Cálculo C - Lista 17

Transformada de Laplace (III): Resolvendo EDO's usando a transformada de Laplace

Resolva o problema de valor inicial usando a transformada de Laplace inversa

1. $y'' + y' - 2y = 0$; $y(0) = 0$, $y'(0) = 2$
2. $y'' - 16y = e^{-2t}$; $y(0) = 1$, $y'(0) = 0$
3. $y'' + 2y' - 3y = te^{t}$; $y(0) = 0$, $y'(0) = 1$
4. $y' = y$; $y(0) = 1$
5. $y'' + 3y' + 2y = \cos 2t$; $y(0) = -1$, $y'(0) = 1$
6. $y'' + a^2 y = 0$; $y(0) = 1$, $y'(0) = 0$
7. $y''' - y' = e^{3t}$; $y(0) = 0$, $y'(0) = 5$, $y''(0) = -2$
8. $y'' + 3y' - 4\int_0^t y\,du = 0$; $y(0) = 2$, $y'(0) = 1$
9. $y'' + 9y = 0$; $y(0) = 0$; $y'(0) = 2$
10. $y'' + \omega^2 y = 0$; $y(0) = A$, $y'(0) = B$, $(\omega \neq 0)$
11. $y'' - 2y' - 3y = 0$; $y(0) = 1$, $y'(0) = 7$
12. $4y'' + y = 0$; $y(0) = 1$, $y'(0) = -2$
13. $y'' - 4y' + 4y = 0$; $y(0) = 0$, $y'(0) = 2$
14. $y'' + 4y' + 4y = 0$; $y(0) = 2$, $y'(0) = -3$
15. $y'' + y' + 1.25y = 0$; $y(0) = 1$, $y'(0) = -0.5$

Lista 17 — Respostas

1. $\frac{2}{3}e^{t} - \frac{2}{3}e^{-2t}$

2. $\frac{25}{48}e^{4t} + \frac{9}{16}e^{-4t} - \frac{1}{12}e^{-2t}$

3. $-\frac{17}{64}e^{-3t} + \frac{17}{64}e^{t} + \left(\frac{1}{8}t^2 - \frac{1}{16}t\right)e^{t}$

4. e^{t}

5. $-\frac{6}{5}e^{-t} + \frac{1}{4}e^{-2t} + \frac{3}{20}\sin 2t - \frac{1}{20}\cos 2t$ ✓

6. $\cos at$

7. $-\frac{19}{8}\cosh t + \frac{39}{8}\sinh t + \frac{7}{3} + \frac{1}{24}e^{3t}$

8. $e^{t} + e^{-2t} + 2t\,e^{-2t}$

9. $\frac{2}{3}\sin 3t$

10. $A\cos\omega t + \frac{B}{\omega}\sin\omega t$

11. $-e^{-t} + 2e^{3t}$

12. $\cos\frac{t}{2} - 4\sin\frac{t}{2}$

13. $2t\,e^{2t}$

14. $(t+2)\,e^{-2t}$

15. $e^{-\frac{t}{2}}\cos t$

Lista 17

1. $y'' + y' - 2y = 0$; $y(0) = 0$, $y'(0) = 2$

$\mathcal{L}(y'') + \mathcal{L}(y') - 2\mathcal{L}(y) = 0$

$s^2\mathcal{L}(y) - s\,y(0) - y'(0) + s\,\mathcal{L}(y) - y(0) - 2\mathcal{L}(y) = 0$

$\mathcal{L}(y)(s^2 + s - 2) - (s+1)\underbrace{y(0)}_{0} - y'(0) = 0$

$\mathcal{L}(y)(s^2 + s - 2) - 2 = 0$

$$\mathcal{L}(y) = \frac{2}{s^2+s-2} = \frac{2}{(s+2)(s-1)}$$

$$\rightarrow y = \mathcal{L}^{-1}\left(\frac{2}{(s+2)(s-1)}\right)$$

$$\frac{2}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$$

$$\frac{2}{(s+2)(s-1)} = \frac{A(s-1) + B(s+2)}{(s+2)(s-1)}$$

$\therefore$ $A(s-1) + B(s+2) = 2$

$(A+B)s - A + 2B = 2$

$\therefore$ $A+B = 0 \Rightarrow A = -B$

$-A + 2B = 2$ $\therefore$ $3B = 2$

$//B = 2/3//$ $\therefore$ $//A = -2/3//$

$$\frac{\quad}{(s+2)(s-1)} = \frac{\quad}{3(s+2)} \quad \frac{\quad}{3(s-1)}$$

$$\mathcal{L}^{-1}\left(\frac{2}{(s+2)(s-1)}\right) = -\frac{2}{3}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + \frac{2}{3}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)$$

$$= \boxed{-\frac{2}{3}e^{-2t} + \frac{2}{3}e^{+t}}$$

---

2. $y'' - 16y = e^{-2t}$; $y(0) = 1$, $y'(0) = 0$

$$\mathcal{L}(y'') - 16\mathcal{L}(y) = \mathcal{L}(e^{-2t})$$

$$s^2\mathcal{L}(y) - s\,y(0) - \cancel{y'(0)}^{\,0} - 16\mathcal{L}(y) = \frac{1}{s+2}$$

$$s^2\mathcal{L}(y) - s - 16\mathcal{L}(y) = \frac{1}{s+2}$$

$$\mathcal{L}(y)(s^2 - 16) = s + \frac{1}{s+2}$$

$$\mathcal{L}(y) = \frac{s^2+2s+1}{(s^2-16)(s+2)}$$

$$\frac{s^2+2s+1}{\underbrace{(s^2-16)}_{(s-4)(s+4)}(s+2)} = \frac{A}{s-4} + \frac{B}{s+4} + \frac{C}{s+2}$$

$$\frac{s^2+2s+1}{(s^2-16)(s+2)} = \frac{A(s+4)(s+2) + B(s-4)(s+2) + C(s^2-16)}{(s^2-16)(s+2)}$$

$\therefore$

$$s^2+2s+1 = A(s^2+6s+8) + B(s^2-2s-8) +$$
$$+ C(s^2-16)$$

$$s^2+2s+1 \equiv s^2(A+B+C) + s(6A-2B) + 8A-8B-16C$$

$$\Longrightarrow \begin{cases} A+B+C=1 \\ 6A-2B=2 \quad \Longrightarrow \quad \underline{B=3A-1} \\ 8A-8B-16C=1 \end{cases}$$

$\therefore$ $A+3A-1+C=1$

$$\| 4A+C=2 \|$$

$$8A-8(3A-1)-16C=1$$

$$-16A+8-16C=1$$

$$-16A-16C=-7$$

$$\| +A+C=\frac{7}{16} \| \Longrightarrow A=\frac{7}{16}-C$$

$\therefore$ $4\left(\frac{7}{16}-C\right)+C=2$

$$\frac{7}{4}-4C+C=2$$

$$-3C=2-\frac{7}{4}$$

$$-3C=\frac{1}{4} \Longrightarrow \boxed{C=-\frac{1}{12}}$$

$$A = \frac{7}{16} - C = \frac{7}{16} + \frac{1}{12} = \frac{1}{4}\left(\frac{7}{4} + \frac{1}{3}\right)$$

$$= \frac{1}{4}\left(\frac{21+4}{12}\right) = \frac{25}{48}$$

$$\therefore \boxed{A = \frac{25}{48}}$$

$$B = 3A - 1 = 3 \times \frac{25}{48} - 1 = \frac{25}{16} - 1$$

$$= \frac{25-16}{16}$$

$$\boxed{B = \frac{9}{16}}$$

$$\therefore \frac{s^2 + 0s + 1}{(s^2-16)(s+2)} = \frac{25}{48}\frac{1}{(s-4)} + \frac{9}{16}\frac{1}{s+4} - \frac{1}{12}\frac{1}{s+2}$$

$$\therefore y = \mathcal{L}^{-1}\left(\frac{s^2+0s+1}{(s^2-16)(s+2)}\right) =$$

$$= \frac{25}{48}\mathcal{L}^{-1}\left(\frac{1}{s-4}\right) + \frac{9}{16}\mathcal{L}^{-1}\left(\frac{1}{s+4}\right) - \frac{1}{12}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)$$

$$= \boxed{\frac{25}{48}e^{4t} + \frac{9}{16}e^{-4t} - \frac{1}{12}e^{-2t}}$$

3. $y'' + 2y' - 3y = te^t \; ; \; y(0) = 0 \; , \; y'(0) = 1$

$\mathcal{L}(y'') + 2\mathcal{L}(y') - 3\mathcal{L}(y) = \mathcal{L}(te^t)$

$s^2\mathcal{L}(y) - s\cancel{y(0)} - y'(0) + 2\left(s\mathcal{L}(y) - \cancel{y(0)}\right) - 3\mathcal{L}(y) = \dfrac{1}{(s-1)^2}$

$\mathcal{L}(y)(s^2 + 2s - 3) - 1 = \dfrac{1}{(s-1)^2}$

$\mathcal{L}(y) = \left[1 + \dfrac{1}{(s-1)^2}\right] \Big/ (s^2 + 2s - 3)$

$= \dfrac{1}{s^2+2s-3} + \dfrac{1}{(s-1)^2(s^2+2s-3)}$

Mas:

$\dfrac{1}{s^2+2s-3} = \dfrac{1}{(s+3)(s-1)} = \dfrac{A}{s+3} + \dfrac{B}{s-1}$

$\dfrac{1}{(s+3)(s-1)} = \dfrac{A(s-1) + B(s+3)}{(s+3)(s-1)}$

$\Longrightarrow \quad 1 = A(s-1) + B(s+3)$

$1 = s(A+B) - A + 3B$

$\Longrightarrow \begin{cases} A+B = 0 & \Longrightarrow \; //A = -B// \\ -A+3B = 1 & \Longrightarrow \; B + 3B = 1 \end{cases}$

$$A = -B = -\frac{1}{4}$$

$$\therefore \quad \left\| \; \frac{1}{s^2+2s-3} = \frac{-1}{4(s+3)} + \frac{1}{4(s-1)} \; \right\|$$

$$\frac{1}{(s-1)^2 \underbrace{(s^2+2s-3)}_{(s+3)(s-1)}} = \frac{1}{(s-1)^3(s+3)}$$

$$= \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{D}{s+3}$$

$$\therefore \quad 1 = A(s-1)^2(s+3) + B(s-1)(s+3) + C(s+3) + D(s-1)^3$$

$\underline{\underline{s=1}}$: $1 = 4C \Longrightarrow \| C = 1/4 \|$

$s=-3$: $1 = D(-4)^3 \Longrightarrow \left\| D = -\frac{1}{64} \right\|$

$\underline{\underline{s=0}}$: $1 = 3A - 3B + 3C - D$

$$1 = 3A - 3B + \frac{3}{4} + \frac{1}{64} \qquad \Bigg| \quad \frac{3}{4_{/16}} + \frac{1}{64} = \frac{49}{64}$$

$$3A - 3B = 1 - \frac{49}{64}$$

$$3A - 3B = \frac{15}{64}$$

$$\left\| A - B = \frac{5}{64} \right\|$$

$\underline{\Delta = -1}$:

$$1 = A(-2)^2 2 + B(-2)2 + C2 + D(-2)^3$$

$$1 = 8A - 4B + 2C - 8D$$

$$1 = 8A - 4B + 2\frac{1}{4} - 8\left(-\frac{1}{64}\right)$$

$$1 = 8A - 4B + \frac{1}{2} + \frac{1}{8}$$

$$8A - 4B = \frac{1}{2} - \frac{1}{8}$$

$$8A - 4B = \frac{3}{8}$$

$$\| 2A - B = \frac{3}{32} \|$$

$$\therefore \begin{cases} A - B = \frac{5}{64} \rightarrow \| A = B + \frac{5}{64} \| \\ 2A - B = \frac{3}{32} \end{cases}$$

$$2\left(B + \frac{5}{64}\right) - B = \frac{3}{32}$$

$$2B + \frac{5}{32} - B = \frac{3}{32}$$

$32 = 2^5$

$12 = 2^2 \cdot 3$

$$B = \frac{3}{32} - \frac{5}{32} = \frac{-2}{32} = -\frac{1}{16}$$

$$\| B = -\frac{1}{16} \|$$

$$\| A = -\frac{1}{16} + \frac{5}{64} = \frac{-4+5}{64} = \frac{1}{64} \|$$

$$\frac{1}{(s-1)^2(s^2+2s-3)} = \frac{1}{64(s-1)} - \frac{1}{16(s-1)^2} + \frac{1}{4(s-1)^3} - \frac{1}{64(s+3)}$$

$$\mathcal{L}(y) = \frac{1}{s^2+2s-3} + \frac{1}{(s-1)^2(s^2+2s-3)}$$

$$= -\frac{1}{4(s+3)} + \frac{1}{4(s-1)} + \frac{1}{64(s-1)} - \frac{1}{16(s-1)^2} + \frac{1}{4(s-1)^3} - \frac{1}{64(s+3)}$$

$$= -\frac{17}{64}\frac{1}{s+3} + \frac{17}{64}\frac{1}{s-1} - \frac{1}{16(s-1)^2} + \frac{1}{4(s-1)^3}$$

$$y = -\frac{17}{64}\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) + \frac{17}{64}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) - \frac{1}{16}\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right) + \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{(s-1)^3}\right)$$

$$= -\frac{17}{64}e^{-3t} + \frac{17}{64}e^{+t} - \frac{1}{16}e^{t}t + \frac{1}{4}e^{t}\frac{1}{2}t^2$$

$$= \boxed{-\frac{17}{64}e^{-3t} + \frac{17}{64}e^{t} + \left(\frac{1}{8}t^2 - \frac{1}{16}t\right)e^{t}}$$

4. $y' = y \quad ; \quad y(0) = 1$

$$\mathcal{L}(y') = \mathcal{L}(y)$$

$$s\mathcal{L}(y) - y(0) = \mathcal{L}(y)$$

$$s\mathcal{L}(y) - 1 = \mathcal{L}(y) \longrightarrow (s-1)\mathcal{L}(y) = 1$$

$$\mathcal{L}(y) = \frac{1}{s-1}$$

$$\therefore \quad y = \mathcal{L}^{-1}\left(\frac{1}{s-1}\right)$$

$$\boxed{y = e^{t}}$$

---

5. $y'' + 3y' + 2y = \cos 2t \quad ; \quad y(0) = -1 \quad , \quad y'(0) = 1$

$$\mathcal{L}(y'') + 3\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(\cos 2t)$$

$$s^2\mathcal{L}(y) - s\,y(0) - y'(0) + 3\big(s\mathcal{L}(y) - y(0)\big) + 2\mathcal{L}(y) = \frac{s}{s^2+4}$$

$$\mathcal{L}(y)\left(s^2 + 3s + 2\right) + s - 1 + 3 = \frac{s}{s^2+4}$$

$$\mathcal{L}(y)\,\underbrace{(s^2+3s+2)}_{(s+1)(s+2)} = -s - 2 + \frac{s}{s^2+4}$$

$$\therefore \quad \mathcal{L}(y) = \frac{-\cancel{(s+2)}}{(s+1)\cancel{(s+2)}} + \frac{s}{(s+1)(s+2)(s^2+4)}$$

Mas $\frac{s}{(s+1)(s+2)(s^2+4)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{Cs+D}{s^2+4}$

$\therefore$

$s = A(s+2)(s^2+4) + B(s+1)(s^2+4) + (Cs+D)(s+1)(s+2)$

$\underline{s=-1}$: $-1 = A\,5 \Rightarrow A = -\frac{1}{5}$

$\underline{s=-2}$: $-2 = B(-1)8 \Rightarrow B = \frac{1}{4}$

$s=0$:

$0 = 8A + 4B + 2D$

$0 = 8(-\frac{1}{5}) + 4\,\frac{1}{4} + 2D$

$\therefore\ 2D = \frac{8}{5} - 1$

$2D = \frac{3}{5} \Rightarrow //D = \frac{3}{10}//$

$\underline{s=1}$: $1 = 15A + 10B + (C+D)6$

$1 = 15(-\frac{1}{5}) + 10\,\frac{1}{4} + (C+D)6$

$1 = -3 + \frac{5}{2} + 6C + 6D$

$4 - \frac{5}{2} = 6C + 6D$

$\frac{3}{2} = 6C + 6D$

$\frac{1}{2} = 2C + 2D \longrightarrow$

$\frac{1}{2} = 2C + \frac{3}{5}$

$2C = \frac{1}{2} - \frac{3}{5} = -\frac{1}{10} \therefore //C = -$

$$\therefore \quad \frac{s}{(s+1)(s+2)(s^2+4)} = \frac{-1}{5(s+1)} + \frac{1}{4(s+2)} + \frac{-\frac{1}{20}s + \frac{3}{10}}{s^2+4}$$

$\therefore$

$$\mathcal{L}(y) = -\frac{1}{s+1} - \frac{1}{5(s+1)} + \frac{1}{4(s+2)} - \frac{1}{20}\frac{s}{s^2+4} + \frac{3}{10}\frac{1}{s^2+4}$$

$$y = -\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{5}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) - \frac{1}{50}\underline{\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right)}$$

$$+ \frac{3}{10}\underline{\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right)}$$

$$= -e^{-t} - \frac{1}{5}e^{-t} + \frac{1}{4}e^{-2t} - \frac{1}{20}\underline{\cos 2t}$$

$$+ \frac{3}{10}\underline{\frac{1}{2}\sin 2t}$$

$$\boxed{y = -\frac{6}{5}e^{-t} + \frac{1}{4}e^{-2t} - \frac{1}{20}\cos 2t + \frac{3}{20}\sin 2t}$$

6. $y'' + a^2 y = 0 \quad , \quad y(0) = 1 \quad , \quad y'(0) = 0$

$$\mathcal{L}(y'') + a^2 \mathcal{L}(y) = 0$$

$$s^2 \mathcal{L}(y) - s\, y(0) - y'(0) + a^2 \mathcal{L}(y) = 0$$

$$(s^2 + a^2)\mathcal{L}(y) - s = 0$$

$$\mathcal{L}(y) = \frac{s}{s^2 + a^2}$$

$$y = \mathcal{L}^{-1}\left(\frac{s}{s^2 + a^2}\right) = \boxed{\cos at}$$

---

7. $y''' - y' = e^{3t} \quad , \quad y(0) = 0 \quad , \quad y'(0) = 5 \quad , \quad y''(0) = -2$

$$\mathcal{L}(y''') - \mathcal{L}(y') = \mathcal{L}(e^{3t})$$

$$s^3 \mathcal{L}(y) - s^2 \cancelto{0}{y(0)} - s\, y'(0) - y''(0) - [s\,\mathcal{L}(y) - y(0)] = \frac{1}{s-3}$$

$$s^3 \mathcal{L}(y) - s5 + 2 - s\,\mathcal{L}(y) = \frac{1}{s-3}$$

$$(s^3 - s)\,\mathcal{L}(y) = 5s - 2 + \frac{1}{s-3}$$

$$\mathcal{L}(y) = \frac{5s}{s(s^2-1)} - \frac{2}{s(s^2-1)} + \frac{1}{s(s^2-1)(s-3)}$$

$$= \frac{5}{s^2-1} - \frac{2}{s(s^2-1)} + \frac{1}{s(s^2-1)(s-3)}$$

7. Cont.

$$\mathcal{L}^{-1}\left(\frac{5}{s^2-1}\right) = 5\,\mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right) = 5\sinh t$$

$$\mathcal{L}^{-1}\left(\frac{-2}{s(s^2-1)}\right) = -2\,\mathcal{L}^{-1}\left(\frac{1}{s(s^2-1)}\right)$$

Mas $\quad \dfrac{1}{s(s^2-1)} = \dfrac{A}{s} + \dfrac{B}{(s-1)} + \dfrac{C}{(s+1)}$

$$\frac{1}{s(s^2-1)} = \frac{A(s-1)(s+1) + Bs(s+1) + Cs(s-1)}{s(s^2-1)}$$

$\therefore$

$$1 = As^2 - A + Bs^2 + Bs + Cs^2 - Cs$$

$$1 = (A+B+C)s^2 + (B-C)s - A$$

$\Rightarrow \quad A+B+C = 0 \Longrightarrow -1+2C=0 \quad \| C = \frac{1}{2} \|$

$\quad B - C = 0 \Rightarrow \underline{\underline{B=C}} \quad \| B = \frac{1}{2} \|$

$\quad \underline{\underline{A=-1}}$

$$\Big\|\ \mathcal{L}^{-1}\left(\frac{-2}{s(s^2-1)}\right) = -2\,\mathcal{L}^{-1}\left(\frac{1}{s(s^2-1)}\right)$$

$$= -2\,\mathcal{L}^{-1}\left(\frac{-1}{s} + \frac{1}{2(s-1)} + \frac{1}{2(s+1)}\right)$$

$$= 2\,\mathcal{L}^{-1}(s) - \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$= 2\ 1 - e^{+t} - e^{-t}$$

$$= 2 - e^{t} - e^{-t} \quad //$$

$$\mathcal{L}^{-1}\left(\frac{1}{s\underbrace{(s^2-1)}_{(s-1)(s+1)}(s-3)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s(s-1)(s+1)(s-3)}\right)$$

Mos

$$\frac{1}{s(s-1)(s+1)(s-3)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} + \frac{D}{s-3}$$

$\therefore$

$$1 = A(s-1)(s+1)(s-3) + B\,s(s+1)(s-3) + C\,s(s-1)(s-3) + D\,s(s-1)(s+1)$$

$\underline{s=1}$:

$$1 = B\,2(-2) \Rightarrow \| B = -\tfrac{1}{4} \|$$

$\underline{\underline{s=3}}$

$$1 = D\,3.2.4 \Rightarrow \| D = \tfrac{1}{24} \|$$

$\underline{\underline{s=-1}}$

$$1 = C(-1)(-2)(-4) \Rightarrow \| C = -\tfrac{1}{8} \|$$

$\underline{\underline{s=0}}$:

$$1 = A(-1)(1)(-3) \Rightarrow \| A = \tfrac{1}{3} \|$$

$$\therefore \quad \frac{1}{s(s-1)(s+1)(s-3)} = \frac{1}{3s} - \frac{1}{4(s-1)} - \frac{1}{8(s+1)} + \frac{1}{24(s-3)}$$

$\Rightarrow$

$$\mathcal{L}^{-1}\left(\frac{1}{s(s-1)(s+1)(s-3)}\right) =$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) - \frac{1}{8}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{24}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right)$$

$$= \frac{1}{3} - \frac{1}{4}e^{t} - \frac{1}{8}e^{-t} + \frac{1}{24}e^{3t}$$

$\therefore$

$$y = \underbrace{\mathcal{L}^{-1}\left(\frac{5s}{s(s^2-1)}\right)} \underbrace{-2\mathcal{L}^{-1}\left(\frac{1}{s(s^2-1)}\right)} + \mathcal{L}^{-1}\left(\frac{1}{s(s-1)(s+1)(s-3)}\right)$$

$$= 5\sinh t + 2 - e^{t} - e^{-t} + \frac{1}{3} - \frac{1}{4}e^{t} - \frac{1}{8}e^{-t} + \frac{1}{24}e^{3t}$$

$$y = 5\sinh t - \frac{5}{4}e^{t} - \frac{9}{8}e^{-t} + \frac{1}{24}e^{3t} + \frac{7}{3}$$

Mas

$$\begin{cases} e^{t} + e^{-t} = 2\cosh t \\ e^{t} - e^{-t} = 2\sinh t \end{cases} \Rightarrow \begin{cases} e^{t} = \cosh t + \sinh t \\ e^{-t} = \cosh t - \sinh t \end{cases}$$

$$\therefore y = 5\sinh t - \frac{5}{4}\underline{(\cosh t + \sinh t)} - \frac{9}{8}\underline{(\cosh t - \sinh t)}$$

$$+ \frac{1}{24}e^{3t} + \frac{7}{3}$$

$$= \left(5 - \frac{5}{4} + \frac{9}{8}\right)\sinh t + \left(-\frac{5}{4} - \frac{9}{8}\right)\cosh t + \frac{e^{3t}}{24} + \frac{7}{3}$$

$\frac{10-10+9}{8}$

$$\therefore \boxed{y = \frac{39}{8}\sinh t - \frac{19}{8}\cosh t + \frac{e^{3t}}{24} + \frac{7}{3}}$$

$-\frac{5}{4} - \frac{9}{8} = -\frac{19}{8}$

8. $y'' + 3y' - 4\int_0^t y\,du = 0 \quad ; \quad y(0) = 2 \quad , \quad y'(0) = 1$

$$\mathcal{L}(y'') + 3\mathcal{L}(y') - 4\underbrace{\mathcal{L}\left(\int_0^t y\,du\right)} = 0$$

$$s^2\mathcal{L}(y) - s\,y(0) - y'(0) + 3\left(s\mathcal{L}(y) - \underbrace{y(0)}\right) - 4\underline{\frac{1}{s}\mathcal{L}(y)} = 0$$

$$\mathcal{L}(y)\left(s^2 + 3s - \frac{4}{s}\right) - 2s - 1 \underbrace{- 6} = 0$$

$$\mathcal{L}(y)\left(s^2 + 3s - \frac{4}{s}\right) = 2s + 7$$

$$\mathcal{L}(y) = \frac{2s}{s^2+3s-\frac{4}{s}} + \frac{7}{s^2+3s-\frac{4}{s}} \qquad (\star)$$

$$= \frac{2s^2}{s^3+3s^2-4} + \frac{7s}{s^3+3s^2-4}$$

Mas

$$s^3+3s^2-4 = 0$$

$s = 1$ é raiz $\Longrightarrow$

$$\begin{array}{l|l} s^3+3s^2-4 & s-1 \\ \cline{2-2} -s^3+s^2 & s^2+4s+4 \\ \hline 4s^2-4 & \\ -4s^2+4s & \\ \hline 4s-4 & \\ -4s+4 & \\ \hline 0 & \end{array}$$

$s^2+4s+4 =$
$= (s+2)^2$

$$\| s^3+3s^2-4 = (s-1)(s+2)^2 \|$$

Ent-5

$$\frac{2s^2}{s^3+3s^2-4} = \frac{2s^2}{(s-1)(s+2)^2} = \frac{A}{s-1} + \frac{B}{(s+2)} + \frac{C}{(s+2)^2}$$

$$\therefore \quad 2s^2 = A(s+2)^2 + B(s-1)(s+2) + C(s-1)$$

$$= A(s^2+4s+4) + B(s^2+s-2) + Cs - C$$

$$\left\{ \begin{aligned} 2s^2 &= s^2(A+B) + s(4A+B+C) + \\ &\quad + 4A - 2B - C \end{aligned} \right.$$

$$\left\{ \begin{aligned} &A+B = 2 \\ &4A+B+C = 0 \\ &4A-2B-C = 0 \end{aligned} \right. \qquad \left. \begin{aligned} &4A+B+C = 0 \\ &4A-2B-C = 0 \end{aligned} \right\} \quad 8A - B = 0$$

$$\therefore \quad A+B = 2 \quad \longrightarrow \quad A + 8A = 2 \quad \therefore \quad \left\| A = \frac{2}{9} \right\|$$

$$8A - B = 0 \quad \Rightarrow \quad B = 8A$$

$$\therefore \quad \left\| B = 8 \cdot \frac{2}{9} = \frac{16}{9} \right\|$$

$$\to \quad C = -4A - B$$

$$= -\frac{8}{9} - \frac{16}{9} = -\frac{24}{9} \quad \therefore \quad \left\| C = -\frac{24}{9} \right\|$$

$\therefore$

$$\frac{2s^2}{s^3+3s^2-4} = \frac{2}{9(s-1)} + \frac{16}{9(s+2)} - \frac{24}{9(s+2)^2}$$

$$\therefore \quad \mathcal{L}^{-1}\left(\frac{2s^2}{s^3+3s^2-4} \right) = \frac{2}{9} \mathcal{L}^{-1}\left(\frac{1}{s-1} \right) + \frac{16}{9} \mathcal{L}^{-1}\left(\frac{1}{s+2} \right) - \frac{24}{9} \mathcal{L}^{-1}\left(\frac{1}{(s+2)^2} \right)$$

$$\mathcal{L}^{-1}\left(\frac{2s^2}{s^3+3s^2-4}\right) = \frac{2}{9}e^{t} + \frac{16}{9}e^{-2t} - \frac{24}{9}e^{-2t}\,t$$

también

$$\frac{7s}{s^3+3s^2-4} = \frac{7s}{(s-1)(s+2)^2} = \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$\frac{7s}{(s-1)(s+2)^2} = \frac{A(s+2)^2 + B(s-1)(s+2) + C(s-1)}{(s-1)(s+2)^2}$$

$$7s = A(s^2+4s+4) + B(s^2+s-2) + C(s-1)$$

$$7s = (A+B)s^2 + (4A+B+C)s + (4A-2B-C)$$

$$\Rightarrow \begin{cases} A+B=0 \\ 4A+B+C=7 \\ 4A-2B-C=0 \end{cases} \qquad \begin{array}{l} A+B=0 \\ 8A-B=7 \\ \hline 9A=7 \Rightarrow A=\frac{7}{9} \end{array}$$

$$B=-A=-\frac{7}{9}$$

$$C = 4A-2B$$

$$C = \frac{28}{9}+\frac{14}{9} = \frac{42}{9} = \frac{14}{3}$$

$$\frac{7s}{s^3+3s^2-4} = \frac{7}{9(s-1)} - \frac{7}{9(s+2)} + \frac{14}{3(s+2)^2}$$

$$\mathcal{L}^{-1}\left(\frac{7s}{s^3+3s^2-4}\right) = \frac{7}{9}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) - \frac{7}{9}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + \frac{14}{3}\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right)$$

$$= \frac{7}{9}e^{t} - \frac{7}{9}e^{-2t} + \frac{14}{3}e^{-2t}t$$

De (*):

$$y = \mathcal{L}^{-1}\left(\frac{2s^2}{s^3+3s^2-4}\right) + \mathcal{L}^{-1}\left(\frac{7s}{s^3+3s^2-4}\right)$$

$$= \frac{2}{9}e^{t} + \frac{16}{9}e^{-2t} - \overset{\frac{8}{3}}{\frac{24}{9}}e^{-2t}t + \frac{7}{9}e^{t} - \frac{7}{9}e^{-2t} + \frac{14}{3}e^{-2t}t$$

$$y = e^{t} + e^{-2t} + 2e^{-2t}t$$

$$\therefore \boxed{y = e^{t} + e^{-2t} + 2t\,e^{-2t}}$$

9. $y'' + 9y = 0$; $y(0) = 0$, $y'(0) = 2$

$\mathcal{L}(y'') + 9\,\mathcal{L}(y) = 0$

$s^2 \mathcal{L}(y) - s\,\cancel{y(0)}^{\,=0} - y'(0) + 9\,\mathcal{L}(y) = 0$

$(s^2 + 9)\,\mathcal{L}(y) - 2 = 0$

$$\mathcal{L}(y) = \frac{2}{s^2+9}$$

$$y = 2\,\mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right)$$

$$= 2\,\frac{1}{3}\sin 3t$$

$$\boxed{y = \frac{2}{3}\sin 3t}$$

---

10. $y'' + \omega^2 y = 0$; $y(0) = A$, $y'(0) = B$ $(\omega \neq 0)$

$\mathcal{L}(y'') + \omega^2 \mathcal{L}(y) = 0$

$s^2 \mathcal{L}(y) - s\,y(0) - y'(0) + \omega^2 \mathcal{L}(y) = 0$

$\mathcal{L}(y)\,(s^2 + \omega^2) - sA - B = 0$

$$\mathcal{L}(y) = \frac{sA + B}{s^2+\omega^2} = A\,\frac{s}{s^2+\omega^2} + B\,\frac{1}{s^2+\omega^2}$$

$$y = A\,\mathcal{L}^{-1}\left(\frac{s}{s^2+\omega^2}\right) + B\,\mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right)$$

$$\boxed{y = A\cos\omega t + \frac{B}{\omega}\sin\omega t}$$

11. $y'' - 2y' - 3y = 0$; $y(0) = 1$, $y'(0) = 7$

$$\mathcal{L}(y'') - 2\mathcal{L}(y') - 3\mathcal{L}(y) = 0$$

$$s^2\mathcal{L}(y) - s\,y(0) - y'(0) - 2\left(s\mathcal{L}(y) - y(0)\right) - 3\mathcal{L}(y) = 0$$

$$s^2\mathcal{L}(y) - s - 7 - 2s\mathcal{L}(y) + 2 - 3\mathcal{L}(y) = 0$$

$$\mathcal{L}(y)\left(s^2 - 2s - 3\right) = s + 5$$

$$\mathcal{L}(y) = \frac{s+5}{s^2-2s-3} = \frac{s+5}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$$

$$\therefore \quad s+5 = A(s+1) + B(s-3)$$

$$s+5 = s(A+B) + A - 3B$$

$$\Rightarrow \quad A+B = 1 \Rightarrow A = 1-B$$

$$A - 3B = 5 \Rightarrow 1 - B - 3B = 5$$

$$1 - 4B = 5$$

$$4B = -4$$

$$\underline{\underline{B = -1}}$$

$$\underline{\underline{A = 2}}$$

$$\mathcal{L}(y) = \frac{2}{s-3} - \frac{1}{s+1}$$

$$y = 2\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$\boxed{y = 2e^{3t} - e^{-t}}$$

12. $4y'' + y = 0 \quad ; \; y(0) = 1 \; , \; y'(0) = -2$

$$4\mathcal{L}(y'') + \mathcal{L}(y) = 0$$

$$4\left[s^2\mathcal{L}(y) - s\,y(0) - y'(0)\right] + \mathcal{L}(y) = 0$$

$$\mathcal{L}(y)\,(4s^2+1) - 4s + 8 = 0$$

$$\mathcal{L}(y) = \frac{4s-8}{4s^2+1} = \frac{s-2}{s^2+\frac{1}{4}}$$

$$= \frac{s}{s^2+\frac{1}{4}} - \frac{2}{s^2+\frac{1}{4}}$$

$$y = \mathcal{L}^{-1}\left(\frac{s}{s^2+\frac{1}{4}}\right) - 2\,\mathcal{L}^{-1}\left(\frac{1}{s^2+\frac{1}{4}}\right)$$

$$\boxed{y = \cos\tfrac{1}{2}t - 4\sin\tfrac{1}{2}t}$$

13. $\mathcal{L}(y'') - 4\mathcal{L}(y') + 4\mathcal{L}(y) = 0$; $y(0) = 0$, $y'(0) = 2$

$s^2\mathcal{L}(y) - s\,y(0) - y'(0) - 4\left(s\mathcal{L}(y) - y(0)\right) + 4\mathcal{L}(y) = 0$

$\mathcal{L}(y)\left(s^2 - 4s + 4\right) - 2 = 0$

$$\mathcal{L}(y) = \frac{2}{s^2-4s+4} = \frac{2}{(s-2)^2}$$

$$y = \mathcal{L}^{-1}\left(\frac{2}{(s-2)^2}\right) = 2e^{+2t}t$$

$$\boxed{y = 2e^{2t}t}$$

---

14. $y'' + 4y' + 4y = 0$; $y(0) = 2$, $y'(0) = -3$

$\mathcal{L}(y'') + 4\mathcal{L}(y') + 4\mathcal{L}(y) = 0$

$\left(s^2\mathcal{L}(y) - s\,y(0) - y'(0)\right) + 4\left(s\mathcal{L}(y) - y(0)\right) + 4\mathcal{L}(y) = 0$

$\mathcal{L}(y)\left(s^2 + 4s + 4\right) - 2s + 3 - 8 = 0$

$$\mathcal{L}(y) : \frac{2s+5}{s^2+4s+4} = \frac{2(s+2)}{(s+2)^2} + \frac{1}{(s+2)^2}$$

$$= 2\frac{1}{(s+2)} + \frac{1}{(s+2)^2}$$

$$y = 2\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right)$$

$$= 2e^{-2t} + e^{-2t}t$$

$$\boxed{y = (t+2)e^{-2t}}$$

---

15. $y'' + y' + 1.25y = 0 \quad ; \quad y(0) = 1 \; , \; y'(0) = -0.5$

$$\mathcal{L}(y'') + \mathcal{L}(y') + 1.25\,\mathcal{L}(y) = 0$$

$$s^2\mathcal{L}(y) - s\,y(0) - y'(0) + s\mathcal{L}(y) - y(0) + 1.25\,\mathcal{L}(y) = 0$$

$$\mathcal{L}(y)\left[s^2 + s + 1.25\right] - s + 0.5 - 1 = 0$$

$$\mathcal{L}(y) = \frac{s + \frac{1}{2}}{s^2 + s + \frac{5}{4}}$$

$1\frac{1}{4} = \frac{5}{4}$

$s^2 + s + \frac{5}{4} = 0$

$s = -1 \pm \sqrt{1 -}$

$$= \frac{s + \frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{5}{4}}$$

$$= \frac{s + \frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + 1}$$

$$y = \mathcal{L}^{-1}\left(\frac{s + \frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + 1}\right) = \boxed{e^{-\frac{1}{2}t}\cos t}$$