Cálculo C - Lista 18

Transformada de Laplace (IV):

Função degrau unitária

Faça um esboço do gráfico da função dada para $t \geq 0$

1. $tu(t-1)$

2. $u(t-1) - 3u(t-4) - 4u(t-5)$

3. $(t^2-4)u(t-4)$

4. $\sin(t-\pi)u(t-\pi)$

5. $t(u(t) - u(t-1)) + (u(t-1) - u(t-3)) +$
   $+ (u(t-3) - u(t-4))(4-t)$

6. $u(t-1) - 2u(t-2) + 2u(t-3) - 2u(t-4) + \ldots$

   Represente as funções em termos da função degrau unitário e encontre a transformada de Laplace

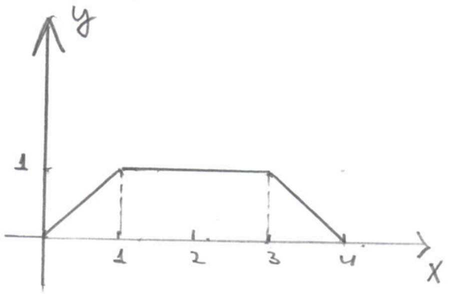
7.

$$f(t) = \begin{cases} 3, & 0 < t < 2 \\ t+1, & 2 < t \end{cases}$$

8.

$$f(t) = \begin{cases} \sin 3t, & 0 < t < \pi \\ 0, & \pi < t \end{cases}$$

9.



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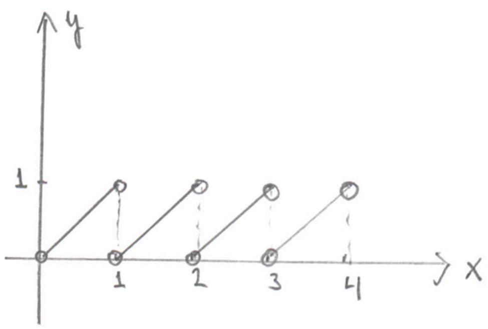

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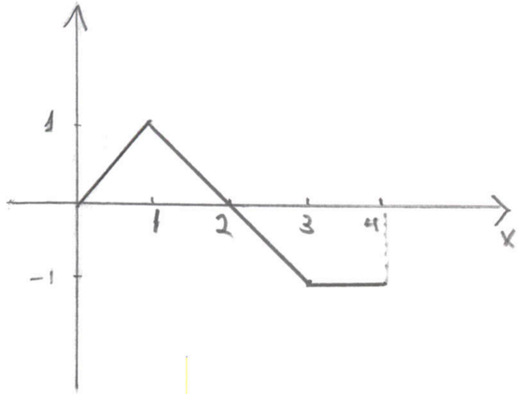


Encontre a transformada de Laplace inversa das funções

12. $\frac{e^{-4s}-e^{-s}}{s^3}$

13. $\frac{e^{-3s}}{s^2-9}$

14. $\frac{e^{-s}}{(s-1)(s-2)}$

15. $\frac{e^{-s}}{(s+1)^3}$

16. $\frac{(s-2)e^{-s}}{s^2-4s+3}$

    Encontre a transformada de Laplace das funções

17. $t\,u(t-2)$

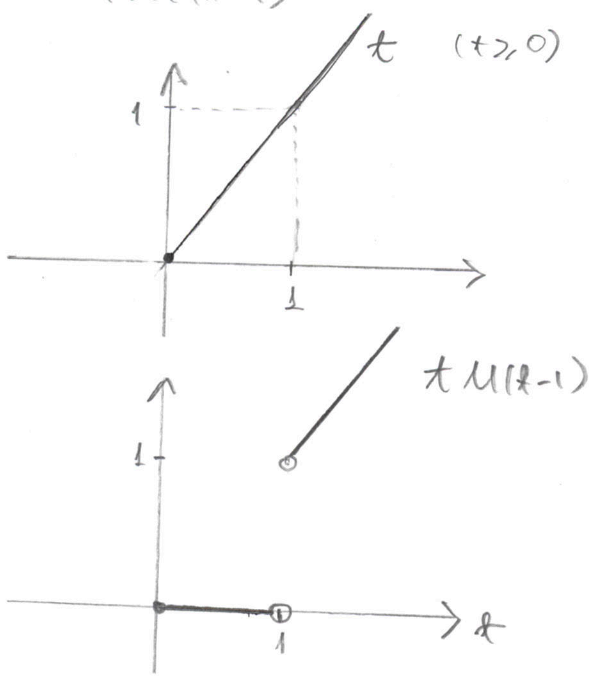
18. $\cos t\; u(t-\pi)$

19. $e^t\, u(t-3)$

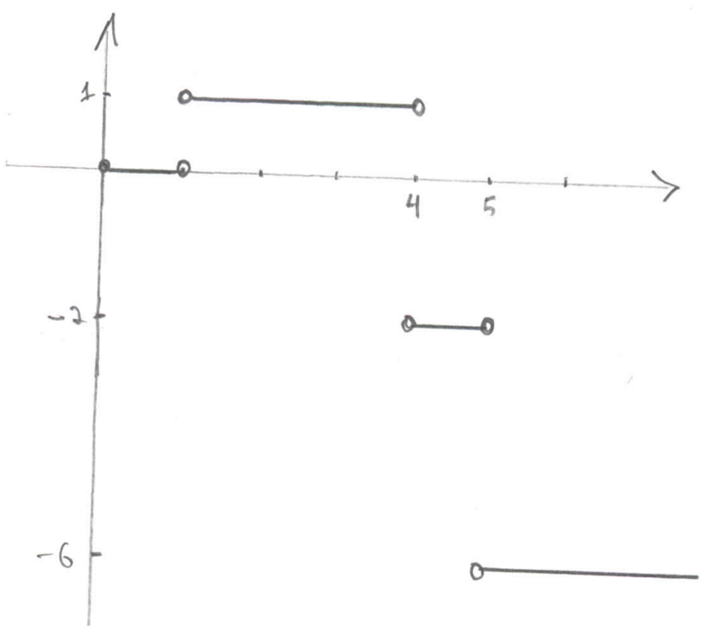
20. $\sin t\; u(t-\pi/2)$

Lista 18 - Respostas

1. $t\,u(t-1)$



2. $u(t-1) - 3u(t-4) - 4u(t-5)$



$\varphi(t) = u(t-1) - 3u(t-4) - 4u(t-5)$

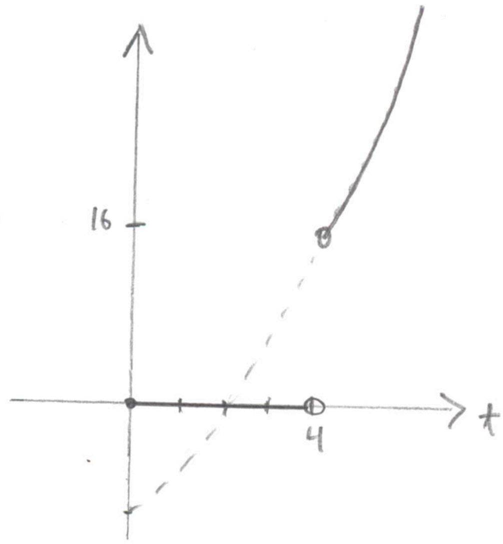
$0 \leq t < 1$: $\varphi(t) = 0$

$1 < t < 4$: $\varphi(t) = 1$

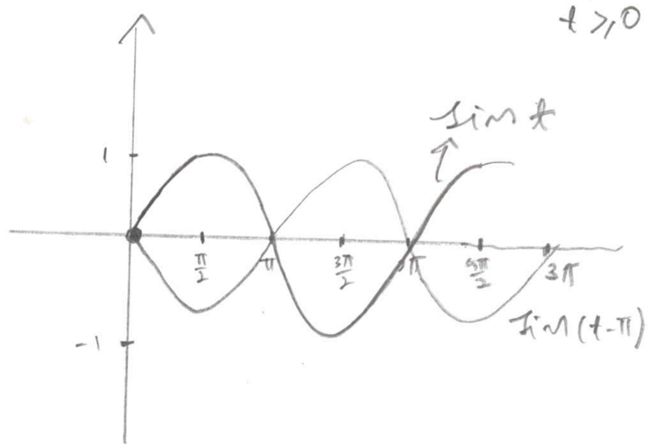
$4 < t < 5$: $\varphi(t) = 1 - 3 = -2$

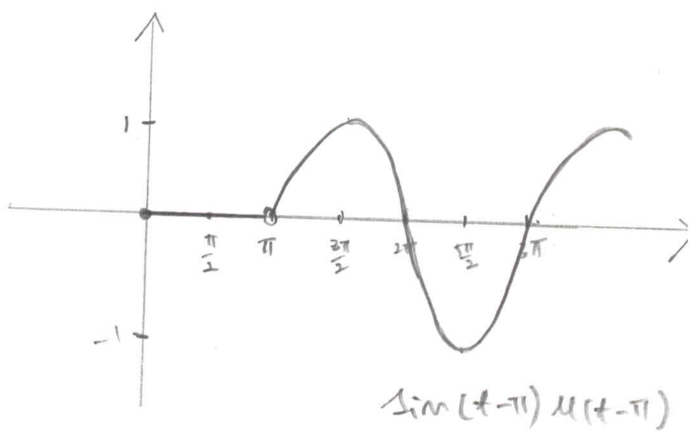
$5 < t$: $\varphi(t) = 1 - 3 - 4 = -6$

3. $(t^2-4)\,u(t-4)$



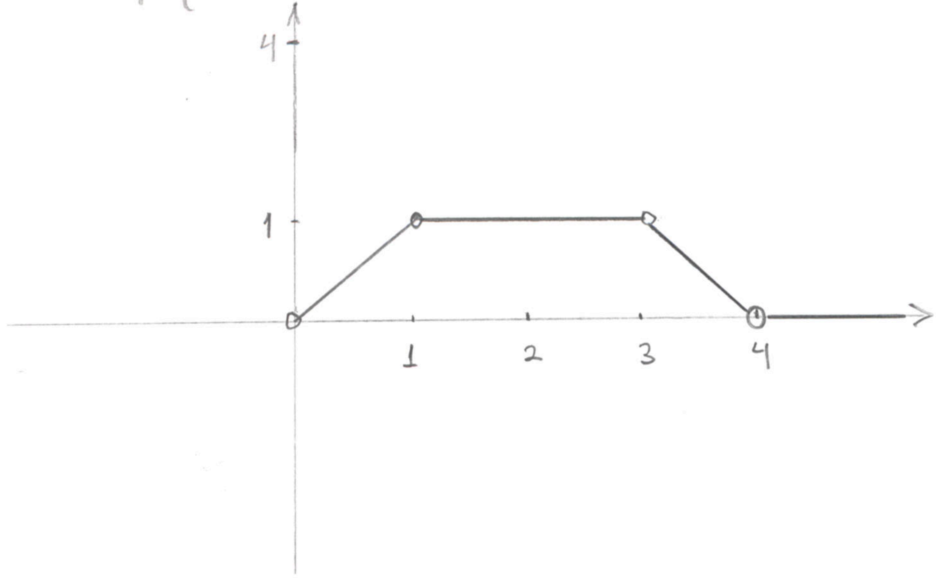
4. $\sin(t-\pi)\,u(t-\pi)$





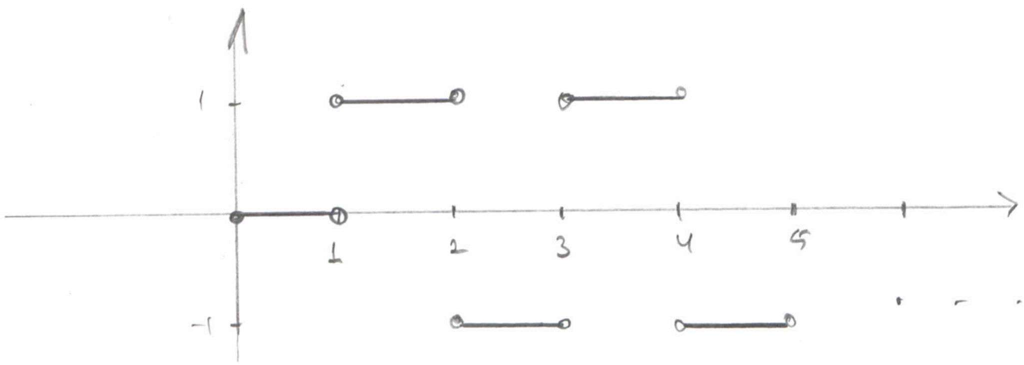
5. $t\left(u(t) - u(t-1)\right) + \left(u(t-1) - u(t-3)\right) +$

$+ \left(u(t-3) - u(t-4)\right)(4-t)$



6. $u(t-1) - 2u(t-2) + 2u(t-3) - 2u(t-4) + \cdots$

$= \left[u(t-1) - u(t-2)\right] - \left[u(t-2) - u(t-3)\right] + \left[u(t-3) - u(t-4)\right] + \cdots$



7. $f(t) = \begin{cases} 3, & 0 < t < 2 \\ t+1, & 2 < t \end{cases}$

$f(t) = 3\left[u(t-0) - u(t-2)\right] + (t+1)\,u(t-2)$

$\mathcal{L}(f(t)) = 3\,\mathcal{L}(u(t-0)) - 3\,\mathcal{L}(u(t-2)) + \underbrace{\mathcal{L}\left[(t+1)\,u(t-2)\right]}$

$= 3\,\frac{1}{s} - 3\,\frac{e^{-2s}}{s} + (*)$

$$\circledast = \mathcal{L}((t+1)\,u(t-2))$$

$$= \mathcal{L}((t-2)\,u(t-2) + 3\,u(t-2))$$

$$= \underbrace{\mathcal{L}((t-2)\,u(t-2))} + 3\,\mathcal{L}(u(t-2))$$

$$= e^{-2s}\,\mathcal{L}(t) + 3\,\frac{e^{-2s}}{s}$$

$$= \frac{e^{-2s}}{s^2} + \frac{3e^{-2s}}{s}$$

$$\therefore \quad \mathcal{L}(f(t)) = \frac{3}{s} - \cancel{\frac{3e^{-2s}}{s}} + \frac{e^{-2s}}{s^2} + \cancel{\frac{3e^{-2s}}{s}}$$

$$\boxed{\mathcal{L}(f(t)) = \frac{3}{s} + \frac{e^{-2s}}{s^2}}$$

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8. $$f(t) = \begin{cases} \sin 3t, & 0 < t < \pi \\ 0, & \pi < t \end{cases}$$

$$f(t) = \sin 3t\,[u(t) - u(t-\pi)]$$

$$\mathcal{L}(f(t)) = \mathcal{L}[\sin 3t\,(u(t) - u(t-\pi))]$$

$$= \underbrace{\mathcal{L}(\sin 3t\,u(t))}_{\circledast} - \underbrace{\mathcal{L}(\sin 3t\,u(t-\pi))}_{\circledast\circledast}$$

$$\mathcal{L}(f(t-a)\,u(t-a)) = e^{-as}\,\mathcal{L}(f)$$

$(*) = \mathcal{L}(\sin 3t\; u(t))$

$$= \mathcal{L}(\sin 3(t-0)\; u(t-0))$$

$$= e^{-0s}\,\mathcal{L}(\sin 3t)$$

$$= \frac{3}{s^2+9}$$

$(**) = \mathcal{L}(\sin 3t\; u(t-\pi))$

$$= \mathcal{L}(-\sin 3(t-\pi)\; u(t-\pi))$$

$$= -e^{-\pi s}\,\mathcal{L}(\sin 3t)$$

$$= -e^{-\pi s}\,\frac{3}{s^2+9}$$

$$\left[\ \sin 3(t-\pi) = \sin 3t\cos 3\pi - \cos 3t\,\cancel{\sin 3\pi}^{\,0} = -\sin 3t\ \right]$$

$$\therefore\quad \mathcal{L}(f(t)) = \frac{3}{s^2+9} + e^{-\pi s}\,\frac{3}{s^2+9}$$

$$= \boxed{\frac{3}{s^2+9}\left(1+e^{-\pi s}\right)}$$

9.

$$f(t) = \begin{cases} t, & 0 < t < 1 \\ 1, & 1 < t < 3 \\ 4-t, & 3 < t < 4 \\ 0, & t > 4 \end{cases}$$

[Desconsidera-se os pontos extremos dos intervalos]

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Aqui

$$f(t) = t\left[u(t) - u(t-1)\right] + 1\left[u(t-1) - u(t-3)\right] + (4-t)\left[u(t-3) - u(t-4)\right]$$

$$\mathcal{L}(f(t)) = \mathcal{L}\left[t(u(t) - u(t-1))\right] + \mathcal{L}(u(t-1) - u(t-3)) + \mathcal{L}\left[(4-t)(u(t-3) - u(t-4))\right]$$

$$= \mathcal{L}(t\,u(t)) - \mathcal{L}(t\,u(t-1)) + \mathcal{L}(u(t-1)) - \mathcal{L}(u(t-3)) + \mathcal{L}(4u(t-3) - t\,u(t-3) + (t-4)\,u(t-4))$$

$$= \overset{①}{\mathcal{L}(t\,u(t))} - \overset{②}{\mathcal{L}(t\,u(t-1))} + \overset{③}{\mathcal{L}(u(t-1))} - \overset{④}{\mathcal{L}(u(t-3))} + 4\overset{⑤}{\mathcal{L}(u(t-3))} - \overset{⑥}{\mathcal{L}(t\,u(t-3))} + \overset{⑦}{\mathcal{L}((t-4)\,u(t-4))}$$

$$① = \mathcal{L}(t\,u(t)) = e^{-0s}\,\mathcal{L}(t) = \frac{1}{s^2}$$

$$② = -\mathcal{L}(t\,u(t-1)) = -\mathcal{L}((t-1)\,u(t-1)) - \mathcal{L}(u(t-1))$$

$$= -e^{-s}\frac{1}{s^2} - \frac{e^{-s}}{s}$$

$$= -\frac{e^{-s}}{s}\left(\frac{1}{s}+1\right)$$

$$③ = \mathcal{L}(u(t-1)) = \frac{e^{-s}}{s}$$

$$④ = -\mathcal{L}(u(t-3)) = -\frac{e^{-3s}}{s}$$

$$⑤ = 4\,\mathcal{L}(u(t-3)) = 4\,\frac{e^{-3s}}{s}$$

$$⑥ = -\mathcal{L}(t\,u(t-3)) = -\mathcal{L}((t-3)\,u(t-3)) - 3\,\mathcal{L}(u(t-3))$$

$$= -\frac{e^{-3s}}{s^2} - \frac{3e^{-3s}}{s}$$

$$= -\frac{e^{-3s}}{s}\left(\frac{1}{s}+3\right)$$

$$⑦ = \mathcal{L}((t-4)\,u(t-4)) = \frac{e^{-4s}}{s^2}$$

$$\therefore\ \mathcal{L}(f(t)) = \frac{1}{s^2} - \frac{e^{-s}}{s}\left(\frac{1}{s}\cancel{+1}\right) \cancel{+\frac{e^{-s}}{s}} - \frac{e^{-3s}}{s}$$

$$+ \cancel{4\frac{e^{-3s}}{s}} - \frac{e^{-3s}}{s}\left(\frac{1}{s}+\cancel{3}\right) + \frac{e^{-4s}}{s^2}$$

$$\mathcal{L}(f(t)) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-3s}}{s^2} + \frac{e^{-4s}}{s^2}$$

$$\boxed{\mathcal{L}(f(t)) = \frac{1}{s^2}\left(1 - e^{-s} - e^{-3s} + e^{-4s}\right)}$$

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10. $$f(t) = \begin{cases} t, & 0 < t < 1 \\ t-1, & 1 < t < 2 \\ t-2, & 2 < t < 3 \\ \dots & \end{cases}$$

(Despreza-se os pontos extremos do intervalo]

$$f(t) = t\left[u(t) - u(t-1)\right] + (t-1)\left[u(t-1) - u(t-2)\right] + (t-2)\left[u(t-2) - u(t-3)\right] + \dots$$

$$= t\,u(t) - u(t-1) + (\cancel{-t}+1 \cancel{+t}-2)\,u(t-2) + (\cancel{-t}+2 \cancel{+t}-3)\,u(t-3) + \dots$$

$$= t\,u(t) - u(t-1) - u(t-2) - u(t-3) + \dots$$

$$\mathcal{L}(f(t)) = \mathcal{L}(t\,u(t)) - \mathcal{L}(u(t-1)) - \mathcal{L}(u(t-2)) - \mathcal{L}(u(t-3)) + \dots$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} \dots$$

$$= \frac{1}{s^2} - \frac{1}{s}\left(\underbrace{e^{-s}}_{a} + e^{-2s} + e^{-3s} + \dots\right)$$

$$, q = e^{-s}$$

$$S_n = a + aq + \dots aq^{n-1}$$

$$qS_n = \quad aq + \dots aq^{n-1} + aq^n$$

$$S_n(1-q) = a - aq^n = a(1-q^n)$$

$$S_n = \frac{a(1-q^n)}{1-q}$$

$$\therefore \quad S_n = \frac{e^{-s}(1-e^{-ns})}{1-e^{-s}}$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{e^{-s}(1-\cancel{e^{-ns}}^{\,0})}{1-e^{-s}}$$

$$= \frac{e^{-s}}{1-e^{-s}}$$

$$\therefore \quad \boxed{\mathcal{L}(f(x)) = \frac{1}{s^2} - \frac{1}{s}\,\frac{e^{-s}}{1-e^{-s}}}$$

11. $f(t) = \begin{cases} t\,, & 0 < t < 1 \\ -t+2\,, & 1 < t < 3 \\ -1\,, & 3 < t < 4 \end{cases}$

[Despreza-se os pontos extremos de cada intervalo]

$$f(t) = t\,[u(t) - u(t-1)] + (2-t)\,[u(t-1) - u(t-3)]$$
$$+ -1\,[u(t-3) - u(t-4)]$$

$$= t\,u(t) + (-t+2-t)\,u(t-1)$$
$$+ (-2+t-1)\,u(t-3) + u(t-4)$$

$$f(t) = t\,u(t) - 2(t-1)\,u(t-1) +$$
$$+ (t-3)\,u(t-3) + u(t-4)$$

$\therefore$

$$\mathcal{L}(f(t)) = \mathcal{L}(t\,u(t)) - 2\,\mathcal{L}((t-1)\,u(t-1))$$
$$+ \mathcal{L}((t-3)\,u(t-3)) + \mathcal{L}(u(t-4))$$

$$= \frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-3s}}{s^2}$$
$$+ \frac{e^{-4s}}{s}$$

$$\boxed{\mathcal{L}(f(t)) = \frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-3s}}{s^2} + \frac{e^{-4s}}{s}}$$

12. $\dfrac{e^{-4s}-e^{-s}}{s^3}$; $\mathcal{L}\left(f(t-a)u(t-a)\right) = e^{-as}\mathcal{L}(f(t))$

$$\mathcal{L}^{-1}\left(\frac{e^{-4s}-e^{-s}}{s^3}\right) = \mathcal{L}^{-1}\left(\frac{e^{-4s}}{s^3}\right) - \mathcal{L}^{-1}\left(\frac{e^{-s}}{s^3}\right)$$

$$= \mathcal{L}^{-1}\left(e^{-4s}\mathcal{L}\left(\frac{t^2}{2}\right)\right) - \mathcal{L}^{-1}\left(e^{-s}\mathcal{L}\left(\frac{t^2}{2}\right)\right)$$

$$\boxed{\mathcal{L}^{-1}\left(\frac{e^{-4s}-e^{-s}}{s^3}\right) = \frac{(t-4)^2}{2}u(t-4) - \frac{(t-1)^2}{2}u(t-1)}$$

13. $\dfrac{e^{-3s}}{s^2-9}$; $\mathcal{L}\left(f(t-a)u(t-a)\right) = e^{-as}\mathcal{L}(f(t))$

$$\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^2-9}\right) = \mathcal{L}^{-1}\left(e^{-3s}\mathcal{L}\left(\frac{1}{3}\sinh 3t\right)\right)$$

$$\boxed{\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^2-9}\right) = \frac{1}{3}\sinh 3(t-3)\,u(t-3)}$$

14. $\dfrac{e^{-s}}{(s-1)(s-2)}$; $\mathcal{L}\left(f(t-a)\,u(t-a)\right) = e^{-as}\mathcal{L}(f(t))$

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{(s-1)(s-2)}\right) =$$

$$\frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

$$\frac{1}{(s-1)(s-2)} = \frac{As - 2A + Bs - B}{(s-1)(s-2)}$$

$$\therefore \quad 1 = (A+B)s - 2A - B$$

$$\Rightarrow \quad A+B=0 \quad \Rightarrow \quad \underline{\underline{A=-B}}$$

$$-2A - B = 1$$

$$-2A + A = 1$$

$$-A = 1 \quad \therefore \quad \underline{\underline{A=-1, \; B=+1}}$$

$\therefore$

$$\frac{1}{(s-1)(s-2)} = \frac{-1}{s-1} + \frac{1}{s-2}$$

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{(s-1)(s-2)}\right) = \mathcal{L}^{-1}\left(\frac{-e^{-s}}{s-1}\right) + \mathcal{L}^{-1}\left(\frac{e^{-s}}{s-2}\right)$$

$$= -\mathcal{L}^{-1}\left(e^{-s}\mathcal{L}(e^{+t})\right) + \mathcal{L}^{-1}\left(e^{-s}\mathcal{L}(e^{2t})\right)$$

$$= -e^{t-1}u(t-1) + e^{2(t-1)}u(t-1)$$

$$= \left[e^{2(t-1)} - e^{(t-1)}\right] u(t-1)$$

15. $\dfrac{e^{-s}}{(s+1)^3}$; $\mathcal{L}\left(f(t-a)u(t-a)\right) = e^{-as}\mathcal{L}\left(f(t)\right)$

$$\mathcal{L}^{-1}\left(\underbrace{\frac{e^{-s}}{(s+1)^3}}\right) = \mathcal{L}^{-1}\left(e^{-s}\mathcal{L}\underbrace{\left(e^{-t}\frac{t^2}{2}\right)}\right)$$

$$= e^{-(t-1)}\frac{(t-1)^2}{2}u(t-1)$$

$$\boxed{\mathcal{L}^{-1}\left(\frac{e^{-s}}{(s+1)^3}\right) = \frac{1}{2}e^{-(t-1)}(t-1)^2u(t-1)}$$

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16. $\dfrac{(s-2)e^{-s}}{s^2-4s+3} = \dfrac{(s-2)e^{-s}}{(s-2)^2-1}$

$$\mathcal{L}^{-1}\left(\frac{(s-2)e^{-s}}{s^2-4s+3}\right) = \mathcal{L}^{-1}\left(\frac{(s-2)e^{-s}}{(s-2)^2-1}\right)$$

$$= \mathcal{L}^{-1}\left(e^{-s}\mathcal{L}\left(e^{2t}\cosh t\right)\right)$$

$$= \boxed{e^{2(t-1)}\cosh(t-1)\,u(t-1)}$$

ou

$$= e^{2(t-1)}\frac{e^{t-1}+e^{-(t-1)}}{2}u(t-1)$$

$$= \boxed{\frac{1}{2}\left(e^{3(t-1)}+e^{(t-1)}\right)u(t-1)}$$

17. $t\,u(t-2)$

$$\mathcal{L}(t\,u(t-2)) = \mathcal{L}((t-2)\,u(t-2) + 2\,u(t-2))$$

$$= \mathcal{L}((t-2)\,u(t-2)) + 2\,\mathcal{L}(u(t-2))$$

$$= e^{-2s}\,\mathcal{L}(t) + 2\,\frac{e^{-2s}}{s}$$

$$= \frac{e^{-2s}}{s^2} + 2\,\frac{e^{-2s}}{s}$$

$$= \boxed{\frac{e^{-2s}}{s}\left(\frac{1}{s} + 2\right)} \quad , \; s > 0$$

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18. $\cos t\;u(t-\pi)$

$$\mathcal{L}(\cos t\;u(t-\pi)) = \mathcal{L}(-\cos(t-\pi)\,u(t-\pi))$$

$$= -\,\mathcal{L}(\cos(t-\pi)\,u(t-\pi))$$

$$= -\,e^{-\pi s}\,\mathcal{L}(\cos t)$$

$$= -\,e^{-\pi s}\,\frac{s}{s^2+1} \quad , \; s > 0$$

$$= \boxed{\frac{-s\,e^{-\pi s}}{s^2+1}} \quad , \; s > 0$$

19. $e^t u(t-3)$

$$\mathcal{L}(e^t u(t-3)) = \mathcal{L}(e^{t-3} e^3 u(t-3))$$

$$= e^3 \mathcal{L}(e^{t-3} u(t-3))$$

$$= e^3 e^{-3s} \mathcal{L}(e^t)$$

$$= e^3 e^{-3s} \frac{1}{s-1}$$

$$= \boxed{\frac{e^{3(1-s)}}{s-1}, \quad s > 1.}$$

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20. $\sin t \, u(t-\frac{\pi}{2})$

$$\mathcal{L}(\sin t \, u(t-\tfrac{\pi}{2})) = \mathcal{L}(\cos(t-\tfrac{\pi}{2}) \, u(t-\tfrac{\pi}{2}))$$

$$= e^{-\frac{\pi}{2}s} \mathcal{L}(\cos t)$$

$$= e^{-\frac{\pi}{2}s} \frac{s}{s^2+1}$$

$$= \boxed{\frac{s e^{-\frac{\pi}{2}s}}{s^2+1}, \quad s > 0}$$