Integral Definida

Ex. $$\int_4^7 |x-5|\, dx$$

Temos

$$(*) \quad |x-5| = \begin{cases} x-5 & \text{se } x-5 \geq 0 \therefore \underline{\underline{x \geq 5}} \\ -x+5 & \text{se } x-5 < 0 \therefore \underline{\underline{x < 5}} \end{cases}$$

Daí, usando a propriedade:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

temos:

$$\int_4^7 |x-5|\,dx = \int_4^5 \underbrace{|x-5|}_{\substack{\text{é negativo} \\ \text{qndo } 4<x<5}}\,dx + \int_5^7 \underbrace{|x-5|}_{\substack{\text{é positivo} \\ \text{qndo. } 5<x<7}}\,dx$$

$$\overset{\text{De }(*)}{=} \int_4^5 (-x+5)\,dx + \int_5^7 (x-5)\,dx$$

$$= \underbrace{\left(-\frac{x^2}{2}+5x\right)\Big|_4^5}\; + \left(\frac{x^2}{2}-5x\right)\Big|_5^7$$

$$= \cancel{-\frac{25}{2}} + \cancel{25} - \left(-\frac{16}{2}+20\right) + \left(\frac{49}{2}-35\right) - \left(\cancel{\frac{25}{2}} - 25\right) = \frac{5}{2}.$$

4. $\displaystyle\int_0^{\frac{\pi}{2}} f(x)\,dx$

$$f(x) = \begin{cases} \sec^2 x, & 0 \leq x \leq \frac{\pi}{4} \\ \operatorname{cosec}^2 x, & \frac{\pi}{4} < x \leq \frac{\pi}{2} \end{cases}$$

Temos :

$$\int_0^{\frac{\pi}{2}} f(x)\,dx = \int_0^{\frac{\pi}{4}} \underbrace{f(x)}_{\substack{f(x) = \sec^2 x \\ \text{qndo. } 0 < x < \frac{\pi}{4}}} dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \underbrace{f(x)}_{\substack{f(x) = \operatorname{cosec}^2 x \\ \text{qndo. } \frac{\pi}{4} < x < \frac{\pi}{2}}} dx$$

$$= \int_0^{\frac{\pi}{4}} \sec^2 x\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \operatorname{cosec}^2 x\,dx$$

$$= \operatorname{tg} x \Big]_0^{\frac{\pi}{4}} + (-\operatorname{cotg} x) \Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \operatorname{tg}\frac{\pi}{4} - \cancelto{0}{\operatorname{tg} 0} + \left(-\cancelto{0}{\operatorname{cotg}\frac{\pi}{2}} - \underbrace{\left(-\operatorname{cotg}\frac{\pi}{4}\right)}_{1}\right)$$

$$= 1 + 1$$

$$= 2$$

Ex. $$\int_{3/2}^{3} \frac{dx}{x\sqrt{4x^2-9}}$$

Lembremos que

$$\int \frac{dx}{x\sqrt{x^2-1}} = \operatorname{arcsec} x$$

Então:

$$\int \frac{dx}{x\sqrt{4x^2-9}} = \int \frac{dx}{x\sqrt{9\left(\frac{4x^2}{9}-1\right)}} = \int \frac{dx}{3x\sqrt{\frac{4}{9}x^2-1}}$$

$$= \frac{1}{3}\int \frac{dx}{x\sqrt{\left(\frac{2x}{3}\right)^2-1}}$$

Seja agora

$$u = \frac{2}{3}x \quad \therefore \quad du = \frac{2}{3}dx$$

$$\therefore \quad \frac{3}{2}du = dx$$

$\therefore$

$$\frac{1}{3}\int \frac{dx}{x\sqrt{\left(\frac{2}{3}x\right)^2-1}} = \frac{1}{3}\int \frac{\cancel{\frac{3}{2}}du}{\cancel{\frac{3}{2}}u\sqrt{u^2-1}}$$

$$= \frac{1}{3}\int \frac{du}{u\sqrt{u^2-1}} = \frac{1}{3}\operatorname{arcsec} u$$

$\rightarrow$

$$= \frac{1}{3} \operatorname{arc\,sec} u$$

$$\downarrow$$

$$= \frac{1}{3} \operatorname{arc\,sec} \frac{2}{3} x$$

$\therefore$

$$\int \frac{dx}{x\sqrt{4x^2-9}} = \frac{1}{3} \operatorname{arc\,sec} \frac{2}{3} x$$

Daí

$$\int_{\frac{3}{2}}^{3} \frac{dx}{x\sqrt{4x^2-9}} = \frac{1}{3} \operatorname{arc\,sec} \frac{2}{3} x \Big]_{\frac{3}{2}}^{3}$$

$$= \frac{1}{3} \operatorname{arc\,sec} \left(\frac{2}{\cancel{3}} \cdot \cancel{3}\right) - \frac{1}{3} \operatorname{arc\,sec} \left(\frac{2}{3} \cdot \frac{3}{2}\right)$$

$$= \frac{1}{3} \operatorname{arc\,sec} 2 - \frac{1}{3} \operatorname{arc\,sec} 1 \quad (*)$$

Mas $y = \operatorname{arc\,sec} x \quad \therefore \begin{cases} \sec y = x \\ y \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2}) \end{cases}$

Daí: $y = \operatorname{arc\,sec} 2 \quad \therefore \sec y = 2 \quad \therefore \cos y = \frac{1}{2}$

$\therefore y = \frac{\pi}{3}$

$\therefore \operatorname{arc\,sec} 2 = \frac{\pi}{3} \quad (**)$

$y = \text{arc sec } 1 \quad \therefore \quad \sec y = 1 \quad \therefore \quad \cos y = 1$

$\therefore \quad y = 0$

$\therefore \quad \text{arc sec } 1 = 0 \quad (***)$

Substituindo $(**)$ e $(***)$ em $(*)$ temos:

$$\int_{\frac{3}{2}}^{3} \frac{dx}{x\sqrt{4x^2-9}} = \frac{1}{3}\,\frac{\pi}{3} - \frac{1}{3}\cdot 0$$

$$= \frac{\pi}{9}$$

Ex. $\displaystyle\int_0^4 \frac{dx}{\sqrt{16-x^2}}$

Lembremos que

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$$

Então

$$\int \frac{dx}{\sqrt{16-x^2}} = \int \frac{dx}{\sqrt{16\left(1-\frac{x^2}{16}\right)}} =$$

$$= \int \frac{dx}{4\sqrt{1-\frac{x^2}{16}}} = \frac{1}{4}\int \frac{dx}{\sqrt{1-\left(\frac{x}{4}\right)^2}}$$

Seja $u = \frac{x}{4} \therefore du = \frac{1}{4}dx \therefore dx = 4du$

$$\therefore \frac{1}{4}\int \frac{dx}{\sqrt{1-\left(\frac{x}{4}\right)^2}} = \frac{1}{\cancel{4}}\int \frac{\cancel{4}du}{\sqrt{1-u^2}} = \int \frac{du}{\sqrt{1-u^2}}$$

$$= \arcsin u$$

$$= \arcsin \frac{x}{4}$$

Daí

$$\int_0^4 \frac{dx}{\sqrt{16-x^2}} = \arcsin \frac{x}{4}\Big]_0^4 =$$

$$= \arcsin 1 - \arcsin 0 \quad (*) \longrightarrow$$

Mas

$$y = \arcsin x \quad \therefore \quad \begin{cases} \sin x = y \\ y \in [-\frac{\pi}{2}, \frac{\pi}{2}] \end{cases}$$

Daí

$$y = \arcsin 1 \quad \therefore \quad \sin y = 1 \quad \therefore \quad y = \frac{\pi}{2} \quad (**)$$

$$y = \arcsin 0 \quad \therefore \quad \sin 0 = y \quad \therefore \quad y = 0 \quad (***)$$

Substituindo (**) e (***) em (*) ;

$$\int_0^4 \frac{dx}{\sqrt{16 - x^2}} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$