Cálculo C - Lista 13

Equações diferenciais lineares de segunda ordem

(I) Equação homogênea com coeficientes constantes: $y'' + ay + by = 0$

1. $y'' + 25y = 0$
2. $y'' - 4y = 0$
3. $4y'' + 36y' + 81y = 0$
4. $16y'' - 8y' + y = 0, \ \ y(1) = 0, \ y'(1) = -\sqrt[4]{e}$
5. $y'' - 2y' + (\pi^2 + 1)y = 0, \ \ y(1/4) = 0, \ y'(1/4) = -\pi\sqrt[4]{4e}$
6. $4y'' + 15y' - 4y = 0, \ \ y(0) = 6, \ y'(0) = -7$
7. $4y'' - 4y' - 3y = 0, \ \ y(-2) = e, \ y'(-2) = -e/2$

(II) Equação não-homogênea com coeficientes constantes: $y'' + ay' + by = r(x)$ (Método dos coeficientes a determinar)

8. $y'' + y = -x - x^2$
9. $y'' - y = e^x$
10. $y'' - 4y' + 3y = e^{3x}$
11. $y'' + y' - 2y = e^x$
12. $y'' + 25y = 5x, \ \ y(0) = 5, \ y'(0) = -4.8$
13. $y'' - 2y' + y = 2x^2 - 8x + 4, \ \ y(0) = 0.3, \ y'(0) = 0.3$

(III) Equação de Euler-Cauchy homogênea

14. $x^2y'' - 20y = 0$
15. $x^2y'' - 7xy' + 16y = 0$
16. $4x^2y'' + 12xy' + 3y = 0$
17. $x^2y'' - xy' - 3y = 0, \ \ y(2) = -1, \ y'(2) = 1/2$

(IV) Equação não-homogênea com coeficientes constantes: $y'' + ay' + by = r(x)$ (Método da variação dos parâmetros)

18. $y'' - 4y' + 4y = e^{2x}/x$
19. $y'' - 2y' + y = e^x \sin x$
20. $y'' + 2y' + y = e^{-x} \cos x$
21. $y'' - 2y' + y = x^{3/2}e^x$
22. $y'' + 4y' + 4y = e^{-2x}/x^2$
23. $y'' - 4y' + 4y = 6x^{-4}e^{2x}$
24. $y'' - 2y' + y = e^x/x^3$

(V) Uma aplicação do método da variação dos parâmetros a equações não homogêneas com coeficientes variáveis: equação de Euler-Cauchy não-homogênea de segunda ordem $x^2y'' + axy' + by = r(x)$

25. $x^2y'' + xy' - y = 4$
26. $x^2y'' - 2xy' + 2y = 1/x^2$
27. $x^2y'' - 2xy' + 2y = x^4$
28. $x^2y'' - xy' = 2x^3e^x$

Cálculo C - Lista 13

(I)

1. $y'' + 25y = 0$

$r^2 + 25 = 0 \Rightarrow r = \pm 5i$

$$\boxed{y_h = A\cos 5x + B\sin 5x}$$

2. $y'' - 4y = 0$

$r^2 - 4 = 0 \Rightarrow r = \pm 2$

$$\boxed{y_h = Ae^{2x} + Be^{-2x}}$$

3. $4y'' + 36y' + 81y = 0$

$4r^2 + 36r + 81 = 0$

$r = -36 \pm \sqrt{1296 - 1296} / 8$

$= \frac{-36}{8} = -\frac{18}{4} = -\frac{9}{2}$

$$\boxed{y_h = Ae^{-\frac{9}{2}x} + Bxe^{-\frac{9}{2}x}}$$

$$\begin{array}{r} 36 \\ \times 36 \\ \hline 216 \\ 108 \\ \hline 1296 \end{array} \qquad \begin{array}{r} 16 \\ \times 81 \\ \hline 16 \\ 128 \\ \hline 1296 \end{array}$$

4. $16y'' - 8y' + y = 0$; $y(1) = 0$
$y'(1) = -\sqrt[4]{e}$

$16\lambda^2 - 8\lambda + 1 = 0$

$\lambda = 8 \pm \sqrt{64 - 64} \,/\, 32$

$\lambda = \frac{8}{32} = \frac{1}{4}$

$\Rightarrow \quad y_H = A e^{\frac{1}{4}x} + Bx e^{\frac{1}{4}x}$

$y_H(1) = A e^{\frac{1}{4}} + B e^{\frac{1}{4}} = 0 \Rightarrow \underline{\underline{A = -B}}$

$y_H'(1) = \frac{A}{4} \cancel{e^{\frac{1}{4}}} + B \cancel{e^{\frac{1}{4}}} + \frac{B}{4} \cancel{e^{\frac{1}{4}}} = -\cancel{\sqrt[4]{e}}$

$\therefore \quad -\cancel{\frac{B}{4}} + B + \cancel{\frac{B}{4}} = -1$

$\underline{\underline{B = -1}} \quad \therefore \quad \underline{\underline{A = +1}}$

$\therefore \quad \boxed{y_H = +e^{\frac{1}{4}x} - x e^{\frac{1}{4}x}}$

5. $y'' - 2y' + (\pi^2+1)y = 0$; $y(\frac{1}{4}) = 0$

$y'(\frac{1}{4}) = -\pi\sqrt[4]{4e}$

$\lambda^2 - 2\lambda + (\pi^2+1) = 0$

$\lambda = 2 \pm \sqrt{4 - 4\pi^2 - 4}\ /2$

$\lambda = \dfrac{2 \pm \sqrt{-4\pi^2}}{2} = \dfrac{2 \pm 2i\pi}{2} = 1 \pm i\pi$

$y_h = e^x(A\cos\pi x + B\sin\pi x)$

$y_h(\frac{1}{4}) = e^{\frac{1}{4}}(A\cos\frac{\pi}{4} + B\sin\frac{\pi}{4}) = 0$

$\therefore \quad A\frac{\sqrt{2}}{2} + B\frac{\sqrt{2}}{2} = 0$

$\therefore \quad \underline{\underline{A = -B}}$

$y_h'(x) = e^x(A\cos\pi x + B\sin\pi x) +$
$+ e^x(-A\pi\sin\pi x + B\pi\cos\pi x)$

$y_h'(\frac{1}{4}) = \cancel{e^{\frac{1}{4}}}(A\frac{\sqrt{2}}{2} + B\frac{\sqrt{2}}{2}) + \cancel{e^{\frac{1}{4}}}(-A\pi\frac{\sqrt{2}}{2} + B\pi\frac{\sqrt{2}}{2})$

$= -\pi\sqrt[4]{4e} = -\pi\sqrt[4]{4}\,\cancel{e^{1/4}}$

$\therefore \quad A\frac{\sqrt{2}}{2} + B\frac{\sqrt{2}}{2} - A\pi\frac{\sqrt{2}}{2} + B\pi\frac{\sqrt{2}}{2} = -\pi\sqrt[4]{4}$

$\cancel{-B\frac{\sqrt{2}}{2}} + \cancel{B\frac{\sqrt{2}}{2}} \underbrace{-(-B)\pi\frac{\sqrt{2}}{2} + B\pi\frac{\sqrt{2}}{2}} = -\pi\sqrt[4]{4}$

$B\cancel{\pi}\sqrt{2} = -\cancel{\pi}\sqrt[4]{4}$

$$B\cancel{\sqrt{2}} = -\sqrt[4]{4} = -(2^2)^{1/4} = -\cancel{2^{1/2}}$$

$\therefore$ $\underline{\underline{B = -1}}$ $\therefore$ $\underline{\underline{A = -B = 1}}$

$$\boxed{y_h = e^x(\cos\pi x - \sin\pi x)}$$

6. $4y'' + 15y' - 4y = 0$; $y(0) = 6$, $y'(0) = -7$

$$4\lambda^2 + 15\lambda - 4 = 0$$

$$\lambda = -15 \pm \sqrt{225 + 64}\,/8$$

$$= -15 \pm \sqrt{289}\,/8$$

$$= \frac{-15 \pm 17}{8} \nearrow \frac{1}{4} \searrow -4$$

$$\begin{array}{r} 17 \\ 17 \\ \hline 119 \\ 17 \\ \hline 289 \end{array}$$

$$y_h = Ae^{\frac{1}{4}x} + Be^{-4x}$$

$y_h(0) = A + B = 6 \implies \underline{\underline{A = 6 - B}}$

$$y_h'(0) = \frac{A}{4} - 4B = -7$$

$$\frac{6-B}{4} - 4B = -7$$

$$6 - B - 16B = -28$$

$$-17B = -34$$

$$//B = 2//$$

$\therefore$ $A = 6 - 2$

$//A = 4//$

$$\therefore \boxed{y_h = 4e^{\frac{x}{4}} + 2e^{-4x}}$$

7. $4y'' - 4y' - 3y = 0$; $y(-2) = e$
$y'(-2) = -\frac{e}{2}$

$4\lambda^2 - 4\lambda - 3 = 0$

$\lambda = 4 \pm \sqrt{16 + 48} \,/ 8$

$= \frac{4 \pm \sqrt{64}}{8} = \frac{4 \pm 8}{8}$ $\nearrow \frac{12}{8} = \frac{3}{2}$ $\searrow -\frac{1}{2}$

$y_h = Ae^{\frac{3}{2}x} + Be^{-\frac{1}{2}x}$

$y_h(-2) = Ae^{-3} + Be = e \Rightarrow B = 1 - Ae^{-4}$

$y'_h(-2) = \frac{3}{2}Ae^{-3} - \frac{B}{2}e = -\frac{e}{2}$

$\therefore \frac{3}{2}Ae^{-3} - \frac{1}{2}(1 - Ae^{-4})e = -\frac{e}{2}$

$\frac{3}{2}Ae^{-3} - \cancel{\frac{e}{2}} + \frac{A}{2}e^{-3} = \cancel{-\frac{e}{2}}$

$2Ae^{-3} = 0 \Rightarrow \underline{\underline{A = 0}}$

$\therefore \underline{\underline{B = 1}}$

$\therefore \boxed{y_h = e^{-\frac{1}{2}x}}$

(II)

8. $y'' + y = -x - x^2$ (*)

y_h : $y'' + y = 0$; $\lambda^2 + 1 = 0$

$\lambda = \pm i$

$\therefore$ $y_h = C_1 \cos x + C_2 \sin x$.

y_p : $r(x) = \underline{-x - x^2}$

$-x \longrightarrow C_0 + C_1 x$

$-x^2 \longrightarrow C_2 + C_3 x + C_4 x^2$

$y_p = C_0 + C_1 x + C_2 + C_3 x + C_4 x^2$

$= (C_0 + C_2) + (C_1 + C_3) x + C_4 x^2$

$// y_p = A_0 + A_1 x + A_2 x^2 //$

(**) $\begin{cases} y'_p = A_1 + 2A_2 x \\ y''_p = 2A_2 \end{cases}$

$\therefore$ (**) $\rightarrow$ (*) :

$\underline{2A_2 + A_0} + A_1 x + A_2 x^2 = -x - x^2$

$\Longrightarrow$

$//A_2 = -1//$

$//A_1 = -1//$

$2A_2 + A_0 = 0 \Rightarrow A_0 = -2A_2 = -2(-1)$

$//A_0 = 2//$

$\therefore \quad //y_p = 2 - x - x^2//$

$$\boxed{y = y_h + y_p = C_1 \cos x + C_2 \sin x + 2 - x - x^2}$$

9. $y'' - y = e^x$

$\underline{y_h}$: $y'' - y = 0 \longrightarrow \lambda^2 - 1 = 0$

$\lambda = \pm 1$

$y_h = A e^x + B e^{-x}$

$\underline{y_p}$ $\quad r(x) = e^x \longleftrightarrow y_h = \underline{A e^x} + B e^{-x}$

$y_p = Cxe^x$

$y'_p = Ce^x + Cxe^x$

$y''_p = Ce^x + Ce^x + Cxe^x$

$= 2Ce^x + Cxe^x$

$y'' - y = e^x$

$\therefore \quad 2ce^x + \cancel{cxe^x} - \cancel{cxe^x} = e^x$

$2ce^x = e^x \Rightarrow c = \frac{1}{2}$

$\therefore \quad // y_p = \frac{1}{2} x e^x //$

$\therefore \quad \boxed{y = y_h + y_p = Ae^x + Be^{-x} + \frac{x}{2} e^x}$

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10. $y'' - 4y' + 3y = e^{3x}$ (*)

y_h : $y'' - 4y' + 3y = 0$

$\lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda = 3$, $\lambda = 1$

$y_h = Ae^x + Be^{3x}$

y_p : $r(x) = e^{3x} \longrightarrow y_h = Ae^x + \underline{Be^{3x}}$

$\therefore \quad y_p = cxe^{3x}$

(**) $\begin{cases} y_p' = ce^{3x} + 3cxe^{3x} \\ y_p'' = 3ce^{3x} + 3ce^{3x} + 9cxe^{3x} \\ \quad = 6ce^{3x} + 9cxe^{3x} \end{cases}$

(**) $\to$ (*)

$$6C e^{3x} + \cancel{9Cx e^{3x}} - 4C e^{3x} - \cancel{12Cx e^{3x}} + \cancel{3Cx e^{3x}} = e^{3x}$$

$$2Ce^{3x} = e^{3x} \implies C = \frac{1}{2}$$

$$\therefore \quad y_p = \frac{1}{2} x e^{3x}$$

$$\therefore \quad \boxed{y = y_h + y_p = Ae^{x} + Be^{3x} + \frac{x}{2} e^{3x}}$$

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11. $y'' + y' - 2y = e^{x}$

y_h : $\quad \lambda^2 + \lambda - 2 = 0 \implies \lambda_1 = -2, \ \lambda_2 = 1$

$$y_h = Ae^{x} + Be^{-2x}$$

y_p : $\quad r(x) = e^{x} \longrightarrow y_h = \underline{Ae^{x}} + Be^{-2x}$

$$y_p = Cxe^{x}$$

$$\begin{cases} y'_p = Ce^{x} + Cxe^{x} \\ y''_p = Ce^{x} + Ce^{x} + Cxe^{x} \\ \quad = 2Ce^{x} + Cxe^{x} \end{cases}$$

11. Cont.

$$2Ce^x + \cancel{Cxe^x} + Ce^x + \cancel{Cxe^x} - 2\cancel{Cxe^x} = e^x$$

$$3Ce^x = e^x \Rightarrow C = \frac{1}{3}$$

$$\therefore \quad // y_p = \frac{1}{3} x e^x //$$

$$\boxed{y = y_h + y_p = Ae^x + Be^{-2x} + \frac{x}{3} e^x}$$

12. $y'' + 25y = 5x \quad ; \quad y(0) = 5 \quad , \quad y'(0) = -4.8$

$\underline{y_h}$: $\quad \lambda^2 + 25 = 0 \Longrightarrow \lambda = \pm 5i$

$$y_h = A\cos 5x + B\sin 5x$$

$\underline{y_p}$: $\left\{ \begin{array}{l} r(x) = 5x \rightarrow y_p = C_0 + C_1 x \\ y'_p = C_1 \quad , \quad y''_p = 0 \end{array} \right.$

$$\therefore \quad 25(C_0 + C_1 x) = 5x$$

$$\Rightarrow \left\{ \begin{array}{l} C_0 = 0 \\ C_1 = \frac{1}{5} \end{array} \right. \quad \therefore // y_p = \frac{1}{5} x //$$

$$\therefore \quad \boxed{y = y_h + y_p = A\cos 5x + B\sin 5x + \frac{x}{5}} \longrightarrow$$

$y(0) = 5 \Rightarrow \underline{A = 5}$

$y'(0) = -4.8 \Rightarrow -5A\cancel{\sin}(5\cdot 0) + 5B\cos(5\cdot 0) + \frac{1}{5} = -4.8$

$5B = -4.8 - 0.2$

$\underline{5B} = -5 = \underline{-1}$

$\therefore \boxed{y = 5\cos 5x - 1 \sin 5x + \frac{8}{5}}$

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13. $y'' - 2y' + y = 2x^2 - 8x + 4$; $y(0) = 0.3$
$y'(0) = 0.3$

$\underline{y_h}$: $y'' - 2y' + y = 0$

$\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda = 1$ (repetida)

$y_h = Ae^x + Bxe^x$

$\underline{y_p}$: $r(x) = 2x^2 - 8x + 4$

$// \; y_p = C_2x^2 + C_1x + C_0 \; //$

$y'_p = 2C_2x + C_1$

$y''_p = 2C_2$

13. cont.

$$2C_2 - 2(2C_2x + C_1) + C_2x^2 + C_1x + C_0 = 2x^2 - 8x + 4$$

$$\therefore (2C_2 + C_0 - 2C_1) + (-4C_2 + C_1)x + C_2x^2 =$$
$$= 2x^2 - 8x + 4$$

$$\begin{cases} 2C_2 + C_0 - 2C_1 = 4 \\ -4C_2 + C_1 = -8 \\ \underline{\underline{C_2 = 2}} \end{cases}$$

$$-4C_2 + C_1 = -8 \implies C_1 = -8 + 4C_2$$
$$= -8 + 8$$
$$/\!/ C_1 = 0 /\!/$$

$$4 + C_0 = 4 \implies \underline{\underline{C_0 = 0}}$$

$$\therefore /\!/ y_p = 2x^2 /\!/$$

$$/\!/ y = y_h + y_p = Ae^x + Bxe^x + 2x^2 /\!/$$

$$y(0) = 0.3 \implies \underline{\underline{A = 0.3}}$$

$$y'(0) = 0.3 \implies A + B = 0.3 \implies \underline{\underline{B = 0}}$$

$$\therefore \boxed{y = 0.3e^x + 2x^2}$$

(II)

14. $x^2y'' - 20y = 0$

$y = x^m$

$x^2 m(m-1)x^{m-2} - 20x^m = 0$

$\therefore$

$m(m-1) - 20 = 0$

$m^2 - m - 20 = 0$

$m = 1 \pm \sqrt{1+80} \;/2$

$= \frac{1 \pm 9}{2} \nearrow 5 \quad \searrow -4$

$\therefore$ $\boxed{y = Ax^5 + Bx^{-4}}$

15. $x^2y'' - 7xy' + 16y = 0$

$x^2 m(m-1)x^{m-2} - 7xmx^{m-1} + 16x^m = 0$

$\therefore$

$m(m-1) - 7m + 16 = 0$

$m^2 - m - 7m + 16 = 0$

$m^2 - 8m + 16 = 0$

$m = 8 \pm \sqrt{64-64} \;/2$

$= \frac{8}{2} = 4$

$y_1 = x^4 \quad ; \quad y_2 = \ln x \, x^4$

$\therefore \boxed{y = Ax^4 + B(\ln x)\, x^4}$

16. $4x^2 y'' + 12x y' + 3y = 0$

$4m(m-1) + 12m + 3 = 0$

$4m^2 - 4m + 12m + 3 = 0$

$4m^2 + 8m + 3 = 0$

$m = -8 \pm \sqrt{64 - 48} \;/8$

$= -8 \pm \sqrt{16} \;/8$

$= \dfrac{-8 \pm 4}{8} \begin{array}{l} \nearrow -\frac{3}{2} \\ \searrow -\frac{1}{2} \end{array}$

$\boxed{y = Ax^{-\frac{1}{2}} + Bx^{-\frac{3}{2}}}$

17.

$$x^2y'' - xy' - 3y = 0 \quad ; \quad y(2) = -1 \, , \; y'(2) = \frac{1}{2}$$

$$m(m-1) - m - 3 = 0$$

$$m^2 - m - m - 3 = 0$$

$$m^2 - 2m - 3 = 0$$

$$m = 2 \pm \sqrt{4+12} \,/\, 2$$

$$= \frac{2 \pm 4}{2} \quad \begin{matrix} \nearrow 3 \\ \searrow -1 \end{matrix}$$

$$y(x) = Ax^3 + Bx^{-1} \quad ; \quad y'(x) = 3Ax^2 - \frac{B}{x^2}$$

$$y(2) = A\,8 + \frac{B}{2} = -1 \implies B = -2 - 16A$$

$$y'(2) = 3A \cdot 4 - \frac{B}{4} = \frac{1}{2}$$

$$\therefore \quad 12A - \frac{B}{4} = \frac{1}{2} \implies 12A - \frac{1}{4}(-2-16A) = \frac{1}{2}$$

$$12A + \frac{1}{2} + 4A = \frac{1}{2}$$

$$16A = 0 \implies A = 0$$

$$B = -2$$

$$\therefore \quad \boxed{y = \frac{-2}{x}}$$

(IV)

18. $y'' - 4y' + 4y = \dfrac{e^{2x}}{x}$

$\underline{y_h}$ $y'' - 4y' + 4y = 0$

$\lambda^2 - 4\lambda + 4 = 0 \rightarrow \lambda = \dfrac{+4 \pm \sqrt{16-16}}{2}$

$= 2$ raiz repetida

$$\boxed{y_h = c_1 e^{2x} + c_2 x e^{2x}}$$

$\underline{y_p}$: $\boxed{y_p = u e^{2x} + v x e^{2x}}$

$y_p' = u' e^{2x} + 2u e^{2x} + v' x e^{2x} + v(e^{2x} + 2x e^{2x})$

$y_p' = (u' e^{2x} + v' x e^{2x}) + 2u e^{2x} + v(e^{2x} + 2x e^{2x})$

$\rightarrow \| u' e^{2x} + v' x e^{2x} = 0 \|$ (*)

$y_p' = 2u e^{2x} + v(e^{2x} + 2x e^{2x})$

$y_p'' = 2u' e^{2x} + 4u e^{2x} + v'(e^{2x} + 2x e^{2x})$

$+ v(2e^{2x} + 2e^{2x} + 4x e^{2x})$

$\therefore$

$y_p'' = 2u' e^{2x} + v'(e^{2x} + 2x e^{2x}) +$

$$\Rightarrow \quad 2u'e^{2x} + v'(e^{2x} + 2xe^{2x}) = \frac{e^{2x}}{x}$$

Temos que resolver as eqs.:

$$(a) \quad u'e^{2x} + v'xe^{2x} = 0$$

$$(b) \quad 2u'e^{2x} + v'(e^{2x} + 2xe^{2x}) = \frac{e^{2x}}{x}$$

$$(a) \Rightarrow \quad u' = -v'x$$

$$(b): \quad -2v'x\cancel{e^{2x}} + v'(\cancel{e^{2x}} + 2x\cancel{e^{2x}}) = \frac{\cancel{e^{2x}}}{x}$$

$$-\cancel{2x}v' + v' + \cancel{2x}v' = \frac{1}{x}$$

$$\| v' = \frac{1}{x} \| \rightarrow v = \ln x$$

$$\| u' = -1 \| \rightarrow u = -x$$

$$\therefore \quad y_p = ue^{2x} + vxe^{2x}$$

$$\boxed{y_p = -xe^{2x} + x\ln x\, e^{2x}}$$

$$\boxed{y = y_H + y_p = C_1e^{2x} + C_2xe^{2x} - xe^{2x} + x\ln x\, e^{2x}}$$

19. $y'' - 2y' + y = e^x \sin x$

$\underline{y_h}$: $y'' - 2y' + y = 0$

$\lambda^2 - 2\lambda + 1 = 0$

$\lambda = 2 \pm \sqrt{4-4} / 2$

$\underline{\underline{\lambda = 1}}$

$y_h = A e^x + B x e^x$

$\underline{y_p}$: $y_p = u e^x + v x e^x$

$\rightarrow y'_p = u' e^x + u e^x + v' x e^x + v (x e^x)'$

$= \underbrace{u' e^x + v' x e^x} + u e^x + v e^x + v x e^x$

$// \; u' e^x + v' x e^x = 0 \; //$ (imponiamo)

$y'_p = u e^x + v e^x + v x e^x$

$y''_p = u' e^x + u e^x + v' e^x + v e^x + v' x e^x +$
$+ v e^x + v x e^x$

$y''_p = \underbrace{u' e^x + v' (1 + x) e^x} + 2 v e^x + v x e^x$

$$u'e^x + v'(1+x)e^x = r(x) = e^x \sin x$$

Temos as duas eqs.:

$$\begin{cases} u'\cancel{e^x} + v'x\,\cancel{e^x} = 0 \Rightarrow \underline{\underline{u' = -xv'}} \\ u'\cancel{e^x} + v'(1+x)\cancel{e^x} = \cancel{e^x}\sin x \end{cases}$$

$$u' + v'(1+x) = \sin x$$

$$-\cancel{xv'} + v'(1+\cancel{x}) = \sin x$$

$$\underline{\underline{v' = \sin x}} \Longrightarrow \underline{\underline{v = -\cos x}}$$

$$\underline{\underline{u' = -x\sin x}} \Longrightarrow u = \int -x\sin x\, dx$$

$$u = -\int x\sin x\, dx = x\cos x - \int \cos x\, dx$$

$$= x\cos x - \sin x$$

$f = x \to df = dx$

$dg = -\sin x\, dx \to g = \cos x$

$$\therefore \underline{\underline{u = x\cos x - \sin x}}$$

$$y_p = (\cancel{x\cos x} - \sin x)e^x - \cos x\,\cancel{x}e^x = \underline{\underline{-\sin x\, e^x}}$$

$$\boxed{y = y_h + y_p = Ae^x + Bxe^x - \sin x\, e^x}$$

20. $y'' + 2y' + y = e^{-x}\cos x$

$\underline{\underline{y_h}}$: $r^2 + 2r + 1 = 0$

$r = -2 \pm \sqrt{4-4} \,/\, 2 = -1$

$y_h = Ae^{-x} + Bxe^{-x}$

$\underline{\underline{y_p}}$: $y_p = ue^{-x} + vxe^{-x}$

$y_p' = u'e^{-x} - ue^{-x} + v'xe^{-x} + ve^{-x} - vxe^{-x}$

$= \underbrace{u'e^{-x} + v'xe^{-x}} - ue^{-x} + ve^{-x} - vxe^{-x}$

$// \; u'\cancel{e^{-x}} + v'x\cancel{e^{-x}} = 0 \; //$

$\therefore$

$y_p' = -ue^{-x} + ve^{-x} - vxe^{-x}$

$y_p'' = -u'e^{-x} + ue^{-x} + v'e^{-x} - ve^{-x}$
$\quad - v'xe^{-x} - ve^{-x} + vxe^{-x}$

$= \underbrace{-u'e^{-x} + v'(1-x)e^{-x}} + ue^{-x} - 2ve^{-x} + vxe^{-x}$

$\therefore$ $-u'\cancel{e^{-x}} + v'(1-x)\cancel{e^{-x}} = \cancel{e^{-x}}\cos x$

$$\begin{cases} u' + x v' = 0 \\ -u' + (1-x) v' = \cos x \end{cases} \longrightarrow u' = -x v'$$

$$\therefore \quad \cancel{x v'} + (1 - \cancel{x}) v' = \cos x$$

$$v' = \cos x \implies v = \sin x$$

$$u' = -x v' = -x \cos x$$

$$\therefore \quad u = \int -x \cos x \, dx = -x \sin x + \int \sin x \, dx$$

$$f = x \to df = dx$$

$$dg = -\cos x \, dx \to g = -\sin x$$

$$= -x \sin x - \cos x$$

$$\therefore \quad u = -x \sin x - \cos x$$

$$y_p = u e^{-x} + v x e^{-x}$$

$$= (\cancel{-x \sin x} - \cos x) e^{-x} + \cancel{\sin x \, x} e^{-x}$$

$$y_p = -\cos x \, e^{-x}$$

$$y = y_h + y_p = A e^{-x} + B x e^{-x} - \cos x \, e^{-x}$$

$$\boxed{y = (A + Bx - \cos x) e^{-x}}$$

21. $y'' - 2y' + y = x^{3/2} e^{x}$

$\underline{y_h}$: $\lambda^2 - 2\lambda + 1 = 0$

$\lambda = 1$ (repetida)

$y_h = Ae^{x} + Bxe^{x}$

$\underline{y_p}$: $y_p = ue^{x} + vxe^{x}$

$y'_p = u'e^{x} + ue^{x} + v'xe^{x} + ve^{x} + vxe^{x}$

$= \underbrace{u'e^{x} + v'xe^{x}} + ue^{x} + ve^{x} + vxe^{x}$

$\therefore$ // $u'\cancel{e^{x}} + v'x\cancel{e^{x}} = 0$ //

$y'_p = ue^{x} + ve^{x} + vxe^{x}$

$y''_p = u'e^{x} + ue^{x} + v'e^{x} + ve^{x} + v'xe^{x}$
$+ ve^{x} + vxe^{x}$

$= \underbrace{u'e^{x} + v'(1+x)e^{x}} + ue^{x} + 2ve^{x} + vxe^{x}$

$\therefore$ $u'\cancel{e^{x}} + v'(1+x)\cancel{e^{x}} = x^{3/2}\cancel{e^{x}}$

// $u' + v'(1+x) = x^{3/2}$ /

$$u' + xv' = 0 \longrightarrow u' = -xv'$$

$$u' + (1+x)v' = x^{3/2}$$

$$\therefore \cancel{-xv'} + (1+\cancel{x})v' = x^{3/2}$$

$$v' = x^{3/2} \longrightarrow // v = \frac{2}{5}x^{5/2} //$$

$$u' = -xv' = -x\,x^{3/2} = -x^{5/2}$$

$$\therefore // u = -\frac{2}{7}x^{7/2} //$$

$$y_p: \; ue^x + vxe^x$$

$$= -\frac{2}{7}x^{7/2}e^x + \frac{2}{5}x^{5/2}x\,e^x$$

$$// y_p = -\frac{2}{7}x^{7/2}e^x + \frac{2}{5}x^{7/2}e^x = \frac{4}{35}x^{7/2}e^x //$$

$$y = y_h + y_p = Ae^x + Bxe^x + \frac{4}{35}x^{7/2}e^x$$

$$\boxed{y = \left(A + Bx + \frac{4}{35}x^{7/2}\right)e^x}$$

22. $y'' + 4y' + 4y = \frac{e^{-2x}}{x^2}$

$\underline{y_h}$: $r^2 + 4r + 4 = 0 \longrightarrow r = -2$ (repetida)

$y_h = A e^{-2x} + B x e^{-2x}$

$\underline{y_p}$: $y_p = u e^{-2x} + v x e^{-2x}$

$y_p' = u' e^{-2x} - 2u e^{-2x} + v' x e^{-2x} +$
$+ v e^{-2x} - 2v x e^{-2x}$

$= \underbrace{u' e^{-2x} + v' x e^{-2x}} - 2u e^{-2x} + v e^{-2x} - 2v x e^{-2x}$

$\Longrightarrow \; // u' \cancel{e^{-2x}} + v' x \cancel{e^{-2x}} = 0 //$

$y_p' = -2u e^{-2x} + v e^{-2x} - 2v x e^{-2x}$

$y_p'' = -2u' e^{-2x} + 4u e^{-2x} + v' e^{-2x} - 2v e^{-2x}$
$- 2v' x e^{-2x} - 2v e^{-2x} + 4v x e^{-2x}$

$= \underline{-2u' e^{-2x} + v' e^{-2x}} - 2v' x e^{-2x} +$
$+ 4u e^{-2x} - 4v e^{-2x} + 4v x e^{-2x}$

$\therefore \; // -2u' \cancel{e^{-2x}} + v' \cancel{e^{-2x}} - 2v' x e^{-2x} = \frac{e^{-2x}}{x^2}$

$u' + v'x = 0 \implies u' = -xv'$

$-2u' + v' - 2v'x = \frac{1}{x^2}$

$\cancel{2xv'} + v' - \cancel{2v'x} = \frac{1}{x^2}$

$$v' = \frac{1}{x^2} \longrightarrow \underline{\underline{v = -\frac{1}{x}}}$$

$$\therefore \quad u' = -x\,v' = -x\frac{1}{x^2}$$

$$u' = -\frac{1}{x} \longrightarrow \underline{\underline{u = -\ln|x|}}$$

$y_p = u e^{-2x} + \underline{v x} e^{-2x}$

$\| \; y_p = -\ln|x|\, e^{-2x} \underline{-} e^{-2x} \; \|$

$$\therefore \quad y = y_h + y_p$$
$$= Ae^{-2x} + Bxe^{-2x} - \ln|x|\, e^{-2x} - e^{-2x}$$

$$= \left((A+1) + Bx - \ln|x| \right) e^{-2x}$$

$$\boxed{y = \left(\tilde{A} + Bx - \ln|x| \right) e^{-2x}}$$

23. $y'' - 4y' + 4y = 6x^{-4} e^{2x}$

$\underline{y_h}$: $y'' - 4y' + 4y = 0$

$\lambda^2 - 4\lambda + 4 = 0 \implies \lambda = +2$ (dupla)

$y_h = A e^{+2x} + B x e^{+2x}$

$\underline{y_p}$: $y_p = u e^{+2x} + v x e^{+2x}$

$y_p' = u' e^{+2x} + 2u e^{+2x} + v' x e^{+2x} +$
$+ v e^{+2x} + 2 v x e^{+2x}$

$= u' e^{+2x} + v' x e^{+2x} + 2u e^{+2x} + v e^{+2x}$
$+ 2 v x e^{+2x}$

$\Rightarrow // \; u' e^{+2x} + v' x e^{+2x} = 0 \; //$

$y_p' = +2u e^{+2x} + v e^{+2x} + 2 v x e^{+2x}$

$y_p'' = \underline{+2u' e^{+2x}} + 4u e^{+2x} \underline{+ v' e^{+2x}}$
$\underline{\underline{+ 2v e^{+2x}}} \underline{+ 2v' x e^{+2x}} + \underline{\underline{2v e^{+2x}}} + 4 v x e^{+2x}$

$= \underbrace{+2u' e^{+2x} + v' e^{+2x} + 2v' x e^{+2x}}$

$+ 4u e^{+2x} \quad 4v e^{+2x} + 4 v x e^{+2x}$

$$+2u' \cancel{e^{+2x}} + v' \cancel{e^{+2x}} + 2v'x \cancel{e^{+2x}} = 6x^{-4} \cancel{e^{2x}}$$

$$// +2u' + v' + 2v'x = 6x^{-4} //$$

$$\begin{cases} u' + v'x = 0 \quad \rightarrow \quad u' = -v'x \\ +2u' + v' + 2v'x = 6x^{-4} \end{cases}$$

$$+2(-v'x) + v' + 2v'x = 6x^{-4}$$

$$\cancel{-2v'x} + v' + \cancel{2v'x} = 6x^{-4}$$

$$v' = 6x^{-4} \Rightarrow v = \frac{6}{-3} x^{-3}$$

$$// v = -2x^{-3} //$$

$$u' = -v'x = -6x^{-4}x$$

$$u' = -6x^{-3} \Rightarrow u = -\frac{6}{-2} x^{-2}$$

$$\underline{\underline{u = 3x^{-2}}}$$

$$// y_p = ue^{2x} + vxe^{2x} = 3x^{-2}e^{2x} + (-2)x^{-3}xe^{2x}$$

$$= (3x^{-2} - 2x^{-2})e^{2x} = x^{-2}e^{2x} //$$

$$\boxed{y = y_h + y_p = (A + Bx + x^{-2})\, e^{2x}}$$

24. $y'' - 2y' + y = e^x / x^3$

$\underline{y_h}$: $\lambda^2 - 2\lambda + 1 = 0 \rightarrow \lambda = +1$

$y_h = Ae^x + Bxe^x$

$\underline{y_p}$: $y_p = ue^x + vxe^x$

$y'_p = \underline{u'e^x} + ue^x + \underline{v'xe^x} + ve^x + vxe^x$

$= \underbrace{(u' + v'x)e^x} + ue^x + ve^x + vxe^x$

$// u' + v'x = 0 //$

$y'_p = ue^x + ve^x + vxe^x$

$y''_p = u'e^x + ue^x + v'e^x + ve^x + v'xe^x$
$+ ve^x + vxe^x$

$= \underbrace{u'e^x + v'e^x + v'x e^x} + ue^x + 2ve^x$
$+ vxe^x$

$\therefore$ $u'e^x + v'e^x + v'xe^x = e^x / x^3$

$\therefore$ $// u' + v' + v'x = \frac{1}{x^3} //$

$$\begin{cases} u' + v'x = 0 \\ u' + v' + v'x = \dfrac{1}{x^3} \end{cases} \longrightarrow u' = -v'x$$

$$\cancel{-v'x} + v' + \cancel{v'x} = \frac{1}{x^3} \Rightarrow v' = \frac{1}{x^3}$$

$$\therefore \; v = -\frac{x^{-2}}{2}$$

$$u' = -v'x$$

$$= -\left(\frac{1}{x^3}\right)x = -x^{-2}$$

$$\therefore \; u = +x^{-1}$$

$$\therefore \; y_p = ue^x + vxe^x$$

$$= \frac{1}{x}e^x - \frac{1}{2x^{\cancel{2}}}\cancel{x}\, e^x$$

$$= \frac{1}{x}e^x - \frac{1}{2x}e^x = \frac{1}{2x}e^x$$

$$\therefore \; y = y_h + y_p = Ae^x + Bxe^x + \frac{1}{2x}e^x$$

$$\boxed{y = \left(A + Bx + \frac{1}{2x}\right)e^x}$$

(V)

25. $x^2y'' + xy' - y = 4$

<u>Eq. homogénea</u>

$$x^2y'' + xy' - y = 0$$

$y = x^m$: $\quad m(m-1) + m - 1 = 0$

$$m^2 - \cancel{m} + \cancel{m} - 1 = 0$$

$$m^2 = 1 \Rightarrow \underline{\underline{m = \pm 1}}$$

$$y_h = Ax + Bx^{-1}$$

<u>Eq. não homogénea</u>

$$x^2y'' + xy' - y = 4$$

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = \frac{4}{x^2}$$

$$y_p = ux + vx^{-1}$$

$$y_p' = u'x + u + v'x^{-1} - \frac{v}{x^2}$$

$$= u'x + v'x^{-1} + u - \frac{v}{x^2}$$

$\therefore$ $//u'x + v'x^{-1} = 0//$ $\quad \therefore y_p' = u - \frac{v}{x^2} \Rightarrow$

$x^2 \qquad x^3$

$$\left\| u' - \frac{v'}{x^2} = \frac{4}{x^2} \right\|$$

$$\therefore \begin{cases} u'x + v'x^{-1} = 0 \longrightarrow u' = -\dfrac{v'}{x^2} \\ u' - \dfrac{v'}{x^2} = \dfrac{4}{x^2} \end{cases}$$

$$\downarrow$$

$$-\frac{v'}{x^2} - \frac{v'}{x^2} = \frac{4}{x^2}$$

$$-\frac{2v'}{\cancel{x^2}} = \frac{4}{\cancel{x^2}} \Rightarrow v' = -2$$

$$\underline{\underline{v = -2x}}$$

$$u' = \frac{-v'}{x^2} = \frac{+2}{x^2} \longrightarrow \underline{\underline{u = -\frac{2}{x}}}$$

$$y_p = -\frac{2}{x}x \quad -2x \quad x^{-1}$$

$$= -2 - 2 = -4 \quad \therefore \quad \underline{\underline{y_p = -4}}$$

$$\boxed{y = y_h + y_p = Ax + Bx^{-1} - 4}$$

26. $x^2y'' - 2xy' + 2y = 1/x^2$

$\underline{y_h}$ $\quad x^2y'' - 2xy' + 2y = 0$

$m(m-1) - 2m + 2 = 0$

$m^2 - m - 2m + 2 = 0$

$m^2 - 3m + 2 = 0 \implies \underline{\underline{m = 1, \ m = 2}}$

$y_h = Ax + Bx^2$

$\underline{y_p}$: $\quad x^2y'' - 2xy' + 2y = \frac{1}{x^2}$

$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = \frac{1}{x^4}$

$y_p = ux + vx^2$

$y_p' = u'x + u + v'x^2 + v2x$

$= u'x + v'x^2 + u + 2vx$

$\implies // u'x + v'x^2 = 0 //$

$y_p' = u + 2vx$

$y_p'' = \underbrace{u' + 2v'x} + 2v$

$\therefore \ // u' + 2v'x = \frac{1}{x^4} //$

$$\begin{cases} u'x + v'x^2 = 0 \Rightarrow u' = -xv' \\ u' + 2v'x = \dfrac{1}{x^4} \end{cases}$$

$$\downarrow$$

$$-xv' + 2v'x = \frac{1}{x^4}$$

$$+v'x = \frac{1}{x^4} \rightarrow v' = \frac{1}{x^5}$$

$$v = -\frac{1}{4}x^{-4}$$

$$u' = -x\underbrace{v'} = -x\left(\frac{1}{x^5}\right)$$

$$= -x^{-4}$$

$$\therefore \quad u = +\frac{1}{3}x^{-3}$$

y_p $$= ux + vx^2 = \frac{1}{3}x^{-2} + -\frac{1}{4}x^{-4}x^2$$

$$= \frac{1}{3}x^{-2} - \frac{1}{4}x^{-2}$$

$$= +\frac{1}{12}x^{-2}$$

$$\therefore \quad \boxed{y = y_h + y_p = Ax + Bx^2 + \frac{1}{12}x^{-2}}$$

27. $x^2y'' - 2xy' + 2y = x^4$

$\underline{y_h}$: $x^2y'' - 2xy' + 2y = 0$

$m(m-1) - 2m + 2 = 0$

$m^2 - m - 2m + 2 = 0$

$m^2 - 3m + 2 = 0 \implies m = 2,\ m = 1$

$y_h = Ax + Bx^2$

$\underline{y_p}$: $x^2y'' - 2xy' + 2y = x^4$

$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = x^2$

$y_p = ux + vx^2$

$y_p' = u'x + u + v'x^2 + 2vx$

$= u'x + v'x^2 + u + 2vx$

$\implies // u'x + v'x^2 = 0 //$

$\therefore\ y_p' = u + 2vx$

$y_p'' = u' + 2v'x + 2v$

$\therefore\ // u' + 2v'x = x^2 //$

$$\begin{cases} u'x + v'x^2 = 0 \\ u' + 2v'x = x^2 \end{cases} \qquad \rightarrow \quad \underline{\underline{u' = -xv'}}$$

$$\downarrow$$

$$-xv' + 2v'x = x^2$$

$$v'x = x^2 \Longrightarrow v' = x$$

$$\therefore \quad \underline{\underline{v = \frac{x^2}{2}}}$$

$$u' = -xv' = -x\,x = -x^2$$

$$\therefore \quad \underline{\underline{u = -\frac{x^3}{3}}}$$

$$y_p = ux + vx^2$$

$$= -\frac{1}{3}x^4 + \frac{x^4}{2} =$$

$$= \frac{x^4}{6}$$

$$\therefore \quad \boxed{y = y_h + y_p = Ax + Bx^2 + \frac{x^4}{6}}$$

28. $x^2y'' - xy' = 2x^3e^x$

$\underline{y_h}$: $x^2y'' - xy' = 0$

$m(m-1) - m = 0$

$m^2 - m - m = 0$

$m^2 - 2m = 0$

$m(m-2) = 0 \implies m = 0,\ \underline{\underline{m = 2}}$

$y_h = Ax^0 + Bx = \underline{\underline{A + Bx^2}}$

$\underline{\underline{y_p}}$: $x^2y'' - xy' = 2x^3e^x$

$\longrightarrow y'' - \frac{1}{x}y' = 2xe^x$

$y_p = u \cdot 1 + vx^2$

$y_p' = u' + v'x^2 + 2vx$

$\implies // u' + v'x^2 = 0 //$

$\therefore y_p' = 2vx$

$y_p'' = 2v'x + 2v$

$\cancel{2}v'x = \cancel{2}xe^x$

$v' = e^x \implies // v = e^x$

$$u' + \underbrace{v'}x^2 = 0$$

$$u' + e^x x^2 = 0$$

$$u' = -x^2 e^x$$

$$u = -\int x^2 e^x \, dx = -\left[x^2 e^x - 2\int x e^x dx \right]$$

①

$$f = x^2 \rightarrow df = 2x\,dx$$

$$dg = e^x dx \rightarrow g = e^x$$

$$① = \int x e^x dx = x e^x - \int e^x dx = x e^x - e^x$$

$$f = x \rightarrow df = dx$$

$$dg = e^x dx \rightarrow g = e^x$$

$$\therefore \quad u = -\left[x^2 e^x - 2(x e^x - e^x) \right]$$

$$u = -\left[x^2 e^x - 2x e^x + 2e^x \right]$$

$$y_p = u \cdot 1 + v x^2 = \cancel{-x^2 e^x} + 2x e^x - 2e^x + \cancel{e^x x^2}$$

$$= 2x e^x - 2e^x$$

$$\therefore \quad y = y_h + y_p$$

$$\boxed{y = A + Bx^2 + 2e^x(x-1)}$$

Respostas - Lista 13

1. $y_h = A\cos 5x + B\sin 5x$

2. $y_h = Ae^{2x} + Be^{-2x}$

3. $y_h = Ae^{-\frac{9}{2}x} + Bxe^{-\frac{9}{2}x}$

4. $y_h = e^{\frac{x}{4}} - xe^{\frac{x}{4}}$

5. $y_h = e^{x}(\cos \pi x - \sin \pi x)$

6. $y_h = 4e^{\frac{x}{4}} + 2e^{-4x}$

7. $y_h = e^{-\frac{x}{2}}$

8. $y = C_1\cos x + C_2\sin x + 2 - x - x^2$

9. $y = Ae^{x} + Be^{-x} + \frac{x}{2}e^{x}$

10. $y = Ae^{x} + Be^{3x} + \frac{x}{2}e^{3x}$

11. $y = Ae^{x} + Be^{-2x} + \frac{x}{3}e^{x}$

12. $y = 5\cos 5x - 1\sin 5x + \frac{x}{5}$

13. $y = 0.3e^{x} + 2x^2$

14. $y = Ax^5 + Bx^{-4}$

15. $y = Ax^4 + B(\ln x)x^4$

16. $y = Ax^{-1/2} + Bx^{-3/2}$

17. $y = \frac{2}{x}$

18. $y = C_1e^{2x} + C_2xe^{2x} \quad xe^{2x} + x\ln x\, e^{2x}$

19. $y = Ae^x + Bxe^x - \sin x \, e^x$

20. $y = (A + Bx - \cos x)\, e^{-x}$

21. $y = \left(A + Bx + \frac{4}{35} x^{7/2}\right) e^x$

22. $y = (A + Bx - \ln|x|)\, e^{-2x}$

23. $y = \left(A + Bx + \frac{1}{x^2}\right) e^{2x}$

24. $y = \left(A + Bx + \frac{1}{2x}\right) e^x$

25. $y = Ax + \frac{B}{x} - 4$

26. $y = Ax + Bx^2 + \frac{1}{12x^2}$

27. $y = Ax + Bx^2 + \frac{x^4}{6}$

28. $y = A + Bx^2 + 2e^x (x-1)$