin(mx+my) - nm\cos(mx+my)$$

Exata

$$\frac{\partial \mu}{\partial x} = n\cos(nx+my) - m\sin(mx+ny)$$

$$\mu = \sin(nx+my) + \cos(mx+ny) + k(y) \quad (*)$$

$$\frac{\partial \mu}{\partial y} = m\cos(nx+my) - n\sin(mx+ny)$$

$(*)$

$$\cancel{m\cos(nx+my)} - n\cancel{\sin(mx+ny)} + \frac{dk}{dy} = m\cos(\cancel{nx+my}) - n\sin(\cancel{mx+ny})$$

$$\therefore \frac{dk}{dy} = 0 \Rightarrow k(y) = cte$$

(Ex)

$\therefore$ (8') $\to$ (8):

$$u = \sin(nx+my) + \cos(mx+my) + C = \text{cte}$$

$$\therefore \boxed{\sin(nx+my) + \cos(mx+my) = C_1}$$

---

19. $$\frac{x\,dx + y\,\boxed{dy}}{\sqrt{(x^2+y^2)(1-x^2-y^2)}} + \left(\frac{1}{y\sqrt{y^2-x^2}} + \frac{e^{\frac{x}{y}}}{y^2}\right)(y\,dx - x\,\boxed{dy}) = 0$$

$$\left(\frac{x}{\sqrt{(x^2+y^2)(1-x^2-y^2)}} + \frac{1}{\sqrt{y^2-x^2}} + \frac{e^{\frac{x}{y}}}{y}\right)dx +$$

$$+\left(\frac{y}{\sqrt{(x^2+y^2)(1-x^2-y^2)}} - \frac{x}{y\sqrt{y^2-x^2}} - \frac{x e^{\frac{x}{y}}}{y^2}\right)dy = 0$$

$$M(x,y) = \frac{x}{\sqrt{(x^2+y^2)(1-x^2-y^2)}} + \frac{1}{\sqrt{y^2-x^2}} + \frac{e^{\frac{x}{y}}}{y}$$

$$\frac{\partial M}{\partial y} = -\frac{x}{\cancel{2}}\,\frac{\cancel{2}y(1-x^2-y^2) + (x^2+y^2)(\cancel{-2}y)}{\left((x^2+y^2)(1-x^2-y^2)\right)^{3/2}} - \frac{1}{\cancel{2}}\,\frac{\cancel{2}y}{(y^2-x^2)^{3/2}} +$$

$$+ \frac{e^{\frac{x}{y}}}{y}\left(-\frac{x}{y^2}\right) - \frac{e^{\frac{x}{y}}}{y^2}$$

$$\frac{\partial N}{\partial x} = \frac{-yx(1-x^2-y^2)+yx(x^2+y^2)}{\left[(x^2+y^2)(1-x^2-y^2)\right]^{3/2}} - \frac{1}{y\sqrt{y^2-x^2}}$$

$$- \frac{x^2}{y(y^2-x^2)^{3/2}} - \frac{ye^{\frac{x}{y}} + xe^{\frac{x}{y}}}{y^3}$$

$$= \frac{2xy(x^2+y^2) - xy}{\left[(x^2+y^2)(1-x^2-y^2)\right]^{3/2}} - \frac{x^2+(y^2-x^2)}{y(y^2-x^2)^{3/2}}$$

$$- \frac{(x+y)}{y^3}e^{\frac{x}{y}}$$

$$\frac{\partial N}{\partial x} = \frac{2xy(x^2+y^2) - xy}{\left[(x^2+y^2)(1-x^2-y^2)\right]^{3/2}} - \frac{y^{\cancel{2}}}{\cancel{y}(y^2-x^2)^{3/2}}$$

$$- \frac{(x+y)}{y^3}e^{\frac{x}{y}} \quad (**)$$

De (*) e (**) tem-se

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \Rightarrow \text{eq. Exacta}$$

$$= \frac{-xy(1-x^2-y^2) + xy(x^2+y^2)}{\left[(x^2+y^2)(1-x^2-y^2)\right]^{3/2}} - \frac{y}{(y^2-x^2)^{3/2}}$$

$$- \frac{x e^{\frac{x}{y}}}{y^3} - \frac{e^{\frac{x}{y}}}{y^2}$$

$$\frac{\partial M}{\partial y} = \frac{2xy(x^2+y^2) - xy}{\left[(x^2+y^2)(1-x^2-y^2)\right]^{3/2}} - \frac{y}{(y^2-x^2)^{3/2}}$$

$$- \frac{\left(x e^{\frac{x}{y}} + y e^{\frac{x}{y}}\right)}{y^3} \qquad (*)$$

$$N(x,y) = \frac{y}{\sqrt{(x^2+y^2)(1-x^2-y^2)}} - \frac{x}{y\sqrt{y^2-x^2}} - \frac{x e^{\frac{x}{y}}}{y^2}$$

$$\frac{\partial N}{\partial x} = -\frac{1}{\cancel{2}} y \, \frac{\cancel{2}x(1-x^2-y^2) + (x^2+y^2)(-\cancel{2}x)}{\left[(x^2+y^2)(1-x^2-y^2)\right]^{3/2}} - \frac{1}{y\sqrt{y^2-x^2}}$$

$$- \frac{x^2}{y(y^2-x^2)^{3/2}} - \frac{e^{\frac{x}{y}}}{y^2} - \frac{x}{y^3} e^{\frac{x}{y}}$$

19: Cont.

$$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{(x^2+y^2)(1-x^2-y^2)}} + \frac{1}{\sqrt{y^2-x^2}} + \frac{e^{\frac{x}{y}}}{y}$$

$$\left\{ u = \underbrace{\int \frac{x}{\sqrt{(x^2+y^2)(1-x^2-y^2)}}\,dx}_{(1)} + \underbrace{\int \frac{dx}{\sqrt{y^2-x^2}}}_{(2)} + \underbrace{\int \frac{e^{\frac{x}{y}}}{y}\,dx}_{(3)} + K(y) \right.$$

Mas

$$z = x^2+y^2 \; ; \quad dz = 2x\,dx$$

$$\therefore \quad (1) = \int \frac{x}{\sqrt{(x^2+y^2)(1-x^2-y^2)}}\,dx = \int \frac{\frac{1}{2}\,dz}{\sqrt{z(1-z)}}$$

$$= \frac{1}{2}\int \frac{dz}{\sqrt{z-z^2}} = \frac{1}{2}\int \frac{dz}{\sqrt{\frac{1}{4}-\left(z-\frac{1}{2}\right)^2}}$$

$$\int \frac{dv}{\sqrt{a^2-v^2}} = \arcsin \frac{v}{a}$$

$$z - \frac{1}{2} = v$$

$$= \frac{1}{2}\int \frac{dv}{\sqrt{\frac{1}{4}-v^2}}$$

$$= \frac{1}{2} \arcsin 2\underline{v}$$

$$= \frac{1}{2} \arcsin (2z-1) = \frac{1}{2} \arcsin \left(2(x^2+y^2)-1\right)$$

$$② = \int \frac{dx}{\sqrt{y^2-x^2}} = \arcsin \frac{x}{y}$$

$$③ = \int \frac{e^{\frac{x}{y}}}{y}\, dx = e^{\frac{x}{y}}$$

$$\therefore\ u = \frac{1}{2}\arcsin\left(2(x^2+y^2)-1\right) + \arcsin \frac{x}{y} + e^{\frac{x}{y}} + k(y)$$

(*)

$$\frac{\partial u}{\partial y} = \frac{y}{\sqrt{(x^2+y^2)(1-x^2-y^2)}} - \frac{x}{y\sqrt{y^2-x^2}} - \frac{x e^{\frac{x}{y}}}{y^2}$$

(*)

$$\frac{1}{\cancel{2}} \frac{1}{\sqrt{1-[2(x^2+y^2)-1]^2}} \cdot (4y) + \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(-\frac{x}{y^2}\right) +$$

$$+ \cancel{e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right)} + \frac{dk}{dy} = \frac{y}{\sqrt{(x^2+y^2)(1-x^2-y^2)}} - \frac{x}{y\sqrt{y^2-x^2}} \cancel{- \frac{x e^{\frac{x}{y}}}{y^2}}$$

$$\frac{2y}{\sqrt{\cancel{1} - 4(x^2+y^2)^2 + 4(x^2+y^2) - \cancel{1}}} \cancel{- \frac{x}{y\sqrt{y^2-x^2}}} + \frac{dk}{dy} =$$

$$= \frac{y}{\sqrt{(x^2+y^2)(1-x^2-y^2)}} \cancel{- \frac{x}{y\sqrt{y^2-x^2}}}$$

$$\frac{2y}{\sqrt{4(x^2+y^2)\left[1-(x^2+y^2)\right]}} + \frac{dk}{dy} = \frac{y}{\sqrt{(x^2+y^2)(1-x^2-y^2)}}$$

$$\therefore \quad \cancel{\frac{y}{\sqrt{(x^2+y^2)(1-x^2-y^2)}}} + \frac{dk}{dy} = \cancel{\frac{y}{\sqrt{(x^2+y^2)(1-x^2-y^2)}}}$$

$$\frac{dk}{dy} = 0$$

$$\therefore \quad k(y) = \text{cte.} \qquad (**)$$

$$(**) \rightarrow (*) :$$

$$u = \frac{1}{2}\arcsin\left(2(x^2+y^2)-1\right) + \arcsin\frac{x}{y} + e^{\frac{x}{y}} + C = C$$

$$\therefore \quad \boxed{\frac{1}{2}\arcsin\left(2(x^2+y^2)-1\right) + \arcsin\frac{x}{y} + e^{\frac{x}{y}} = C_1}$$

20. Eq. Separável : $g(y)\, y' = f(x)$ (*)

$\therefore \quad g(y)\, y' - f(x) = 0$

$\therefore \quad g(y)\, dy - f(x)\, dx = 0$

$\Leftrightarrow$

$N(x,y)\, dy + M(x,y)\, dx$

$\therefore \quad N = g(y) \rightarrow \dfrac{\partial N}{\partial x} = 0$

$M = -f(x) \rightarrow \dfrac{\partial M}{\partial y} = 0$

$=$ Eq (*) é exata

$\therefore$ Eq. Separável é exata

Uma eq. exata não é necessariamente separável.

---

21. $(ax + by)\, dx + (kx + ly)\, dy = 0$

$M = ax + by \rightarrow \dfrac{\partial M}{\partial y} = b$

$N = kx + ly \rightarrow \dfrac{\partial N}{\partial x} = k$

Eq. exata: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow \boxed{b = k}$

Neste caso:

$(ax + by)\, dx + (kx + ly)\, dy = 0$

$$\frac{\partial u}{\partial x} = ax + by \implies u = \frac{a x^2}{2} + bxy + k(y) \quad (*)$$

$$\frac{\partial u}{\partial y} = bx + ly$$

$$(*) \quad \cancel{bx} + \frac{dk}{dy} = \cancel{bx} + ly$$

$$k = \frac{l y^2}{2} + C \quad (**)$$

$$(**) \rightarrow (*) : \quad u = \frac{a}{2}x^2 + bxy + \frac{l}{2}y^2 + C$$

$$\therefore \quad \boxed{\frac{a}{2}x^2 + bxy + \frac{l}{2}y^2 = C_1}$$

---

22. $$\underbrace{\left(f(x) + g(y)\right)}_{M} dx + \underbrace{\left(h(x) + p(y)\right)}_{N} dy = 0$$

$$M(x,y) = f(x) + g(y) \longrightarrow \frac{\partial M}{\partial y} = \frac{dg}{dy}$$

$$N(x,y) = h(x) + p(y) \longrightarrow \frac{\partial N}{\partial x} = \frac{dh}{dx}$$

Exata $\implies \frac{dg}{dy} = \frac{dh}{dx}$

Eq. é exata se $\frac{d\,g(y)}{dy} = \frac{dh}{dx}$

$$\therefore \quad \boxed{\begin{array}{l} g(y) = ay \\ \hline h(x) = ax \end{array}}$$

23.
$$\underbrace{f(x,y)}_{M}\,dx + \underbrace{g(x)\,h(y)}_{N}\,dy = 0$$

Exata $\Rightarrow$ $\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial x}\left(g(x)\,h(y)\right)$

$$\boxed{\frac{\partial f(x,y)}{\partial y} = \frac{d\,g(x)}{dx}\,h(y)}$$

---

24. a) $3x^{-4}y\,dx = x^{-3}dy$

Sep. Variáveis:

$$3x^{-4}y\,dx = x^{-3}dy$$

$$\frac{3x^{-4}}{x^{-3}}\,dx = \frac{1}{y}\,dy$$

$$\frac{3}{x}\,dx = \frac{1}{y}\,dy$$

$$3\ln|x| = \ln|y| + C$$

$$\ln\frac{|x|^3}{|y|} = C \Rightarrow \left|\frac{x^3}{y}\right| = e^C$$

$$\frac{x^3}{y} = C_1$$

$$\underline{\underline{y = C_2 x^3}}$$

Eq. exata

$$\underbrace{3x^{-4}y}\,dx - x^{-3}dy = 0$$

$$d(-x^{-3}y) = 0$$

$$-x^{-3}y = C$$

$$y = -Cx^3$$

$$\underline{y = C_1 x^3}$$

b) $2x\,dx + x^{-2}(x\,dy - y\,dx) = 0$

Sep. variáveis : ?

$$2x\,dx + \underbrace{x^{-2}(x\,dy - y\,dx)} = 0$$

$$2x\,dx + d\left(\frac{y}{x}\right) = 0$$

$$x^2 + \frac{y}{x} = C$$

$$\underline{y = Cx - x^3}$$

Eq. exata

$$2x\,dx + \frac{dy}{x} - \frac{y}{x^2}\,dx = 0$$

$$\underbrace{\left(2x - \frac{y}{x^2}\right)}_{M} dx + \underbrace{\frac{dy}{x}}_{N} = 0$$

$$\frac{\partial u}{\partial x} = 2x - \frac{y}{x^2} \longrightarrow u = x^2 + \frac{y}{x} + k(y) \quad (*)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x}$$

$(*)$

$$\cancel{\frac{1}{x}} + \frac{dk}{dy} = \cancel{\frac{1}{x}} \Rightarrow k = \text{cte} \quad (**)$$

$(**) \rightarrow (*)$; $u(x,y) = x^2 + \frac{y}{x} + C$

$$\therefore \quad x^2 + \frac{y}{x} = C_1$$

$$x^3 + y = C_1 x$$

$$\underline{\underline{y = C_1 x - x^3}}$$

Cálculo C – Lista 10 – Resposta

1. $yx = C$

2. $y^2 x = C$

3. $x(e^x + e^y) = C$

4. $\cosh x \cos y = C$

5. $r e^{-\theta} = C$

6. $x^4 + x^2 y^2 + y^4 = C$

7. $y^4 + 3x^2 y^2 + x^3 = C$

8. $\sqrt{x^2+y^2} + \frac{x}{y} + \ln|xy| = C$

9. $x^3 \operatorname{tg} y + \frac{y^3}{x^2} + y^4 = C$

10. $x^3 y + x^2 - y^2 = Cxy$

11. $\frac{\sin^2 x}{y} + \frac{x^2+y^2}{2} = C$

2. $x^3 + y^3 - x^2 - xy + y^2 = C$

3. $y\sqrt{1+x^2} + x^2 y - y \ln x = C$

4. $\sqrt{x^2+y^2} + \frac{y}{x} = C$

5. $\ln|xy| + x\sin y - y\cos x = C$

6. $\operatorname{tg} xy - \cos x - \cos y = C$

7. $y = x$

8. $\sin(nx+my) + \cos(mx+ny) = C$

19. $\frac{1}{2}\arcsin(2(x^2+y^2)-1) + \arcsin\frac{x}{y} + e^{\frac{x}{y}} = C$

21. $\begin{cases} b = K \\ \frac{a}{2}x^2 + bxy + \frac{\ell}{2}y^2 = C \end{cases}$

22. $\boxed{\begin{array}{l} g(y) = ay \\ h(x) = ax \end{array}}$

$(a = \text{cte})$

23. $\frac{\partial f}{\partial y}(x,y) = \frac{dg}{dx}(x)\, h(y)$

24. a) $y = Cx^3$

b) $y = Cx - x^3$