Cálculo C - Lista 11

Equações exatas - fator integrante

Mostre que a função dada é um fator integrante da equação e resolva a equação. Verifique também que equações são de variáveis separáveis e resolva-as aplicando esse método. Compare os resultados.

1. $2y\,dx + x\,dy = 0;\quad x$
2. $x\,dy - y\,dx = 0;\quad 1/x^2$
3. $\sin y\,dx + \cos y\,dy = 0;\quad e^x$
4. $y^2\,dx + (1+xy)dy = 0;\quad e^{xy}$

Para cada uma das equações a seguir verifique que a condição

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \equiv f(x)$$

é satisfeita. Resolva então a equação usando um fator integrante $F(x) = e^{\int f(x)dx}$.

5. $2\,dx - e^{y-x}dy = 0$
6. $x\cosh y\,dy - \sinh y\,dx = 0$
7. $(y+1)dx - (x+1)dy = 0$
8. $(x+y^2)\,dx - 2xy\,dy = 0$
9. $(x\cos y - y\sin y)\,dy + (x\sin y + y\cos y)\,dx = 0$

Para cada uma das equações a seguir verifique que a condição

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \equiv g(y)$$

é satisfeita. Resolva então a equação usando um fator integrante $F(y) = e^{\int g(y)dy}$.

10. $\cos x\,dx + \sin x\,dy = 0$
11. $2\cosh x\cos y\,dx = \sinh x\sin y\,dy$
12. $y\,dx + (3+3x-y)\,dy = 0$
13. $2x\tan y\,dx + \sec^2 y\,dy = 0$
14. $y(1+xy)\,dx - x\,dy = 0$

Resolva as equações usando um fator integrante do tipo $F(x)$ ou $F(y)$.

15. $(2\cos y + 4x^2)\,dx = x\sin y dy$
16. $\frac{y}{x}\,dx + (y^3 - \ln x)\,dy = 0$
17. $(3xe^y + 2y)dx + (x^2e^y + x)\,dy = 0$
18. **Mostre que se a equação** $M dx + N dy = 0$ for tal que

$$\frac{1}{xM - yN}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(xy)$$

então ela admite um fator integrante do tipo $e^{\int f(u)du}$ onde $u = xy$.

19. **Use o método do exercício anterior para resolver a equação**

$$(y^2 + xy + 1)\,dx + (x^2 + xy + 1)\,dy = 0$$

20. Resolva

$$(2y^2 + 4x^2y)\,dx + (4xy + 3x^3)\,dy = 0$$

sabendo que existe um fator integrante da forma $F(x,y) = x^a y^b$ com a, b constantes.

Lista 11 - Resposta

1. $y = \frac{c}{x^2}$

2. $y = cx$

3. $\sin y = c e^{-x}$

4. $e^{xy} y = c$

5. $2e^x - e^y = c$

6. $\sinh y = cx$

7. $y = c(x+1) - 1$

8. $-\frac{y^2}{x} + \ln|x| = c$

9. $e^x (x \sin y + y \cos y - \sin y) = c$

10. $e^y \sin x = c$

11. $\frac{2 \sinh x \cos y}{|\cos y|^{1/2}} = c$

12. $y^3 (x+1) - \frac{y^4}{4} = c$

13. $\operatorname{tg} y = c e^{-x^2}$

14. $\frac{x}{y} + \frac{x^2}{2} = c$

15. $x^2 \cos y + x^4 = c$

16. $\frac{y^2}{2} + \frac{\ln x}{y} = c$

17. $x^3 e^y + x^2 y = c$

19. $e^{xy}(x+y) = c$

20. $x^2 y^4 + x^4 y^3 = c$

Cálculo C - Lista 11

1.
$$\begin{cases} 2y\,dx + x\,dy = 0 \\ F(x,y) = x \end{cases}$$

$$\underbrace{2xy\,dx + x^2dy}_{} = 0$$

$$d(x^2y) = 0$$

$$x^2y = c$$

$$\boxed{y = \frac{c}{x^2}}$$

Por separação de variáveis:

$$\frac{2}{x}dx + \frac{1}{y}dy = 0$$

$$\ln|y| = -2\ln|x| + c$$

$$\ln|y| + \ln|x|^2 = c$$

$$\ln(|y|x^2) = c$$

$$|y|x^2 = e^c$$

$$// \; y = \frac{c_1}{x^2} \; //$$

2.
$$\begin{cases} x\,dy - y\,dx = 0 \\ F(x,y) = \frac{1}{x^2} \end{cases}$$

$$\frac{1}{x^2}x\,dy - \frac{1}{x^2}y\,dx = 0$$

$$\frac{1}{x}dy - \frac{y}{x^2}dx = 0$$

$$d\left(\frac{y}{x}\right) = 0$$

$$\frac{y}{x} = c$$

$$\boxed{y = cx}$$

Por separação de variáveis

$$\frac{1}{y}dy = \frac{1}{x}dx$$

$$\ln|y| = \ln|x| + c$$

$$\ln\left|\frac{y}{x}\right| = c$$

$$\left|\frac{y}{x}\right| = e^c$$

$$\frac{y}{x} = \pm e^c$$

$$// \; y = c_1x \; //$$

3.

$$\begin{cases} \sin y\, dx + \cos y\, dy = 0 \\ F(x,y) = e^{x} \end{cases}$$

$$e^{x}\sin y\, dx + e^{x}\cos y\, dy = 0$$

$$d(e^{x}\sin y) = 0$$

$$//\ e^{x}\sin y = C\ //$$

Por sep. variables:

$$dx + \cot g\, y\, dy = 0$$

$$\cot g\, y\, dy = -dx$$

$$\ln|\sin y| = -x + C$$

$$|\sin y| = e^{-x} e^{C}$$

$$|\sin y| = C_1 e^{-x}$$

$$\sin y = \pm C_1 e^{-x}$$

$$//\ \sin y = C_2 e^{-x}\ //$$

4.

$$\begin{cases} y^2 dx + (1+xy)\, dy = 0 \\ F(x,y) = e^{xy} \end{cases}$$

$$e^{xy} y^2 dx + e^{xy}(1+xy)\, dy = 0$$

$$d(e^{xy} y) = 0$$

$$//\ e^{xy} y = C\ //$$

5.

$$2dx - e^{y-x} dy = 0$$

$$\begin{cases} M = 2 \\ N = -e^{y-x} \end{cases}$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) =$$

$$= \frac{1}{-e^{y-x}}\left(-e^{y-x}\right)$$

$$= +1 = f(x)$$

$$\therefore\ F(x) = e^{\int f(x)dx} = e^{+x}$$

$$\therefore\ e^{+x}(2dx - e^{y-x}dy) = 0$$

$$2e^{+x}dx - e^{y}dy = 0$$

5. Cont.:

$$d(2e^{x} - e^{y}) = 0$$

$\therefore$

$$\boxed{2e^{x} - e^{y} = c}$$

6.

$$x\cosh y\, dy - \sinh y\, dx = 0$$

$N = x\cosh y$

$M = -\sinh y$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) =$$

$$= \frac{1}{x\cosh y}\left(-\cosh y - \cosh y\right)$$

$$= \frac{-2\cosh y}{x\cosh y}$$

$$= \frac{-2}{x} = f(x)$$

$$\therefore F(x) = e^{\int f(x)dx} = e^{-2\int \frac{dx}{x}}$$

$$= e^{-2\ln|x|}$$

$$= e^{\ln|x|^{-2}}$$

$$= \frac{1}{}$$

$$\frac{1}{x^2}\, x\cosh y\, dy - \frac{1}{x^2}\sinh y\, dx = 0$$

$$\frac{1}{x}\cosh y\, dy - \frac{1}{x^2}\sinh y\, dx = 0$$

$$d\left(\frac{\sinh y}{x}\right) = 0$$

$$\frac{\sinh y}{x} = C$$

$$\boxed{\sinh y = Cx}$$

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7. $(y+1)\,dx - (x+1)\,dy = 0$

$M(x,y) = y+1$

$N(x,y) = -(x+1)$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) =$$

$$= \frac{1}{-(x+1)}\left(1 - (-1)\right)$$

$$= \frac{2}{-(x+1)}$$

$$\therefore F(x) = e^{\int \frac{2}{-(x+1)}dx}$$

$$= e^{-2\ln|x+1|}$$

$$\ln|x+1|^{-2}$$

$$F(x) = |x+1|^{-2}$$
$$= \frac{1}{(x+1)^2}$$

$$\frac{1}{(x+1)^2}(y+1)\,dx - \frac{\cancel{(x+1)}}{(x+1)^2}\,dy = 0$$

$$d\left[-\frac{(y+1)}{(x+1)}\right] = 0$$

$$-\frac{(y+1)}{(x+1)} = C$$

$$y+1 = -C(x+1)$$

$$\boxed{y = C_1(x+1) - 1}$$

$$\frac{1}{x^2}(x+y^2)\,dx - \frac{1}{x^2}2xy\,dy = 0$$

$$\frac{x+y^2}{x^2}\,dx - \frac{2y}{x}\,dy = 0$$

$$\frac{\partial u}{\partial y} = -\frac{2y}{x} \Rightarrow$$
$$\Rightarrow u(x,y) = -\frac{y^2}{x} + k(x)$$

$$\frac{\partial u}{\partial x} = \frac{x+y^2}{x^2}$$

$$\cancel{\frac{y^2}{x^2}} + \frac{dk}{dx} = \frac{x+\cancel{y^2}}{x^2}$$

$$\frac{dk}{dx} = \frac{1}{x}$$

$$k = \ln|x| + C$$

8. $$\underbrace{(x+y^2)}_{M}\,dx\ \underbrace{-2xy}_{N}\,dy = 0$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) =$$

$$= \frac{1}{-2xy}\left(2y - (-2y)\right)$$

$$= \frac{4\cancel{y}}{-2x\cancel{y}} = -\frac{2}{x}$$

$$F(x) = e^{\int -\frac{2}{x}\,dx} = e^{-2\ln|x|}$$
$$= e^{\ln|x|^{-2}} = \frac{1}{x^2}$$

$$u(x,y) = -\frac{y^2}{x} + \ln|x| + C$$

$$\boxed{-\frac{y^2}{x} + \ln|x| = C_1}$$

9.

$(x\cos y - y\sin y)dy +$

$+(x\sin y + y\cos y)dx = 0$

$M = x\sin y + y\cos y$

$N = x\cos y - y\sin y$

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) =$

$= \frac{1}{x\cos y - y\sin y}\left(x\cos y + \cancel{\cos y} - y\sin y - \cancel{\cos y}\right)$

$= \frac{x\cos y - y\sin y}{x\cos y - y\sin y}$

$= 1$

$= f(x)$

$F(x) = e^{\int f(x)dx} = e^x$

$\int e^x(x\cos y - y\sin y)dy +$

$+e^x(x\sin y + y\cos y)dx = 0$

$\frac{\partial u}{\partial y} = e^x(x\cos y - y\sin y)$

$\therefore$

$u(x,y) = +e^x x\sin y$

$- e^x\int y\sin y\,dy + k(x)$

$\int y\sin y\,dy = -y\cos y + \int\cos y\,dy$

$u = y \rightarrow du = dy$

$dv = \sin y\,dy \rightarrow v = -\cos y$

$= -y\cos y + \sin y$

$\therefore$

$u(x,y) = +e^x x\sin y - e^x(-y\cos y + \sin y)$

$+ k(x)$

$u(x,y) = +xe^x\sin y + e^x y\cos y$

$- e^x\sin y + k(x)$ (*)

$\frac{\partial u}{\partial x} = e^x x\sin y + e^x y\cos y$

(*)

$+\cancel{e^x\sin y} + xe^x\sin y + \cancel{e^x y\cos y}$

$-\cancel{e^x\sin y} + \frac{dk}{dx} = \cancel{e^x x\sin y} + \cancel{e^x y\cos y}$

$\frac{dk}{dx} = 0 \Rightarrow k = C$ (**)

9. Cont.

(**) → (*) ;

$u(x,y) = x\,e^x \sin y + e^x y \cos y$

$\quad - e^x \sin y + C \quad = cte$

$\therefore$

$$\boxed{e^x(x \sin y + y\cos y - \sin y) = C_1}$$

10.

$$\underbrace{\cos x}_{M}\,dx + \underbrace{\sin x}_{N}\,dy = 0$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) =$$

$$= \frac{1}{\cos x}(\cos x - 0) = 1$$

$$= g(y)$$

$$F(y) = e^{\int g(y)dy} = e^{y}$$

$\therefore$

$$e^{y}\cos x\,dx + e^{y}\sin x\,dy = 0$$

$$d(e^{y}\sin x) = 0$$

$$\boxed{e^{y}\sin x = C}$$

1.

$$\underbrace{2\cosh x\cos y}_{M}\,dx - \underbrace{\sinh x\sin y}_{N}\,dy = 0$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) =$$

$$= \frac{1}{2\cosh x\cos y}\left(-\cosh x\sin y - (-2\cosh x\sin y)\right)$$

$$= \frac{1}{\ldots}(\cosh x\sin y) = \frac{1}{2}\operatorname{tg} y = g(y)$$

$\therefore$

$$F(y) = e^{\int g(y)dy} = e^{\int \frac{1}{2}\operatorname{tg} y\,dy}$$

$$= e^{\frac{1}{2}\ln|\sec y|}$$

$$= e^{\ln|\sec y|^{1/2}}$$

$$= |\sec y|^{1/2}$$

$$|\sec y|^{1/2}\,2\cosh x\cos y\,dx -$$

$$-|\sec y|^{1/2}\sinh x\sin y\,dy = 0$$

$\therefore$

$$2\cosh x\frac{\cos y}{|\cos y|^{1/2}}dx - \sinh x\frac{\sin y}{|\cos y|^{1/2}}dy = 0$$

$$\frac{\partial u}{\partial x} = 2\cosh x\frac{\cos y}{|\cos y|^{1/2}}$$

$$u = 2\sinh x\frac{\cos y}{|\cos y|^{1/2}} + k(y) \quad (*)$$

$$\frac{\partial u}{\partial y} = -\sinh x\frac{\sin y}{|\cos y|^{1/2}}$$

$(*)$

$$2\sinh x\frac{d}{dy}\frac{\cos y}{|\cos y|^{1/2}} + \frac{dk}{dy} = -\frac{\sinh x\sin y}{|\cos y|^{1/2}}$$

$(**)$

Mas

$$\frac{d}{dy}\frac{\cos y}{|\cos y|^{1/2}} = \frac{-\sin y}{|\cos y|^{1/2}} + \cos y \frac{d}{dy}|\cos y|^{-1/2}$$

$$= \frac{-\sin y}{|\cos y|^{1/2}} + \cos y\left(-\frac{1}{2}|\cos y|^{-3/2} \cdot \underbrace{\frac{d}{dy}|\cos y|}\right)$$

$$= \frac{-\sin y}{|\cos y|^{1/2}} - \frac{1}{2}\frac{\cancel{\cos y}}{|\cos y|^{3/2}} \cdot \underbrace{\frac{|\cos y|}{\cancel{\cos y}}(-\sin y)}$$

$$= \frac{-\sin y}{|\cos y|^{1/2}} + \frac{1}{2}\frac{|\cos y|}{|\cos y|^{3/2}}\sin y$$

$$= \frac{-\sin y}{|\cos y|^{1/2}} + \frac{1}{2}\frac{\sin y}{|\cos y|^{1/2}}$$

$$= -\frac{1}{2}\frac{\sin y}{|\cos y|^{1/2}}$$

$\therefore$

(**) :

$$\cancel{2}\sinh x\left(-\frac{1}{\cancel{2}}\frac{\sin y}{|\cos y|^{1/2}}\right) + \frac{dh}{dy} =$$

$$= -\frac{\sinh x \sin y}{|\cos y|^{1/2}}$$

$$\cancel{-\frac{\sinh x \sin y}{|\cos y|^{1/2}}} + \frac{dh}{dy} =$$

$$= \cancel{-\frac{\sinh x \sin y}{|\cos y|^{1/2}}}$$

$$\therefore \quad \frac{dh}{dy} = 0$$

$$h = C \quad (\text{***})$$

$$\therefore \quad (\text{***}) \to (\text{*})$$

$$u(x,y) = 2\sinh x \frac{\cos y}{|\cos y|^{1/2}} + C$$

$\therefore$

$$\boxed{\frac{2\sinh x \cos y}{|\cos y|^{1/2}} = C_1}$$

$\therefore$ Equivalentemente:

$$4\sinh^2 x \frac{\cos^2 y}{|\cos y|} = C \qquad \frac{x^2}{|x|} = |x|$$

$$\sinh^2 x\,|\cos y| = C$$

$$//\ \sinh^2 x \cos y = C_1\ //$$

12.

$$\underbrace{y\,dx}_{M} + \underbrace{(3+3x-y)}_{N}\,dy = 0$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) =$$

$$= \frac{1}{y}\left(3-1\right) = \frac{2}{y} = g(y)$$

$\therefore$

$$F(y) = e^{\int \frac{2}{y}dy}$$

$$= e^{2\ln|y|} = e^{\ln|y|^2}$$

$$= |y|^2 = \underline{\underline{y^2}}$$

$\therefore$

$$y^2 y\,dx + y^2(3+3x-y)\,dy = 0$$

$$y^3 dx + (3y^2+3y^2x-y^3)\,dy = 0$$

$$\frac{\partial u}{\partial x} = y^3 \Rightarrow u = y^3x + k(y)$$

(*)

$$\frac{\partial u}{\partial y} = 3y^2 + 3y^2x - y^3$$

$$\cancel{3y^2x} + \frac{dk}{dy} = 3y^2 + \cancel{3y^2x} - y^3$$

$$\frac{dk}{dy} = 3y^2 - y^3$$

$$k(y) = y^3 - \frac{1}{4}y^4 + C \quad (**)$$

(**) $\to$ (*) ;

$$u(x,y) = y^3x + y^3 - \frac{1}{4}y^4 + C$$

$\therefore$

$$\boxed{y^3(x+1) - \frac{y^4}{4} = C_1}$$

13.

$$\underbrace{2x\,\mathrm{tg}\,y\,dx}_{M} + \underbrace{\sec^2 y}_{N}\,dy = 0$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) =$$

$$= \frac{1}{2x\,\mathrm{tg}\,y}\left(0 - 2x\sec^2 y\right)$$

$$= -\frac{\sec^2 y}{\mathrm{tg}\,y} = g(y)$$

$$F(y) = e^{-\int \frac{\sec^2 y}{\mathrm{tg}\,y}dy}$$

$$= e^{-\ln|\mathrm{tg}\,y|}$$

$$= e^{\ln|\mathrm{cotg}\,y|}$$

$\therefore$ $F(y) = |\mathrm{cotg}\,y|$

$//F(y) = \mathrm{cotg}\,y//$ a menos de uma constante

$\therefore$

13. cont.

$$2x \cot y \underbrace{\tan y}\, dx + \cot y \sec^2 y\, dy = 0$$

$$2x\, dx + \frac{\sec^2 y}{\tan y}\, dy = 0$$

$$d\left(x^2 + \ln|\tan y| \right) = 0$$

$$x^2 + \ln|\tan y| = C$$

$$\ln|\tan y| = C - x^2$$

$$|\tan y| = e^{C-x^2}$$

$$\tan y = \pm e^{C-x^2}$$

$$= \underbrace{\pm e^{C}}\, e^{-x^2}$$

$$\boxed{\tan y = C_1 e^{-x^2}}$$

14.
$$\underbrace{y(1+xy)}_{M}\, dx \underbrace{- x}_{N}\, dy = 0$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) =$$

$$= \frac{1}{y(1+xy)}\left(-1 - (1 + 2xy) \right)$$

$$= \frac{1}{y(1+xy)}\left(-2 - 2xy \right)$$

$$= \frac{1}{y\cancel{(1+xy)}} (-2)\cancel{(1+xy)}$$

$$= -\frac{2}{y} = g(y)$$

$$f(y) = e^{\int -\frac{2}{y} dy}$$

$$= e^{-2\ln|y|}$$

$$= e^{\ln|y|^{-2}}$$

$$\underline{\underline{f(y)}} = |y|^{-2} = \underline{\underline{\frac{1}{y^2}}}$$

$\therefore$

$$\frac{1}{y^2}\, y(1+xy)\, dx - \frac{x}{y^2}\, dy = 0$$

$$\frac{1}{y}(1+xy)\, dx - \frac{x}{y^2}\, dy = 0$$

$$\left(\frac{1}{y} + x \right) dx - \frac{x}{y^2}\, dy = 0$$

$$d\left(\frac{x}{y} + \frac{1}{2}x^2 \right) = 0$$

$$\boxed{\frac{x}{y} + \frac{x^2}{2} = C}$$

15.

$$(2\cos y + 4x^2)\,dx = x\sin y\,dy$$

$$\underbrace{(2\cos y + 4x^2)}_{M}\,dx - \underbrace{x\sin y}_{N}\,dy = 0$$

$\therefore$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) =$$

$$= \frac{1}{2\cos y + 4x^2}\left(-\sin y - (-2\sin y)\right)$$

$$= \frac{1}{2\cos y + 4x^2}(\sin y)$$

$$= F(x,y) \quad ?$$

$\therefore$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) =$$

$$= \frac{1}{-x\sin y}\left(-2\sin y - (-\sin y)\right)$$

$$= \frac{1}{-x\cancel{\sin y}}(-\cancel{\sin y})$$

$$= \frac{1}{x} = f(x)$$

$\therefore$

$$F(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|}$$

$$= |x|$$

$\therefore$ $F(x) = \pm x$ (a menos de uma cte).

$\therefore$

$$x(2\cos y + 4x^2)\,dx = x^2\sin y\,dy$$

$$(2x\cos y + 4x^3)\,dx - x^2\sin y\,dy = 0$$

$$d(x^2\cos y + x^4) = 0$$

$$\boxed{x^2\cos y + x^4 = C}$$

16.

$$\underbrace{\frac{y}{x}}_{M}\,dx + \underbrace{(y^3 - \ln x)}_{N}\,dy = 0$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) =$$

$$= \frac{x}{y}\left(-\frac{1}{x} - \frac{1}{x}\right) = \frac{\cancel{x}}{y}\left(\frac{-2}{\cancel{x}}\right)$$

$$= -\frac{2}{y} = g(y)$$

$\therefore$

16. Cont.

$$F(y) = e^{\int -\frac{2}{y} dy}$$
$$= e^{-2\ln|y|}$$
$$= e^{\ln|y|^{-2}}$$
$$= |y|^{-2}$$

$$f(y) = \frac{1}{y^2}$$

$$\frac{1}{y^2}\left(\frac{y}{x}dx + (y^3 - \ln x)dy\right) = 0$$

$$\frac{1}{yx}dx + \left(y - \frac{\ln x}{y^2}\right)dy = 0$$

$$d\left(\frac{y^2}{2} + \frac{\ln x}{y}\right) = 0$$

$$\therefore \boxed{\frac{y^2}{2} + \frac{\ln x}{y} = C}$$

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17. $\underbrace{(3xe^y + 2y)}_{M}dx + \underbrace{(x^2e^y + x)}_{N}dy = 0$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) =$$

$$= \frac{1}{x^2e^y + x}\left(3xe^y + 2 - 2xe^y - 1\right)$$

$$= \frac{1}{x^2e^y + x}\left(xe^y + 1\right)$$

$$= \frac{1}{x\cancel{(xe^y+1)}}\cancel{(xe^y+1)}$$

$$= \frac{1}{x} = f(x)$$

$$\therefore F(x) = e^{\int f(x)dx}$$
$$= e^{\int \frac{1}{x}dx}$$
$$= e^{\ln|x|}$$
$$F(x) = |x|$$

$\therefore F(x) = \pm x$ (factor integrante está definido a menos de una cte).

$$\therefore x(3xe^y + 2y)dx + x(x^2e^y + x)dy = 0$$

$$(3x^2e^y + 2xy)dx + (x^3e^y + x^2)dy = 0$$

$$d(x^3e^y + x^2y) = 0$$

$$\boxed{x^3e^y + x^2y = C}$$

18. $M\,dx + N\,dy = 0$ (*)

Seja $F(x,y)$ um fator integrante de (*).

$\therefore$

$$\underbrace{FM\,dx + FN\,dy}_{du(x,y)\,=\,0} = 0 \quad \text{é exata}$$

$\therefore$

$$\frac{\partial (FM)}{\partial y} = \frac{\partial (FN)}{\partial x}$$

$\therefore$

$$\frac{\partial F}{\partial y} M + F \frac{\partial M}{\partial y} = \frac{\partial F}{\partial x} N + F \frac{\partial N}{\partial x}$$

$(\div F)$

$$\frac{M}{F}\frac{\partial F}{\partial y} + \frac{\partial M}{\partial y} = \frac{N}{F}\frac{\partial F}{\partial x} + \frac{\partial N}{\partial x}$$

$$\frac{M}{F}\frac{\partial F}{\partial y} - \frac{N}{F}\frac{\partial F}{\partial x} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$M\frac{\partial}{\partial y}\ln|F| - N\frac{\partial}{\partial x}\ln|F| = \underbrace{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}$$

hipótese

$$M\frac{\partial}{\partial y}\ln|F| - N\frac{\partial}{\partial x}\ln|F| = (xM - yN)f(xy)$$

Da arbitrariedade de M e N põe-se

$$M\left(\frac{\partial}{\partial y}\ln|F| - xf\right) - N\left(\frac{\partial}{\partial x}\ln|F| - yf\right) = 0$$

$\therefore$

$\therefore$

$$\begin{cases} \dfrac{\partial}{\partial y} \ln|F(xy)| = x f(xy) & (*) \\[2ex] \dfrac{\partial}{\partial x} \ln|F(xy)| = y f(xy) & (**) \end{cases}$$

$$(*) : \ln|F| = \int x f(xy)\, dy + k(x) \qquad \underline{\underline{(**)}}$$

$$\Leftarrow$$

$$\therefore \frac{\partial \ln|F|}{\partial x} = \frac{\partial}{\partial x}\left(\int x f(xy)\, dy + k(x) \right)$$

$$= \underbrace{\int f(xy)\,dy}_{1} + \underbrace{\int x \frac{\partial f}{\partial x}\, dy}_{2} + \frac{dk}{dx}$$

$$1 : \int f(xy)\,dy = \int f(u) \frac{du}{x} = \frac{1}{x} \int f(u)\,du$$

$u = xy \rightarrow du = x\,dy$

($x = $ cte
na integral)

$$2 = \int x \frac{\partial f(xy)}{\partial x}\, dy = \int x \frac{\partial (xy)}{\partial x} \frac{\partial f(xy)}{\partial (xy)}\, dy$$

$$= \int x\, y \frac{df}{du}(u)\, dy \qquad \Big/ u = xy$$

$$= \int u \frac{df}{du}(u) \frac{du}{x}$$

$$= \frac{1}{x} \int u \frac{df}{du}\, du \qquad \text{int. partes}$$

$$= \frac{1}{x} \int u\, df = \frac{1}{x}\left(u f(u) - \int f(u)\,du \right.$$

18. cont.

$$② = \int x \frac{\partial f}{\partial x}(xy)\, dy = \frac{1}{x}\left(xy\, f(xy) - \int f(u)\, du \right)$$

$$= y f(xy) - \frac{1}{x}\int f(u)\, du$$

$\therefore$

$$\frac{\partial}{\partial x} \ln |F| = ① + ② + \frac{dk}{dx}$$

$$= \frac{1}{x}\cancel{\int f(u)\, du} + y f(xy) - \frac{1}{x}\cancel{\int f(u)\, du} + \frac{dk}{dx}$$

$\therefore$

$$\frac{\partial}{\partial x} \ln |F| = y f(xy) + \frac{dk}{dx}$$

Mas de $(**)$

$$\frac{\partial}{\partial x} \ln |F| = y f(xy)$$

$\Rightarrow \frac{dk}{dx} = 0$

$\therefore \underline{\underline{k = cte}}$ $(***)$

$(***) \to (**)$: $\quad u = xy$

$$\ln |F(x,y)| = \int \underbrace{x\, f(xy)\, dy}_{du} + C$$

$$\ln |F(x,y)| = \int f(u)\, du + C$$

$$\therefore \quad |F(x,y)| = e^{C} e^{\int f(u)\, du}$$

$$\boxed{F(x,y) = C_1\, e^{\int f(u)\, du}}$$

19.

$$\underbrace{(y^2+xy+1)}_{M}\,dx + \underbrace{(x^2+xy+1)}_{N}\,dy = 0$$

Aqui:

$$\frac{1}{xM-yN}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) =$$

$$= \frac{1}{x(y^2+xy+1)-y(x^2+xy+1)}\left(2x+y-2y-x\right)$$

$$= \frac{1}{\cancel{xy^2}+\cancel{x^2y}+x-\cancel{yx^2}-\cancel{xy^2}-y}\left(x-y\right)$$

$$= \frac{1}{\cancel{x-y}}\left(\cancel{x-y}\right)$$

$$= 1 = f(xy)$$

$$\therefore\quad F(x,y) = e^{\int f(u)\,du} = e^{\int du} = e^{u}$$

$$\therefore\quad // F(x,y) = e^{xy} // \qquad (u = xy)$$

$$e^{xy}(y^2+xy+1)\,dx + e^{xy}(x^2+xy+1)\,dy = 0$$

$$\left(y^2e^{xy}+xy\,e^{xy}+e^{xy}\right)dx + \left(e^{xy}x^2+e^{xy}xy+e^{xy}\right)dy =$$

19. Cont.

$$\frac{\partial M}{\partial x} = y^2 e^{xy} + xy\, e^{xy} + e^{xy}$$

$$M = \int \left(y^2 e^{xy} + xy\, e^{xy} + e^{xy} \right) dx + k(y)$$

$$= y^2 \frac{e^{xy}}{y} + \underbrace{y \int x e^{xy} dx}_{*} + \frac{e^{xy}}{y} + k(y)$$

$$* = \int x e^{xy} dx = \frac{x}{y} e^{xy} - \int \frac{e^{xy}}{y} dx$$

$u = x \rightarrow du = dx$

$dv = e^{xy} dx \rightarrow v = \dfrac{e^{xy}}{y}$

$$= \frac{x}{y} e^{xy} - \frac{1}{y^2} e^{xy}$$

$\therefore$

$$M = y e^{xy} + y\left(\frac{x}{y} e^{xy} - \frac{1}{y^2} e^{xy} \right) + \frac{e^{xy}}{y} + k(y)$$

$$M = y e^{xy} + x e^{xy} - \cancel{\frac{1}{y} e^{xy}} + \cancel{\frac{e^{xy}}{y}} + k(y)$$

$$M = x e^{xy} + y e^{xy} + k(y) \quad **$$

$$\frac{\partial M}{\partial y} = e^{xy} x^2 + e^{xy} xy + e^{xy}$$

$$** \quad \cancel{x^2 e^{xy}} + \cancel{e^{xy}} + \cancel{yx\, e^{xy}} + \frac{dk}{dy} = \cancel{x^2 e^{xy}} + \cancel{xy\, e^{xy}} + \cancel{e^{xy}}$$

$\therefore k = cte$

$(**) \to (**)$:

$$M(x,y) = xe^{x+y} + ye^{x+y} + C$$

$$\therefore \quad xe^{x+y} + ye^{x+y} = C_1$$

$$\boxed{e^{x+y}(x+y) = C_1}$$

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20. $$(2y^2 + 4x^2y)\,dx + (4xy + 3x^3)\,dy = 0$$

$F(x,y) = x^a y^b$ é fator integrante

$$x^a y^b(2y^2 + 4x^2y)\,dx + x^a y^b(4xy + 3x^3)\,dy = 0$$

$$(2y^{2+b}x^a + 4x^{2+a}y^{1+b})\,dx + (4x^{1+a}y^{1+b} + 3x^{3+a}y^b)\,dy =$$

$$\frac{\partial M}{\partial x} = 2y^{2+b}x^a + 4x^{2+a}y^{1+b}$$

$$\therefore \quad M = 2y^{2+b}\frac{x^{a+1}}{a+1} + y^{1+b}\,4\,\frac{x^{3+a}}{3+a} + k(y) \quad (*)$$

$$\frac{\partial M}{\partial y} = 4x^{1+a}y^{1+b} + 3x^{3+a}y^b$$

$$(*)\quad (4+2b)y^{1+b}\frac{x^{a+1}}{a+1} + (1+b)y^b\frac{4x^{3+a}}{3+a} + \frac{dk}{dy} = 4x^{1+a}y^{1+b} + 3x^{3+a}y^b$$

$$\left.\begin{aligned}&\frac{4+2b}{a+1}\, y^{1+b} x^{a+1} + \frac{4(1+b)}{3+a}\, y^{b} x^{3+a} + \frac{dk}{dy} = \\ &= 4x^{1+a} y^{1+b} + 3x^{3+a} y^{b}\end{aligned}\right] \quad (**)$$

$$\therefore \begin{cases} \dfrac{4+2b}{a+1} = 4 \Rightarrow 4+2b = 4a+4 & (1) \\[2ex] \dfrac{4(1+b)}{3+a} = 3 \Rightarrow 4+4b = 9+3a & (2) \end{cases}$$

$2\times(1) - (2)$:

$$\begin{aligned} 8+4b &= 8a+8 \\ 4+4b &= 9+3a \\ \hline 4 &= 5a-1 \Rightarrow 5a = 5 \end{aligned}$$

$$\underline{\underline{a=1}}$$

$$4+2b = 4a+4$$

$$\cancel{4}+2b = 4\cdot 1+\cancel{4} \Rightarrow \underline{\underline{b=2}}$$

Neste caso (**) nos dá : $\frac{dk}{dy} = 0 \Rightarrow k = \text{cte}$

e de (*) $u(x,y)$ fica

$$u(x,y) = \frac{2y^4x^2}{2} + y^3\frac{4x^4}{4} + C$$

$$u(x,y) = x^2y^4 + x^4y^3 + C \Longrightarrow$$

$$\boxed{x^2y^4 + x^4y^3 = C_1}$$