EXAMPLE 5 Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

Resources / Module 5 / Max and Min / Start of Max and Min

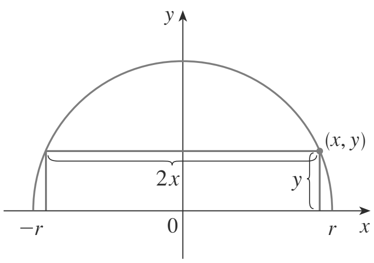


FIGURE 9

SOLUTION 1 Let's take the semicircle to be the upper half of the circle $x^2 + y^2 = r^2$ with center the origin. Then the word *inscribed* means that the rectangle has two vertices on the semicircle and two vertices on the x-axis as shown in Figure 9.

Let (x, y) be the vertex that lies in the first quadrant. Then the rectangle has sides of lengths $2x$ and y, so its area is

$$A = 2xy$$

To eliminate y we use the fact that (x, y) lies on the circle $x^2 + y^2 = r^2$ and so $y = \sqrt{r^2 - x^2}$. Thus

$$A = 2x\sqrt{r^2 - x^2}$$

The domain of this function is $0 \leqslant x \leqslant r$. Its derivative is

$$A' = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}$$

which is 0 when $2x^2 = r^2$, that is, $x = r/\sqrt{2}$ (since $x \geqslant 0$). This value of x gives a maximum value of A since $A(0) = 0$ and $A(r) = 0$. Therefore, the area of the largest inscribed rectangle is

$$A\left(\frac{r}{\sqrt{2}}\right) = 2\frac{r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}} = r^2$$

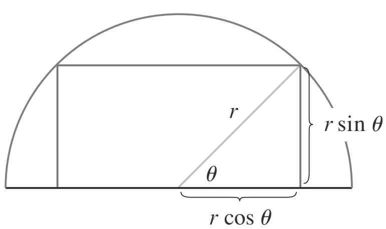


FIGURE 10

SOLUTION 2 A simpler solution is possible if we think of using an angle as a variable. Let θ be the angle shown in Figure 10. Then the area of the rectangle is

$$A(\theta) = (2r\cos\theta)(r\sin\theta) = r^2(2\sin\theta\,\cos\theta) = r^2\sin 2\theta$$

We know that $\sin 2\theta$ has a maximum value of 1 and it occurs when $2\theta = \pi/2$. So $A(\theta)$ has a maximum value of r^2 and it occurs when $\theta = \pi/4$.

Notice that this trigonometric solution doesn't involve differentiation. In fact, we didn't need to use calculus at all.

4.7 Exercises

1. Consider the following problem: Find two numbers whose sum is 23 and whose product is a maximum.

(a) Make a table of values, like the following one, so that the sum of the numbers in the first two columns is always 23. On the basis of the evidence in your table, estimate the answer to the problem.

First number	Second number	Product
1	22	22
2	21	42
3	20	60
.	.	.
.	.	.
.	.	.

(b) Use calculus to solve the problem and compare with your answer to part (a).

2. Find two numbers whose difference is 100 and whose product is a minimum.

3. Find two positive numbers whose product is 100 and whose sum is a minimum.

4. Find a positive number such that the sum of the number and its reciprocal is as small as possible.

5. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

6. Find the dimensions of a rectangle with area 1000 m^2 whose perimeter is as small as possible.

7. Consider the following problem: A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?
(a) Draw several diagrams illustrating the situation, some with shallow, wide pens and some with deep, narrow pens. Find the total areas of these configurations. Does it appear that there is a maximum area? If so, estimate it.
(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
(c) Write an expression for the total area.
(d) Use the given information to write an equation that relates the variables.
(e) Use part (d) to write the total area as a function of one variable.
(f) Finish solving the problem and compare the answer with your estimate in part (a).

8. Consider the following problem: A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.
(a) Draw several diagrams to illustrate the situation, some short boxes with large bases and some tall boxes with small bases. Find the volumes of several such boxes. Does it appear that there is a maximum volume? If so, estimate it.
(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
(c) Write an expression for the volume.
(d) Use the given information to write an equation that relates the variables.
(e) Use part (d) to write the volume as a function of one variable.
(f) Finish solving the problem and compare the answer with your estimate in part (a).

9. A farmer wants to fence an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence?

10. A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.

11. If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

12. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.

13. Do Exercise 12 assuming the container has a lid that is made from the same material as the sides.

14. (a) Show that of all the rectangles with a given area, the one with smallest perimeter is a square.
(b) Show that of all the rectangles with a given perimeter, the one with greatest area is a square.

15. Find the point on the line $y = 4x + 7$ that is closest to the origin.

16. Find the point on the line $6x + y = 9$ that is closest to the point $(-3, 1)$.

17. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

18. Find, correct to two decimal places, the coordinates of the point on the curve $y = \tan x$ that is closest to the point $(1, 1)$.

19. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.

20. Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.

21. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side L if one side of the rectangle lies on the base of the triangle.

22. Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 8 - x^2$.

23. Find the dimensions of the isosceles triangle of largest area that can be inscribed in a circle of radius r.

24. Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 cm and 4 cm if two sides of the rectangle lie along the legs.

25. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

26. A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest possible volume of such a cylinder.

27. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible surface area of such a cylinder.

28. A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus, the diameter of the semicircle is equal to the width of the rectangle. See Exercise 52 on page 24.) If the perimeter of the window is 30 ft, find the dimensions of the window so that the greatest possible amount of light is admitted.

29. The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of printed material on the poster is fixed at 384 cm^2, find the dimensions of the poster with the smallest area.

30. A poster is to have an area of 180 in^2 with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions will give the largest printed area?

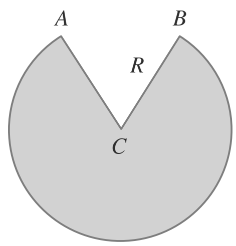
31. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) A minimum?

32. Answer Exercise 31 if one piece is bent into a square and the other into a circle.

33. A cylindrical can without a top is made to contain V cm^3 of liquid. Find the dimensions that will minimize the cost of the metal to make the can.

34. A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

35. A cone-shaped drinking cup is made from a circular piece of paper of radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.



36. A cone-shaped paper drinking cup is to be made to hold 27 cm^3 of water. Find the height and radius of the cup that will use the smallest amount of paper.

37. A cone with height h is inscribed in a larger cone with height H so that its vertex is at the center of the base of the larger cone. Show that the inner cone has maximum volume when $h = \frac{1}{3}H$.

38. For a fish swimming at a speed v relative to the water, the energy expenditure per unit time is proportional to v^3. It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current u $(u < v)$, then the time required to swim a distance L is $L/(v - u)$ and the total energy E required to swim the distance is given by

$$E(v) = av^3 \cdot \frac{L}{v - u}$$

where a is the proportionality constant.
(a) Determine the value of v that minimizes E.
(b) Sketch the graph of E.

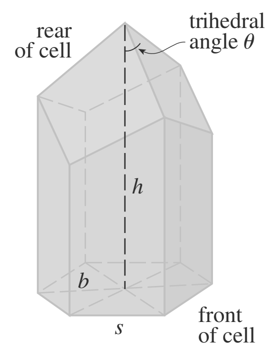
Note: This result has been verified experimentally; migrating fish swim against a current at a speed 50% greater than the current speed.

39. In a beehive, each cell is a regular hexagonal prism, open at one end with a trihedral angle at the other end. It is believed that bees form their cells in such a way as to minimize the surface area for a given volume, thus using the least amount of wax in cell construction. Examination of these cells has shown that the measure of the apex angle θ is amazingly consistent. Based on the geometry of the cell, it can be shown that the surface area S is given by

$$S = 6sh - \tfrac{3}{2}s^2 \cot\theta + \left(3s^2\sqrt{3}/2\right)\csc\theta$$

where s, the length of the sides of the hexagon, and h, the height, are constants.
(a) Calculate $dS/d\theta$.
(b) What angle should the bees prefer?
(c) Determine the minimum surface area of the cell (in terms of s and h).

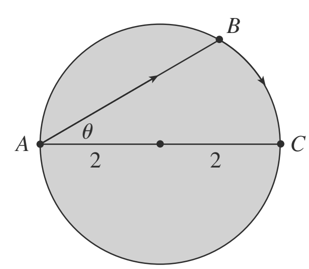
Note: Actual measurements of the angle θ in beehives have been made, and the measures of these angles seldom differ from the calculated value by more than 2°.



40. A boat leaves a dock at 2:00 P.M. and travels due south at a speed of 20 km/h. Another boat has been heading due east at 15 km/h and reaches the same dock at 3:00 P.M. At what time were the two boats closest together?

41. Solve the problem in Example 4 if the river is 5 km wide and point B is only 5 km downstream from A.

42. A woman at a point A on the shore of a circular lake with radius 2 mi wants to arrive at the point C diametrically opposite A on the other side of the lake in the shortest possible time. She can walk at the rate of 4 mi/h and row a boat at 2 mi/h. How should she proceed?



43. The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. If two light sources, one three times as strong as the other, are placed 10 ft apart, where should an object be placed on the line between the sources so as to receive the least illumination?

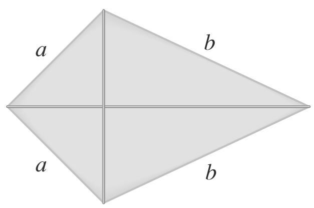
44. Find an equation of the line through the point (3, 5) that cuts off the least area from the first quadrant.

45. Let a and b be positive numbers. Find the length of the shortest line segment that is cut off by the first quadrant and passes through the point (a, b).

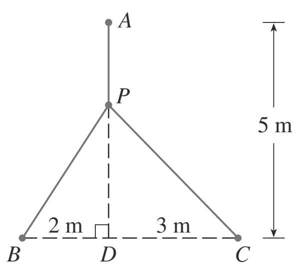
46. At which points on the curve $y = 1 + 40x^3 - 3x^5$ does the tangent line have the largest slope?

47. Show that of all the isosceles triangles with a given perimeter, the one with the greatest area is equilateral.

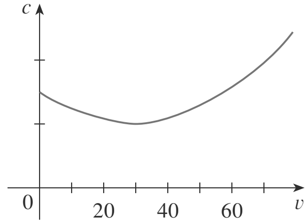
CAS **48.** The frame for a kite is to be made from six pieces of wood. The four exterior pieces have been cut with the lengths indicated in the figure. To maximize the area of the kite, how long should the diagonal pieces be?



49. A point P needs to be located somewhere on the line AD so that the total length L of cables linking P to the points A, B, and C is minimized (see the figure). Express L as a function of $x = |AP|$ and use the graphs of L and dL/dx to estimate the minimum value.



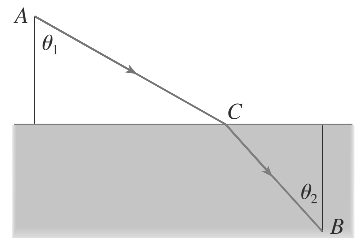
50. The graph shows the fuel consumption c of a car (measured in gallons per hour) as a function of the speed v of the car. At very low speeds the engine runs inefficiently, so initially c decreases as the speed increases. But at high speeds the fuel consumption increases. You can see that $c(v)$ is minimized for this car when $v \approx 30$ mi/h. However, for fuel efficiency, what must be minimized is not the consumption in gallons per hour but rather the fuel consumption in gallons *per mile.* Let's call this consumption G. Using the graph, estimate the speed at which G has its minimum value.



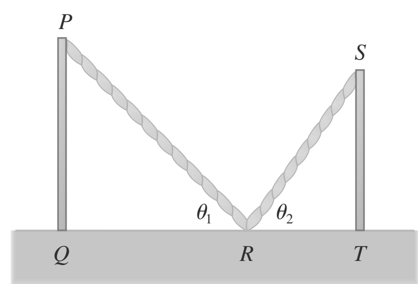
51. Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to Fermat's Principle, a ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimizes the time taken. Show that

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}$$

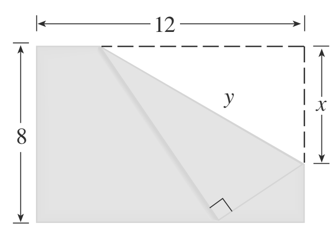
where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are as shown. This equation is known as Snell's Law.



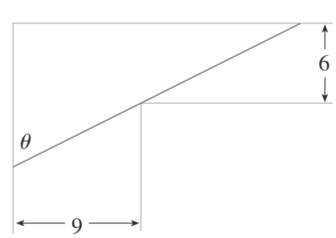
52. Two vertical poles PQ and ST are secured by a rope PRS going from the top of the first pole to a point R on the ground between the poles and then to the top of the second pole as in the figure. Show that the shortest length of such a rope occurs when $\theta_1 = \theta_2$.



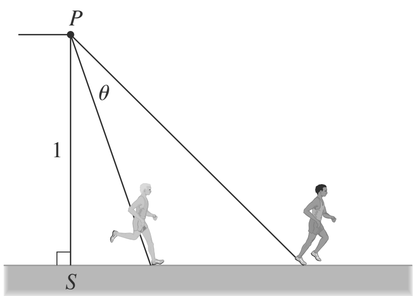
53. The upper right-hand corner of a piece of paper, 12 in. by 8 in., as in the figure, is folded over to the bottom edge. How would you fold it so as to minimize the length of the fold? In other words, how would you choose x to minimize y?



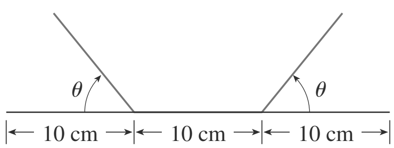
54. A steel pipe is being carried down a hallway 9 ft wide. At the end of the hall there is a right-angled turn into a narrower hallway 6 ft wide. What is the length of the longest pipe that can be carried horizontally around the corner?



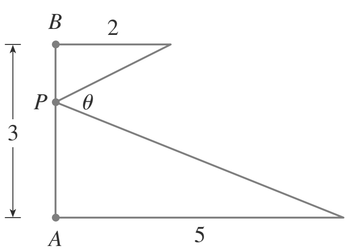
55. An observer stands at a point P, one unit away from a track. Two runners start at the point S in the figure and run along the track. One runner runs three times as fast as the other. Find the maximum value of the observer's angle of sight θ between the runners. [*Hint:* Maximize $\tan \theta$.]



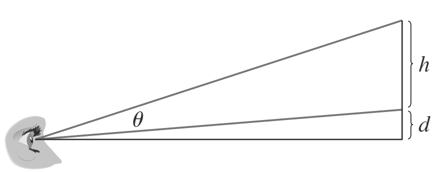
56. A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle θ. How should θ be chosen so that the gutter will carry the maximum amount of water?



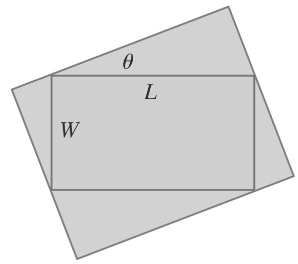
57. Where should the point P be chosen on the line segment AB so as to maximize the angle θ?



58. A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer (as in the figure). How far from the wall should the observer stand to get the best view? (In other words, where should the observer stand so as to maximize the angle θ subtended at his eye by the painting?)



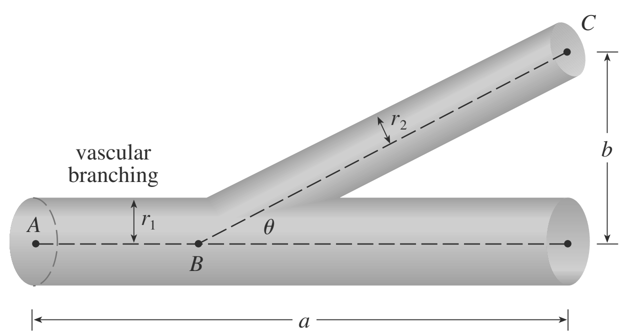
59. Find the maximum area of a rectangle that can be circumscribed about a given rectangle with length L and width W.



60. The blood vascular system consists of blood vessels (arteries, arterioles, capillaries, and veins) that convey blood from the heart to the organs and back to the heart. This system should work so as to minimize the energy expended by the heart in pumping the blood. In particular, this energy is reduced when the resistance of the blood is lowered. One of Poiseuille's Laws gives the resistance R of the blood as

$$R = C\frac{L}{r^4}$$

where L is the length of the blood vessel, r is the radius, and C is a positive constant determined by the viscosity of the blood. (Poiseuille established this law experimentally, but it also follows from Equation 8.4.2.) The figure shows a main blood vessel with radius r_1 branching at an angle θ into a smaller vessel with radius r_2.



(a) Use Poiseuille's Law to show that the total resistance of the blood along the path ABC is

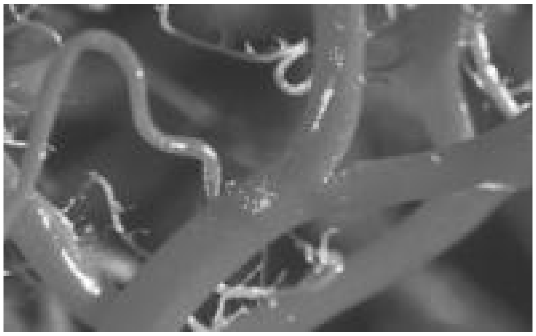
$$R = C\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right)$$

where a and b are the distances shown in the figure.

(b) Prove that this resistance is minimized when

$$\cos\theta = \frac{r_2^4}{r_1^4}$$

(c) Find the optimal branching angle (correct to the nearest degree) when the radius of the smaller blood vessel is two-thirds the radius of the larger vessel.



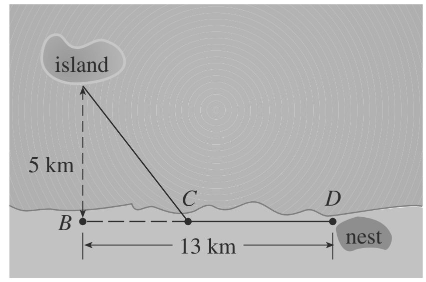
61. Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours. It is believed that more energy is required to fly over water than land because air generally rises over land and falls over water during the day. A bird with these tendencies is released from an island that is 5 km from the nearest point B on a straight shoreline, flies to a point C on the shoreline, and then flies along the shoreline to its nesting area D. Assume that the bird instinctively chooses a path that will minimize its energy expenditure. Points B and D are 13 km apart.

(a) In general, if it takes 1.4 times as much energy to fly over water as land, to what point C should the bird fly in order to minimize the total energy expended in returning to its nesting area?

(b) Let W and L denote the energy (in joules) per kilometer flown over water and land, respectively. What would a large value of the ratio W/L mean in terms of the bird's flight? What would a small value mean? Determine the ratio W/L corresponding to the minimum expenditure of energy.

(c) What should the value of W/L be in order for the bird to fly directly to its nesting area D? What should the value of W/L be for the bird to fly to B and then along the shore to D?

(d) If the ornithologists observe that birds of a certain species reach the shore at a point 4 km from B, how many times more energy does it take a bird to fly over water than land?



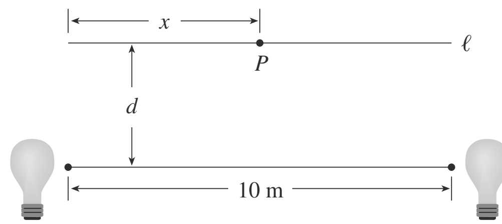
62. Two light sources of identical strength are placed 10 m apart. An object is to be placed at a point P on a line ℓ parallel to the line joining the light sources and at a distance d meters from it (see the figure). We want to locate P on ℓ so that the intensity of illumination is minimized. We need to use the fact that the intensity of illumination for a single source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source.

(a) Find an expression for the intensity $I(x)$ at the point P.

(b) If $d = 5$ m, use graphs of $I(x)$ and $I'(x)$ to show that the intensity is minimized when $x = 5$ m, that is, when P is at the midpoint of ℓ.

(c) If $d = 10$ m, show that the intensity (perhaps surprisingly) is *not* minimized at the midpoint.

(d) Somewhere between $d = 5$ m and $d = 10$ m there is a transitional value of d at which the point of minimal illumination abruptly changes. Estimate this value of d by graphical methods. Then find the exact value of d.



APPLIED PROJECT

The Shape of a Can

In this project we investigate the most economical shape for a can. We first interpret this to mean that the volume V of a cylindrical can is given and we need to find the height h and radius r that minimize the cost of the metal to make the can (see the figure). If we disregard any waste metal in the manufacturing process, then the problem is to minimize the surface area of the cylinder. We solved this problem in Example 2 in Section 4.7 and we found that $h = 2r$; that is, the height should be the same as the diameter. But if you go to your cupboard or your supermarket with a ruler, you will discover that the height is usually greater than the diameter and the ratio h/r varies from 2 up to about 3.8. Let's see if we can explain this phenomenon.

1. **(a)**

First Number	Second Number	Product
1	22	22
2	21	42
3	20	60
4	19	76
5	18	90
6	17	102
7	16	112
8	15	120
9	14	126
10	13	130
11	12	132

We needn't consider pairs where the first number is larger than the second, since we can just interchange the numbers in such cases. The answer appears to be 11 and 12 , but we have considered only integers in the table.

(b) Call the two numbers x and y . Then $x+y=23$, so $y=23-x$. Call the product P . Then $P=xy=x(23-x)=23x-x^2$, so we wish to maximize the function $P(x)=23x-x^2$. Since $P'(x)=23-2x$, we see that $P'(x)=0 \Leftrightarrow x=\frac{23}{2}=11.5$. Thus, the maximum value of P is $P(11.5)=(11.5)^2=132.25$ and it occurs when $x=y=11.5$.

Or: Note that $P''(x)=-2<0$ for all x , so P is everywhere concave downward and the local maximum at $x=11.5$ must be an absolute maximum.

2. The two numbers are $x+100$ and x . Minimize $f(x)=(x+100)x=x^2+100x$. $f'(x)=2x+100=0 \Rightarrow x=-50$. Since $f''(x)=2>0$, there is an absolute minimum at $x=-50$. The two numbers are 50 and -50 .

3. The two numbers are x and $\frac{100}{x}$, where $x>0$. Minimize $f(x)=x+\frac{100}{x}$. $f'(x)=1-\frac{100}{x^2}=\frac{x^2-100}{x^2}$.

The critical number is $x=10$. Since $f'(x)<0$ for $0<x<10$ and $f'(x)>0$ for $x>10$, there is an absolute minimum at $x=10$. The numbers are 10 and 10 .

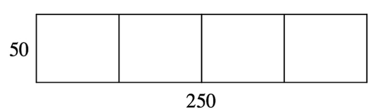
4. Let $x>0$ and let $f(x)=x+1/x$. We wish to minimize $f(x)$. Now $f'(x)=1-\frac{1}{x^2}=\frac{1}{x^2}\left(x^2-1\right)=\frac{1}{x^2}(x+1)(x-1)$, so the only critical number in $(0,\infty)$ is 1 .

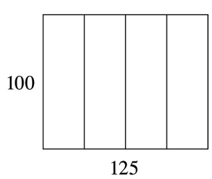
$f'(x)<0$ for $0<x<1$ and $f'(x)>0$ for $x>1$, so f has an absolute minimum at $x=1$, and $f(1)=2$.
Or: $f''(x)=2/x^3>0$ for all $x>0$, so f is concave upward everywhere and the critical point $(1,2)$ must correspond to a local minimum for f .

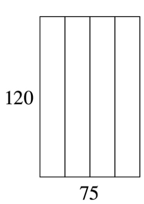
5. If the rectangle has dimensions x and y , then its perimeter is $2x+2y=100$ m, so $y=50-x$. Thus, the area is $A=xy=x(50-x)$. We wish to maximize the function $A(x)=x(50-x)=50x-x^2$, where $0<x<50$. Since $A'(x)=50-2x=-2(x-25)$, $A'(x)>0$ for $0<x<25$ and $A'(x)<0$ for $25<x<50$. Thus, A has an absolute maximum at $x=25$, and $A(25)=25^2=625$ m^2 . The dimensions of the rectangle that maximize its area are $x=y=25$ m. (The rectangle is a square.)

6. If the rectangle has dimensions x and y , then its area is $xy=1000$ m^2 , so $y=1000/x$. The perimeter $P=2x+2y=2x+2000/x$. We wish to minimize the function $P(x)=2x+2000/x$ for $x>0$.
$P'(x)=2-2000/x^2=\left(2/x^2\right)\left(x^2-1000\right)$, so the only critical number in the domain of P is $x=\sqrt{1000}$.
$P''(x)=4000/x^3>0$, so P is concave upward throughout its domain and $P\left(\sqrt{1000}\right)=4\sqrt{1000}$ is an absolute minimum value. The dimensions of the rectangle with minimal perimeter are
$x=y=\sqrt{1000}=10\sqrt{10}$ m.
(The rectangle is a square.)

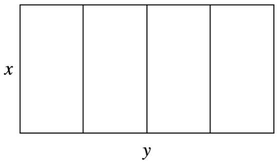
7. **(a)**



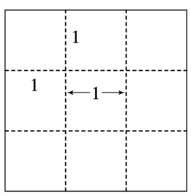


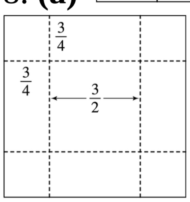


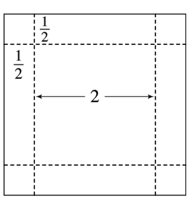
The areas of the three figures are 12 , 500 , 12 , 500 , and 9000 ft^2 . There appears to be a maximum area of at least 12 , 500 ft^2 .
(b) Let x denote the length of each of two sides and three dividers.
Let y denote the length of the other two sides.



(c) Area $A=$ length $\times$ width $=y\cdot x$
(d) Length of fencing $=750 \Rightarrow 5x+2y=750$
(e) $5x+2y=750 \Rightarrow y=375-\frac{5}{2}x \Rightarrow A(x)=\left(375-\frac{5}{2}x\right)x=375x-\frac{5}{2}x^2$
(f) $A^{\prime}(x)=375-5x=0 \Rightarrow x=75$. Since $A^{\prime\prime}(x)=-5<0$ there is an absolute maximum when $x=75$. Then $y=\frac{375}{2}=187.5$. The largest area is $75\left(\frac{375}{2}\right)=14,062.5 \text{ ft}^2$. These values of x and y are between the values in the first and second figures in part (a). Our original estimate was low.

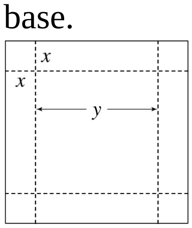
8. **(a)** 





The volumes of the resulting boxes are 1 , 1.6875 , and 2 ft^3 . There appears to be a maximum volume of at least 2 ft^3 .

(b) Let x denote the length of the side of the square being cut out. Let y denote the length of the base.



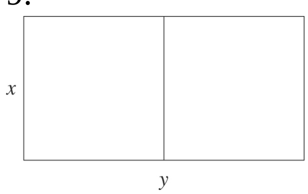
(c) Volume $V=$ length $\times$ width $\times$ height $\Rightarrow V=y\cdot y\cdot x=xy^2$

(d) Length of cardboard $=3 \Rightarrow x+y+x=3 \Rightarrow y+2x=3$

(e) $y+2x=3 \Rightarrow y=3-2x \Rightarrow V(x)=x(3-2x)^2$

(f) $V(x)=x(3-2x)^2 \Rightarrow$
$V'(x)=x\cdot 2(3-2x)(-2)+(3-2x)^2\cdot 1=(3-2x)[-4x+(3-2x)]=(3-2x)(-6x+3)$,
so the critical numbers are $x=\frac{3}{2}$ and $x=\frac{1}{2}$. Now $0\le x\le \frac{3}{2}$ and $V(0)=V\left(\frac{3}{2}\right)=0$, so the maximum is $V\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)(2)^2=2\ \text{ft}^3$, which is the value found from our third figure in part (a).

9.

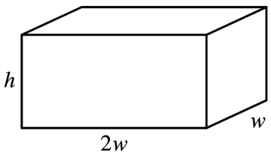


$xy=1.5\times 10^6$, so $y=1.5\times 10^6/x$. Minimize the amount of fencing, which is $3x+2y=3x+2\left(1.5\times 10^6/x\right)=3x+3\times 10^6/x=F(x)$. $F'(x)=3-3\times 10^6/x^2=3\left(x^2-10^6\right)/x^2$. The critical number is $x=10^3$ and $F'(x)<0$ for $0<x<10^3$ and $F'(x)>0$ if $x>10^3$, so the absolute minimum occurs when $x=10^3$ and $y=1.5\times 10^3$. The field should be 1000 feet by 1500 feet with the middle fence parallel to the short side of the field.

10. Let b be the length of the base of the box and h the height. The volume is $32,000=b^2h \Rightarrow h=32,000/b^2$. The surface area of the open box is $S=b^2+4hb=b^2+4(32,000/b^2)b=b^2+4(32,000)/b$. So $S'(b)=2b-4(32,000)/b^2=2\left(b^3-64,000\right)/b^2=0 \Leftrightarrow b=\sqrt[3]{64,000}=40$. This gives an absolute minimum since $S'(b)<0$ if $0<b<40$ and $S'(b)>0$ if $b>40$. The box should be $40\times 40\times 20$.

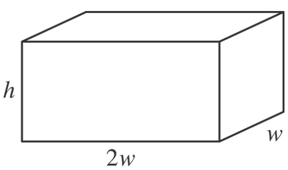
11. Let b be the length of the base of the box and h the height. The surface area is $1200=b^2+4hb\Rightarrow h=\left(1200-b^2\right)/(4b)$. The volume is $V=b^2h=b^2\left(1200-b^2\right)/4b=300b-b^3/4\Rightarrow V'(b)=300-\frac{3}{4}b^2$. $V'(b)=0\Rightarrow 300=\frac{3}{4}b^2 \Rightarrow b^2=400 \Rightarrow b=\sqrt{400}=20$. Since $V'(b)>0$ for $0<b<20$ and $V'(b)<0$ for $b>20$, there is an absolute maximum when $b=20$ by the First Derivative Test for Absolute Extreme Values (see page 280). If $b=20$, then $h=\left(1200-20^2\right)/(4\cdot 20)=10$, so the largest possible volume is $b^2h=(20)^2(10)=4000\ \text{cm}^3$.

12.



$V=lwh \Rightarrow 10=(2w)(w)h=2w^2h$, so $h=5/w^2$. The cost is $10\left(2w^2\right)+6\left[2(2wh)+2(hw)\right]=20w^2+36wh$, so $C(w)=20w^2+36w\left(5/w^2\right)=20w^2+180/w$. $C'(w)=40w-180/w^2=40\left(w^3-\frac{9}{2}\right)\Big/w^2 \Rightarrow w=\sqrt[3]{\frac{9}{2}}$ is the critical number. There is an absolute minimum for C when $w=\sqrt[3]{\frac{9}{2}}$ since $C'(w)<0$ for $0<w<\sqrt[3]{\frac{9}{2}}$ and $C'(w)>0$ for $w>\sqrt[3]{\frac{9}{2}}$. $C\left(\sqrt[3]{\frac{9}{2}}\right)=20\left(\sqrt[3]{\frac{9}{2}}\right)^2+\frac{180}{\sqrt[3]{9/2}}\approx\$163.54$.

13.



$10=(2w)(w)h=2w^2h$, so $h=5/w^2$. The cost is $C(w)=10\left(2w^2\right)+6\left[2(2wh)+2hw\right]+6\left(2w^2\right)=$ $32w^2+36wh=32w^2+180/w$ $C'(w)=64w-180/w^2=4\left(16w^3-45\right)\big/w^2\Rightarrow w=\sqrt[3]{\frac{45}{16}}$ is the critical number. $C'(w)<0$ for $0<w<\sqrt[3]{\frac{45}{16}}$ and $C'(w)>0$ for $w>\sqrt[3]{\frac{45}{16}}$. The minimum cost is $C\left(\sqrt[3]{\frac{45}{16}}\right)=32(2.8125)^{2/3}+180\big/\sqrt{2.8125}\approx\$191.28$.

14. **(a)** Let the rectangle have sides x and y and area A , so $A=xy$ or $y=A/x$. The problem is to minimize the perimeter $=2x+2y=2x+2A/x=P(x)$. Now $P'(x)=2-2A/x^2=2\left(x^2-A\right)/x^2$. So the critical number is $x=\sqrt{A}$. Since $P'(x)<0$ for $0<x<\sqrt{A}$ and $P'(x)>0$ for $x>\sqrt{A}$, there is an absolute minimum at $x=\sqrt{A}$. The sides of the rectangle are $\sqrt{A}$ and $A/\sqrt{A}=\sqrt{A}$, so the rectangle is a square.

(b) Let p be the perimeter and x and y the lengths of the sides, so $p=2x+2y\Rightarrow 2y=p-2x\Rightarrow y=\frac{1}{2}p-x$. The area is $A(x)=x\left(\frac{1}{2}p-x\right)=\frac{1}{2}px-x^2$. Now $A'(x)=0\Rightarrow \frac{1}{2}p-2x=0\Rightarrow 2x=\frac{1}{2}p\Rightarrow x=\frac{1}{4}p$. Since $A''(x)=-2<0$, there is an absolute maximum for A when $x=\frac{1}{4}p$ by the Second Derivative Test. The sides of the rectangle are

$\frac{1}{4}p$ and $\frac{1}{2}p-\frac{1}{4}p=\frac{1}{4}p$, so the rectangle is a square.

15. The distance from a point (x,y) on the line $y=4x+7$ to the origin is $\sqrt{(x-0)^2+(y-0)^2}=\sqrt{x^2+y^2}$. However, it is easier to work with the *square* of the distance; that is, $D(x)=\left(\sqrt{x^2+y^2}\right)^2=x^2+y^2=x^2+(4x+7)^2$. Because the distance is positive, its minimum value will occur at the same point as the minimum value of D .

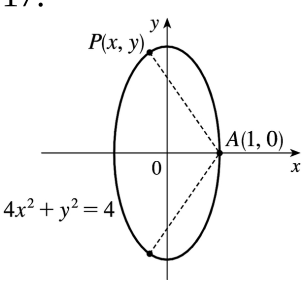
$D'(x)=2x+2(4x+7)(4)=34x+56$, so $D'(x)=0\Leftrightarrow x=-\frac{28}{17}$.

$D''(x)=34>0$, so D is concave upward for all x . Thus, D has an absolute minimum at $x=-\frac{28}{17}$. The point closest to the origin is $(x,y)=\left(-\frac{28}{17},4\left(-\frac{28}{17}\right)+7\right)=\left(-\frac{28}{17},\frac{7}{17}\right)$.

16. The square of the distance from a point (x,y) on the line $y=-6x+9$ to the point $(-3,1)$ is $D(x)=(x+3)^2+(y-1)^2=(x+3)^2+(-6x+8)^2=37x^2-90x+73$. $D'(x)=74x-90$, so $D'(x)=0\Leftrightarrow x=\frac{45}{37}$.

$D''(x)=74>0$, so D is concave upward for all x . Thus, D has an absolute minimum at $x=\frac{45}{37}$. The point on the line closest to $(-3,1)$ is $\left(\frac{45}{37},\frac{63}{37}\right)$.

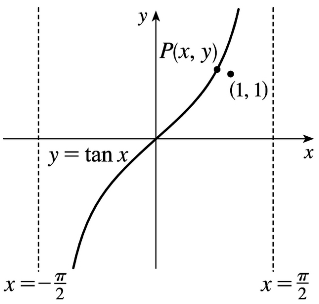
17.



From the figure, we see that there are two points that are farthest away from $A(1,0)$. The distance d from A to an arbitrary point $P(x,y)$ on the ellipse is $d=\sqrt{(x-1)^2+(y-0)^2}$ and the square of the distance is $S=d^2=x^2-2x+1+y^2=x^2-2x+1+\left(4-4x^2\right)=-3x^2-2x+5$. $S'=-6x-2$ and $S'=0\Rightarrow x=-\frac{1}{3}$. Now $S''=-6<0$, so we know that S has a maximum at $x=-\frac{1}{3}$. Since $-1\le x\le 1$, $S(-1)=4$, $S\left(-\frac{1}{3}\right)=\frac{16}{3}$, and $S(1)=0$, we see that the maximum distance is $\sqrt{\frac{16}{3}}$. The corresponding y –values are

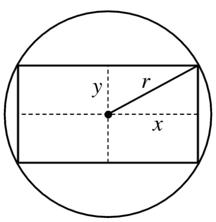
$y=\pm\sqrt{4-4\left(-\frac{1}{3}\right)^2}=\pm\sqrt{\frac{32}{9}}=\pm\frac{4}{3}\sqrt{2}\approx\pm1.89$. The points are $\left(-\frac{1}{3},\pm\frac{4}{3}\sqrt{2}\right)$.

18.



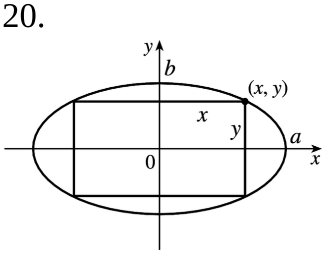
The distance d from $(1,1)$ to an arbitrary point $P(\mathrm{x,y})$ on the curve $y=\tan x$ is $d=\sqrt{(x-1)^2+(y-1)^2}$ and the square of the distance is $S=d^2=(x-1)^2+(\tan x-1)^2$. $S'=2(x-1)+2(\tan x-1)\sec^2 x$. Graphing S' on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ gives us a zero at $x\approx0.82$, and so $\tan x\approx1.08$. The point on $y=\tan x$ that is closest to $(1,1)$ is approximately $(0.82,1.08)$.

19.



The area of the rectangle is $(2x)(2y)=4xy$. Also $r^2=x^2+y^2$ so $y=\sqrt{r^2-x^2}$, so the area is $A(x)=4x\sqrt{r^2-x^2}$. Now $A'(x)=4\left(\sqrt{r^2-x^2}-\frac{x^2}{\sqrt{r^2-x^2}}\right)=4\frac{r^2-2x^2}{\sqrt{r^2-x^2}}$. The critical number is $x=\frac{1}{\sqrt{2}}r$. Clearly this gives a maximum. $y=\sqrt{r^2-\left(\frac{1}{\sqrt{2}}r\right)^2}=\sqrt{\frac{1}{2}r^2}=\frac{1}{\sqrt{2}}r=x$, which tells us that the rectangle is a square. The dimensions are $2x=\sqrt{2}r$ and $2y=\sqrt{2}r$.

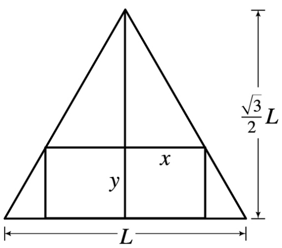
20.



The area of the rectangle is $(2x)(2y)=4xy$. Now

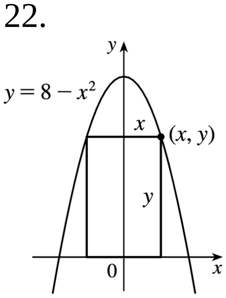
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ gives $y=\frac{b}{a}\sqrt{a^2-x^2}$, so we maximize $A(x)=4\frac{b}{a}x\sqrt{a^2-x^2}$ $A'(x)=$ $\frac{4b}{a}\left[x\cdot\frac{1}{2}\left(a^2-x^2\right)^{-1/2}(-2x)+\left(a^2-x^2\right)^{1/2}\cdot 1\right]=\frac{4b}{a}\left(a^2-x^2\right)^{-1/2}\left[-x^2+a^2-x^2\right]=\frac{4b}{a\sqrt{a^2-x^2}}\left[a^2-2x^2\right]$ So the critical number is $x=\frac{1}{\sqrt{2}}a$, and this clearly gives a maximum. Then $y=\frac{1}{\sqrt{2}}b$, so the maximum area is $4\left(\frac{1}{\sqrt{2}}a\right)\left(\frac{1}{\sqrt{2}}b\right)=2ab$.

21.



The height h of the equilateral triangle with sides of length L is $\frac{\sqrt{3}}{2}L$, since $h^2+(L/2)^2=L^2\Rightarrow$ $h^2=L^2-\frac{1}{4}L^2=\frac{3}{4}L^2\Rightarrow h=\frac{\sqrt{3}}{2}L$. Using similar triangles, $\frac{\frac{\sqrt{3}}{2}L-y}{x}=\frac{\frac{\sqrt{3}}{2}L}{L/2}=\sqrt{3}\Rightarrow\sqrt{3}x=\frac{\sqrt{3}}{2}L-y\Rightarrow$ $y=\frac{\sqrt{3}}{2}L-\sqrt{3}x\Rightarrow y=\frac{\sqrt{3}}{2}(L-2x)$. The area of the inscribed rectangle is $A(x)=(2x)y=\sqrt{3}x(L-2x)=\sqrt{3}Lx-2\sqrt{3}x^2$, where $0\leq x\leq L/2$. Now $0=A'(x)=\sqrt{3}L-4\sqrt{3}x\Rightarrow$ $x=\sqrt{3}L\big/\left(4\sqrt{3}\right)=L/4$. Since $A(0)=A(L/2)=0$, the maximum occurs when $x=L/4$, and $y=\frac{\sqrt{3}}{2}L-\frac{\sqrt{3}}{4}L=\frac{\sqrt{3}}{4}L$, so the dimensions are $L/2$ and $\frac{\sqrt{3}}{4}L$.

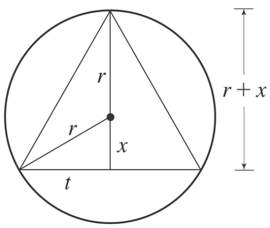
22.



The rectangle has area $A(x)=2xy=2x\left(8-x^2\right)=16x-2x^3$, where $0\leq x\leq 2\sqrt{2}$. Now $A'(x)=16-6x^2=0\Rightarrow$

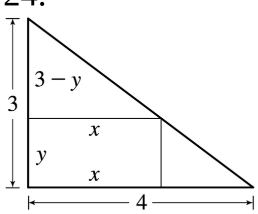
$x=2\sqrt{\frac{2}{3}}$. Since $A(0)=A\left(2\sqrt{2}\right)=0$, there is a maximum when $x=2\sqrt{\frac{2}{3}}$. Then $y=\frac{16}{3}$, so the rectangle has dimensions $4\sqrt{\frac{2}{3}}$ and $\frac{16}{3}$.

23.



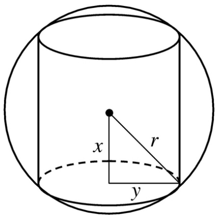
The area of the triangle is $A(x)=\frac{1}{2}(2t)(r+x)=t(r+x)=\sqrt{r^2-x^2}(r+x)$. $0=A'(x)=r\frac{-2x}{2\sqrt{r^2-x^2}}+\sqrt{r^2-x^2}+x\frac{-2x}{2\sqrt{r^2-x^2}}=-\frac{x^2+rx}{\sqrt{r^2-x^2}}+\sqrt{r^2-x^2}\Rightarrow\frac{x^2+rx}{\sqrt{r^2-x^2}}=\sqrt{r^2-x^2}\Rightarrow x^2+rx=r^2-x^2$ $\Rightarrow 0=2x^2+rx-r^2=(2x-r)(x+r)\Rightarrow x=\frac{1}{2}r$ or $x=-r$. Now $A(r)=0=A(-r)\Rightarrow$ the maximum occurs where $x=\frac{1}{2}r$, so the triangle has height $r+\frac{1}{2}r=\frac{3}{2}r$ and base $2\sqrt{r^2-\left(\frac{1}{2}r\right)^2}=2\sqrt{\frac{3}{4}r^2}=\sqrt{3}r$.

24.



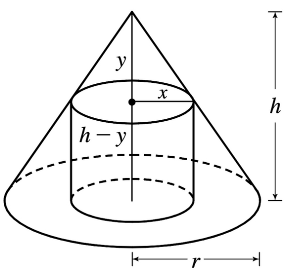
The rectangle has area xy . By similar triangles $\frac{3-y}{x}=\frac{3}{4}\Rightarrow -4y+12=3x$ or $y=-\frac{3}{4}x+3$. So the area is $A(x)=x\left(-\frac{3}{4}x+3\right)=-\frac{3}{4}x^2+3x$ where $0\le x\le 4$. Now $0=A'(x)=-\frac{3}{2}x+3\Rightarrow x=2$ and $y=\frac{3}{2}$. Since $A(0)=A(4)=0$, the maximum area is $A(2)=2\left(\frac{3}{2}\right)=3\text{ cm}^2$.

25.



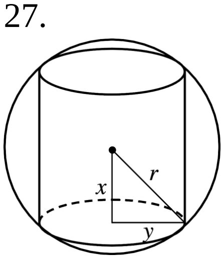
The cylinder has volume $V=\pi y^2(2x)$. Also $x^2+y^2=r^2 \Rightarrow y^2=r^2-x^2$, so $V(x)=\pi\left(r^2-x^2\right)(2x)=2\pi\left(r^2x-x^3\right)$, where $0\leq x\leq r$. $V'(x)=2\pi\left(r^2-3x^2\right)=0\Rightarrow x=r/\sqrt{3}$. Now $V(0)=V(r)=0$, so there is a maximum when $x=r/\sqrt{3}$ and $V\left(r/\sqrt{3}\right)=\pi\left(r^2-r^2/3\right)\left(2r/\sqrt{3}\right)=4\pi r^3/\left(3\sqrt{3}\right)$.

26.



By similar triangles, $y/x=h/r$, so $y=hx/r$. The volume of the cylinder is $\pi x^2(h-y)=\pi hx^2-(\pi h/r)x^3=V(x)$. Now $V'(x)=2\pi hx-(3\pi h/r)x^2=\pi hx(2-3x/r)$. So $V'(x)=0\Rightarrow x=0$ or $x=\frac{2}{3}r$. The maximum clearly occurs when $x=\frac{2}{3}r$ and then the volume is $\pi hx^2-(\pi h/r)x^3=\pi hx^2(1-x/r)=\pi\left(\frac{2}{3}r\right)^2h\left(1-\frac{2}{3}\right)=\frac{4}{27}\pi r^2h$.

27.



The cylinder has surface area 2(area of the base)+(lateral surface area)$=2\pi$(radius)$^2+2\pi$(radius)(height)$=2\pi y^2+2\pi y(2x)$. Now $x^2+y^2=r^2\Rightarrow y^2=r^2-x^2\Rightarrow y=\sqrt{r^2-x^2}$, so the surface area is $S(x)=$ $2\pi\left(r^2-x^2\right)+4\pi x\sqrt{r^2-x^2}$, $0\leq x\leq r=2\pi r^2-2\pi x^2+4\pi\left(x\sqrt{r^2-x^2}\right)$ Thus, $S'(x)$ $=0-4\pi x+4\pi\left[x\cdot\frac{1}{2}\left(r^2-x^2\right)^{-1/2}(-2x)+\left(r^2-x^2\right)^{1/2}\cdot 1\right]$

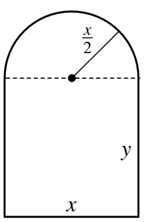
$=4\pi\left[-x-\dfrac{x^2}{\sqrt{r^2-x^2}}+\sqrt{r^2-x^2}\right]=4\pi\cdot\dfrac{-x\sqrt{r^2-x^2}-x^2+r^2-x^2}{\sqrt{r^2-x^2}}$

$S'(x)=0\Rightarrow x\sqrt{r^2-x^2}=r^2-2x^2\ (\,*\,)\Rightarrow\left(x\sqrt{r^2-x^2}\right)^2=\left(r^2-2x^2\right)^2\Rightarrow x^2\left(r^2-x^2\right)=r^4-4r^2x^2+4x^4\Rightarrow$ $r^2x^2-x^4=r^4-4r^2x^2+4x^4\Rightarrow 5x^4-5r^2x^2+r^4=0$. This is a quadratic equation in x^2 . By the quadratic formula, $x^2=\dfrac{5\pm\sqrt{5}}{10}r^2$, but we reject the root with the + sign since it doesn't satisfy (*). So $x=\sqrt{\dfrac{5-\sqrt{5}}{10}}\,r$. Since $S(0)=S(r)=0$, the maximum surface area occurs at the critical number and $x^2=\dfrac{5-\sqrt{5}}{10}r^2\Rightarrow y^2=r^2-\dfrac{5-\sqrt{5}}{10}r^2=\dfrac{5+\sqrt{5}}{10}r^2\Rightarrow$ the surface area is

$2\pi\left(\dfrac{5+\sqrt{5}}{10}\right)r^2+4\pi\sqrt{\dfrac{5-\sqrt{5}}{10}}\sqrt{\dfrac{5+\sqrt{5}}{10}}r^2=\pi r^2\left[2\cdot\dfrac{5+\sqrt{5}}{10}+4\dfrac{\sqrt{\left(5-\sqrt{5}\right)\left(5+\sqrt{5}\right)}}{10}\right]$

$=\pi r^2\left[\dfrac{5+\sqrt{5}}{5}+\dfrac{2\sqrt{20}}{5}\right]=\pi r^2\left[\dfrac{5+\sqrt{5}+2\cdot 2\sqrt{5}}{5}\right]=\pi r^2\left[\dfrac{5+5\sqrt{5}}{5}\right]=\pi r^2\left(1+\sqrt{5}\right)$.

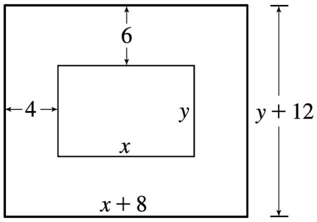
28.



Perimeter=30$\Rightarrow 2y+x+\pi\left(\dfrac{x}{2}\right)=30\Rightarrow y=\dfrac{1}{2}\left(30-x-\dfrac{\pi x}{2}\right)=15-\dfrac{x}{2}-\dfrac{\pi x}{4}$. The area is the area of the rectangle plus the area of the semicircle, or $xy+\dfrac{1}{2}\pi\left(\dfrac{x}{2}\right)^2$, so

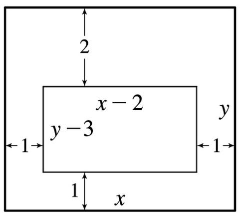
$A(x)=x\left(15-\dfrac{x}{2}-\dfrac{\pi x}{4}\right)+\dfrac{1}{8}\pi x^2=15x-\dfrac{1}{2}x^2-\dfrac{\pi}{8}x^2$. $A'(x)=15-\left(1+\dfrac{\pi}{4}\right)x=0\Rightarrow x=\dfrac{15}{1+\pi/4}=\dfrac{60}{4+\pi}$. $A''(x)=-\left(1+\dfrac{\pi}{4}\right)<0$, so this gives a maximum. The dimensions are $x=\dfrac{60}{4+\pi}$ ft and $y=15-\dfrac{30}{4+\pi}-\dfrac{15\pi}{4+\pi}=\dfrac{60+15\pi-30-15\pi}{4+\pi}=\dfrac{30}{4+\pi}$ ft, so the height of the rectangle is half the base.

29.



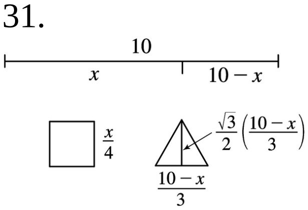
$xy=384 \Rightarrow y=384/x$. Total area is $A(x)=(8+x)(12+384/x)=12(40+x+256/x)$, so $A^{/}(x)=12\left(1-256/x^2\right)=0 \Rightarrow x=16$. There is an absolute minimum when $x=16$ since $A^{/}(x)<0$ for $0<x<16$ and $A^{/}(x)>0$ for $x>16$. When $x=16$, $y=384/16=24$, so the dimensions are 24 cm and 36 cm.

30.



$xy=180$, so $y=180/x$. The printed area is $(x-2)(y-3)=(x-2)(180/x-3)=186-3x-360/x=A(x)$. $A^{/}(x)=-3+360/x^2=0$ when $x^2=120 \Rightarrow x=2\sqrt{30}$. This gives an absolute maximum since $A^{/}(x)>0$ for $0<x<2\sqrt{30}$ and $A^{/}(x)<0$ for $x>2\sqrt{30}$. When $x=2\sqrt{30}$, $y=180/(2\sqrt{30})$, so the dimensions are $2\sqrt{30}$ in. and $90/\sqrt{30}$ in.

31.



Let x be the length of the wire used for the square. The total area is

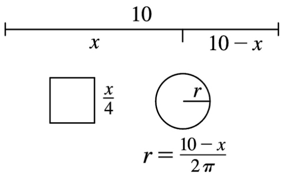
$$A(x)=\left(\frac{x}{4}\right)^2+\frac{1}{2}\left(\frac{10-x}{3}\right)\frac{\sqrt{3}}{2}\left(\frac{10-x}{3}\right)=\frac{1}{16}x^2+\frac{\sqrt{3}}{36}(10-x)^2,\ 0\leq x\leq 10$$

$A^{/}(x)=\frac{1}{8}x-\frac{\sqrt{3}}{18}(10-x)=0 \Leftrightarrow \frac{9}{72}x+\frac{4\sqrt{3}}{72}x-\frac{40\sqrt{3}}{72}=0 \Leftrightarrow x=\frac{40\sqrt{3}}{9+4\sqrt{3}}$. Now $A(0)=\left(\frac{\sqrt{3}}{36}\right)100\approx 4.81$, $A(10)=\frac{100}{16}=6.25$ and $A\left(\frac{40\sqrt{3}}{9+4\sqrt{3}}\right)\approx 2.72$, so

(a) The maximum area occurs when $x=10$ m, and all the wire is used for the square.

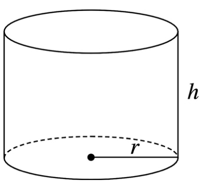
(b) The minimum area occurs when $x=\frac{40\sqrt{3}}{9+4\sqrt{3}}\approx 4.35$ m.

32.



Total area is $A(x)=\left(\frac{x}{4}\right)^2+\pi\left(\frac{10-x}{2\pi}\right)^2=\frac{x^2}{16}+\frac{(10-x)^2}{4\pi}$, $0\leq x\leq 10$.
$A'(x)=\frac{x}{8}-\frac{10-x}{2\pi}=\left(\frac{1}{2\pi}+\frac{1}{8}\right)x-\frac{5}{\pi}=0\Rightarrow x=40/(4+\pi)$. $A(0)=25/\pi\approx 7.96$, $A(10)=6.25$, and $A(40/(4+\pi))\approx 3.5$, so the maximum occurs when $x=0$ m and the minimum occurs when $x=40/(4+\pi)$ m.

33.

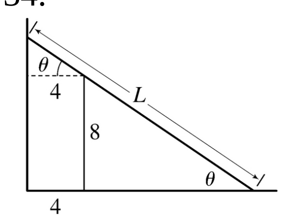


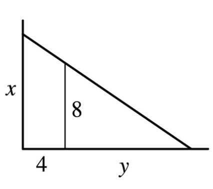
The volume is $V=\pi r^2 h$ and the surface area is $S(r)=\pi r^2+2\pi rh=\pi r^2+2\pi r\left(\frac{V}{\pi r^2}\right)=\pi r^2+\frac{2V}{r}$.

$S'(r)=2\pi r-\frac{2V}{r^2}=0\Rightarrow 2\pi r^3=2V\Rightarrow r=\sqrt[3]{\frac{V}{\pi}}$ cm.

This gives an absolute minimum since $S'(r)<0$ for $0<r<\sqrt[3]{\frac{V}{\pi}}$ and $S'(r)>0$ for $r>\sqrt[3]{\frac{V}{\pi}}$. When $r=\sqrt[3]{\frac{V}{\pi}}$, $h=\frac{V}{\pi r^2}=\frac{V}{\pi(V/\pi)^{2/3}}=\sqrt[3]{\frac{V}{\pi}}$ cm.

34.



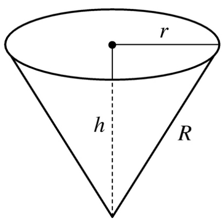


$L=8\theta+4\sec\theta$, $0<\theta<\dfrac{\pi}{2}$, $\dfrac{dL}{d\theta}=-8\theta\cot\theta+4\sec\theta\tan\theta=0$ when $\sec\theta\tan\theta=2\theta\cot\theta \Leftrightarrow \tan^3\theta=2\Leftrightarrow$ $\tan\theta=\sqrt[3]{2}\Leftrightarrow\theta=\tan^{-1}\sqrt[3]{2}$.

$dL/d\theta<0$ when $0<\theta<\tan^{-1}\sqrt[3]{2}$, $dL/d\theta>0$ when $\tan^{-1}\sqrt[3]{2}<\theta<\dfrac{\pi}{2}$, so L has an absolute minimum when $\theta=\tan^{-1}\sqrt[3]{2}$, and the shortest ladder has length $L=8\dfrac{\sqrt{1+2^{2/3}}}{2^{1/3}}+4\sqrt{1+2^{2/3}}\approx16.65$ ft.

Another method: Minimize $L^2=x^2+(4+y)^2$, where $\dfrac{x}{4+y}=\dfrac{8}{y}$.

35.



$h^2+r^2=R^2\Rightarrow V=\dfrac{\pi}{3}r^2h=\dfrac{\pi}{3}\left(R^2-h^2\right)h=\dfrac{\pi}{3}\left(R^2h-h^3\right)$. $V'(h)=\dfrac{\pi}{3}\left(R^2-3h^2\right)=0$ when $h=\dfrac{1}{\sqrt{3}}R$. This gives an absolute maximum, since $V'(h)>0$ for $0<h<\dfrac{1}{\sqrt{3}}R$ and $V'(h)<0$ for $h>\dfrac{1}{\sqrt{3}}R$. The maximum volume is $V\left(\dfrac{1}{\sqrt{3}}R\right)=\dfrac{\pi}{3}\left(\dfrac{1}{\sqrt{3}}R^3-\dfrac{1}{3\sqrt{3}}R^3\right)=\dfrac{2}{9\sqrt{3}}\pi R^3$.

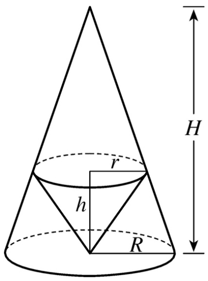
36. The volume and surface area of a cone with radius r and height h are given by $V=\dfrac{1}{3}\pi r^2h$ and $S=\pi r\sqrt{r^2+h^2}$. We'll minimize $A=S^2$ subject to $V=27$. $V=27\Rightarrow\dfrac{1}{3}\pi r^2h=27\Rightarrow r^2=\dfrac{81}{\pi h}$ **(1)** .

$A=\pi^2r^2(r^2+h^2)=\pi^2\left(\dfrac{81}{\pi h}\right)\left(\dfrac{81}{\pi h}+h^2\right)=\dfrac{81^2}{h^2}+81\pi h$, so $A'=0\Rightarrow\dfrac{-2\cdot81^2}{h^3}+81\pi=0\Rightarrow81\pi=\dfrac{2\cdot81^2}{h^3}\Rightarrow$

$h^3=\dfrac{162}{\pi}\Rightarrow h=\sqrt[3]{\dfrac{162}{\pi}}=3\sqrt[3]{\dfrac{6}{\pi}}\approx3.722$. From **(1)** , $r^2=\dfrac{81}{\pi h}=\dfrac{81}{\pi\cdot3\sqrt[3]{6/\pi}}=\dfrac{27}{\sqrt[3]{6\pi^2}}\Rightarrow$

$r=\dfrac{3\sqrt{3}}{\sqrt[6]{6\pi^2}}\approx2.632$. $A''=6\cdot81^2/h^4>0$, so A and hence S has an absolute minimum at these values of r and h .

37.



By similar triangles, $\frac{H}{R}=\frac{H-h}{r}$ **(1)** . The volume of the inner cone is $V=\frac{1}{3}\pi r^2 h$, so we'll solve **(1)** for h . $\frac{Hr}{R}=H-h \Rightarrow h=H-\frac{Hr}{R}=\frac{HR-Hr}{R}=\frac{H}{R}(R-r)$ **(2)** .

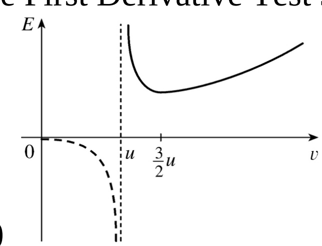
Thus, $V(r)=\frac{\pi}{3}r^2\cdot\frac{H}{R}(R-r)=\frac{\pi H}{3R}(Rr^2-r^3) \Rightarrow V^{\prime}(r)=\frac{\pi H}{3R}(2Rr-3r^2)=\frac{\pi H}{3R}r(2R-3r)$.

$V^{\prime}(r)=0 \Rightarrow r=0$ or $2R=3r \Rightarrow r=\frac{2}{3}R$ and from **(2)** , $h=\frac{H}{R}\left(R-\frac{2}{3}R\right)=\frac{H}{R}\left(\frac{1}{3}R\right)=\frac{1}{3}H$.

$V^{\prime}(r)$ changes from positive to negative at $r=\frac{2}{3}R$, so the inner cone has a maximum volume of $V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(\frac{2}{3}R\right)^2\left(\frac{1}{3}H\right)=\frac{4}{27}\cdot\frac{1}{3}\pi R^2 H$, which is approximately 15% of the volume of the larger cone.

38. **(a)** $E(v)=\frac{aLv^3}{v-u} \Rightarrow E^{\prime}(v)=aL\frac{(v-u)3v^2-v^3}{(v-u)^2}=0$ when $2v^3=3uv^2 \Rightarrow 2v=3u \Rightarrow v=\frac{3}{2}u$.

The First Derivative Test shows that this value of v gives the minimum value of E .

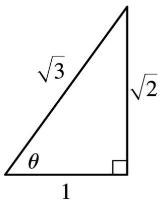
(b) 

39. $S=6sh-\frac{3}{2}s^2\cot\theta+3s^2\frac{\sqrt{3}}{2}\ \theta$

(a) $\frac{dS}{d\theta}=\frac{3}{2}s^2\ {}^2\theta-3s^2\frac{\sqrt{3}}{2}\ \theta\cot\theta$ or $\frac{3}{2}s^2\ \theta\left(\ \theta-\sqrt{3}\cot\theta\right)$.

(b) $\frac{dS}{d\theta}=0$ when $\theta-\sqrt{3}\cot\theta=0 \Rightarrow \frac{1}{\sin\theta}-\sqrt{3}\frac{\cos\theta}{\sin\theta}=0 \Rightarrow \cos\theta=\frac{1}{\sqrt{3}}$. The First Derivative Test shows that the minimum surface area occurs when $\theta=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)\approx 55^{\circ}$.

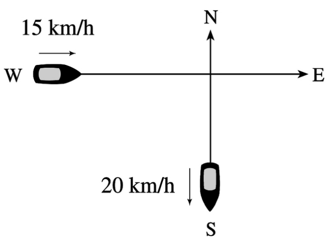
(c)



If $\cos\theta = \frac{1}{\sqrt{3}}$, then $\cot\theta = \frac{1}{\sqrt{2}}$ and $\theta = \frac{\sqrt{3}}{\sqrt{2}}$, so the surface area is $S <=$

$$6sh-\frac{3}{2}s^2\frac{1}{\sqrt{2}}+3s^2\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{\sqrt{2}}=6sh-\frac{3}{2\sqrt{2}}s^2+\frac{9}{2\sqrt{2}}s^2=6sh+\frac{6}{2\sqrt{2}}s^2=6s\left(h+\frac{1}{2\sqrt{2}}s\right)$$

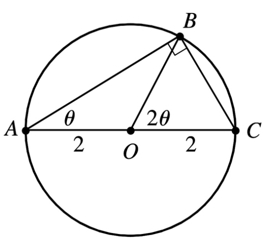
40.



Let t be the time, in hours, after 2:00 P.M. The position of the boat heading south at time t is $(0,-20t)$. The position of the boat heading east at time t is $(-15+15t,0)$. If $D(t)$ is the distance between the boats at time t , we minimize $f(t)=[D(t)]^2=20^2t^2+15^2(t-1)^2$. $f'(t)=800t+450(t-1)=1250t-450=0$ when $t=\frac{450}{1250}=0.36$ h. 0.36 h $\times\ \frac{60\text{min}}{\text{h}}=21.6$ min $=21$ min 36 s. Since $f''(t)>0$, this gives a minimum, so the boats are closest together at 2:21:36 P.M.

41. Here $T(x)=\frac{\sqrt{x^2+25}}{6}+\frac{5-x}{8}$, $0\le x\le 5 \Rightarrow T'(x)=\frac{x}{6\sqrt{x^2+25}}-\frac{1}{8}=0 \Leftrightarrow 8x=6\sqrt{x^2+25} \Leftrightarrow$ $16x^2=9\left(x^2+25\right) \Leftrightarrow x=\frac{15}{\sqrt{7}}$. But $\frac{15}{\sqrt{7}}>5$, so T has no critical number. Since $T(0)\approx 1.46$ and $T(5)\approx 1.18$, he should row directly to B .

42.

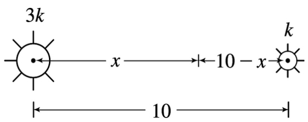


In isosceles triangle AOB , $\angle O=180^\circ-\theta-\theta$, so $\angle BOC=2\theta$. The distance rowed is $4\cos\theta$ while the distance walked is the length of arc $BC=2(2\theta)=4\theta$. The time taken is given by

$T(\theta)=\dfrac{4\cos\theta}{2}+\dfrac{4\theta}{4}=2\cos\theta+\theta$, $0\leq\theta\leq\dfrac{\pi}{2}$. $T^{\prime}(\theta)=-2\sin\theta+1=0\Leftrightarrow\sin\theta=\dfrac{1}{2}\Rightarrow\theta=\dfrac{\pi}{6}$.

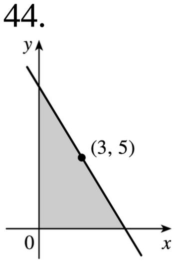
Check the value of T at $\theta=\dfrac{\pi}{6}$ and at the endpoints of the domain of T ; that is, $\theta=0$ and $\theta=\dfrac{\pi}{2}$. $T(0)=2$, $T\left(\dfrac{\pi}{6}\right)=\sqrt{3}+\dfrac{\pi}{6}\approx2.26$, and $T\left(\dfrac{\pi}{2}\right)=\dfrac{\pi}{2}\approx1.57$. Therefore, the minimum value of T is $\dfrac{\pi}{2}$ when $\theta=\dfrac{\pi}{2}$; that is, the woman should walk all the way. Note that $T^{\prime\prime}(\theta)=-2\cos\theta<0$ for $0\leq\theta<\dfrac{\pi}{2}$, so $\theta=\dfrac{\pi}{6}$ gives a maximum time.

43.



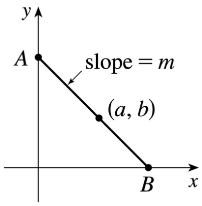
The total illumination is $I(x)=\dfrac{3k}{x^{2}}+\dfrac{k}{(10-x)^{2}}$, $0<x<10$. Then $I^{\prime}(x)=\dfrac{-6k}{x^{3}}+\dfrac{2k}{(10-x)^{3}}=0\Rightarrow$ $6k(10-x)^{3}=2kx^{3}\Rightarrow3(10-x)^{3}=x^{3}\Rightarrow\sqrt[3]{3}(10-x)=x\Rightarrow10\sqrt[3]{3}-\sqrt[3]{3}x=x\Rightarrow10\sqrt[3]{3}=x+\sqrt[3]{3}x\Rightarrow$ $10\sqrt[3]{3}=\left(1+\sqrt[3]{3}\right)x\Rightarrow x=\dfrac{10\sqrt[3]{3}}{1+\sqrt[3]{3}}\approx5.9$ ft. This gives a minimum since $I^{\prime\prime}(x)>0$ for $0<x<10$.

44.



The line with slope m (where $m<0$) through $(3,5)$ has equation $y-5=m(x-3)$ or $y=mx+(5-3m)$. The $y-$ intercept is $5-3m$ and the $x-$ intercept is $-5/m+3$. So the triangle has area $A(m)=\dfrac{1}{2}(5-3m)(-5/m+3)=15-25/(2m)-\dfrac{9}{2}m$. Now $A^{\prime}(m)=\dfrac{25}{2m^{2}}-\dfrac{9}{2}=0\Leftrightarrow m^{2}=\dfrac{25}{9}\Rightarrow m=-\dfrac{5}{3}$ (since $m<0$). $A^{\prime\prime}(m)=-\dfrac{25}{m^{3}}>0$, so there is an absolute minimum when $m=-\dfrac{5}{3}$. Thus, an equation of the line is $y-5=-\dfrac{5}{3}(x-3)$ or $y=-\dfrac{5}{3}x+10$.

45.



Every line segment in the first quadrant passing through (a,b) with endpoints on the x – and y –axes satisfies an equation of the form $y-b=m(x-a)$, where $m<0$. By setting $x=0$ and then $y=0$, we find its endpoints, $A(0,b-am)$ and $B\left(a-\frac{b}{m},0\right)$. The distance d from A to B is given by

$$d=\sqrt{\left[\left(a-\frac{b}{m}\right)-0\right]^2+[0-(b-am)]^2} .$$

It follows that the square of the length of the line segment, as a function of m , is given by

$$S(m)=\left(a-\frac{b}{m}\right)^2+(am-b)^2=a^2-\frac{2ab}{m}+\frac{b^2}{m^2}+a^2m^2-2abm+b^2 \text{ . Thus,}$$

$$S'(m)=\frac{2ab}{m^2}-\frac{2b^2}{m^3}+2a^2m-2ab=\frac{2}{m^3}\left(abm-b^2+a^2m^4-abm^3\right)$$

$$=\frac{2}{m^3}\left[b(am-b)+am^3(am-b)\right]=\frac{2}{m^3}(am-b)\left(b+am^3\right)$$

Thus, $S'(m)=0 \Leftrightarrow m=b/a$ or $m=-\sqrt[3]{\frac{b}{a}}$. Since $b/a>0$ and $m<0$, m must equal $-\sqrt[3]{\frac{b}{a}}$. Since $\frac{2}{m^3}<0$, we see that $S'(m)<0$ for $m<-\sqrt[3]{\frac{b}{a}}$ and $S'(m)>0$ for $m>-\sqrt[3]{\frac{b}{a}}$. Thus, S has its absolute minimum value when $m=-\sqrt[3]{\frac{b}{a}}$. That value is

$$S\left(-\sqrt[3]{\frac{b}{a}}\right)=\left(a+b\sqrt[3]{\frac{a}{b}}\right)^2+\left(-a\sqrt[3]{\frac{b}{a}}-b\right)^2=\left(a+\sqrt[3]{ab^2}\right)^2+\left(\sqrt[3]{a^2b}+b\right)^2$$

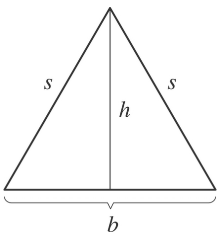
$$=a^2+2a^{4/3}b^{2/3}+a^{2/3}b^{4/3}+a^{4/3}b^{2/3}+2a^{2/3}b^{4/3}+b^2=a^2+3a^{4/3}b^{2/3}+3a^{2/3}b^{4/3}+b^2$$

The last expression is of the form $x^3+3x^2y+3xy^2+y^3\left[=(x+y)^3\right]$ with $x=a^{2/3}$ and $y=b^{2/3}$, so we can write it as $(a^{2/3}+b^{2/3})^3$ and the shortest such line segment has length $\sqrt{S}=(a^{2/3}+b^{2/3})^{3/2}$.

46. $y=1+40x^3-3x^5 \Rightarrow y'=120x^2-15x^4$, so the tangent line to the curve at $x=a$ has slope $m(a)=120a^2-15a^4$. Now $m'(a)=240a-60a^3=-60a(a^2-4)=-60a(a+2)(a-2)$, so $m'(a)>0$ for $a<-2$,

and $0<a<2$, and $m'(a)<0$ for $-2<a<0$ and $a>2$. Thus, m is increasing on $(-\infty,-2)$, decreasing on $(-2,0)$, increasing on $(0,2)$, and decreasing on $(2,\infty)$. Clearly, $m(a)\to-\infty$ as $a\to\pm\infty$, so the maximum value of $m(a)$ must be one of the two local maxima, $m(-2)$ or $m(2)$. But both $m(-2)$ and $m(2)$ equal $120\cdot 2^2-15\cdot 2^4=480-240=240$. So 240 is the largest slope, and it occurs at the points $(-2,-223)$ and $(2,225)$. *Note:* $a=0$ corresponds to a local *minimum* of m .

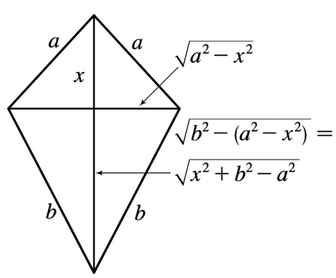
47.



Here $s^2=h^2+b^2/4$, so $h^2=s^2-b^2/4$. The area is $A=\frac{1}{2}b\sqrt{s^2-b^2/4}$. Let the perimeter be p , so $2s+b=p$ or $s=(p-b)/2\Rightarrow A(b)=\frac{1}{2}b\sqrt{(p-b)^2/4-b^2/4}=b\sqrt{p^2-2pb}/4$. Now

$$A'(b)=\frac{\sqrt{p^2-2pb}}{4}-\frac{bp/4}{\sqrt{p^2-2pb}}=\frac{-3pb+p^2}{4\sqrt{p^2-2pb}}$$

. Therefore, $A'(b)=0\Rightarrow-3pb+p^2=0\Rightarrow b=p/3$. Since $A'(b)>0$ for $b<p/3$ and $A'(b)<0$ for $b>p/3$, there is an absolute maximum when $b=p/3$. But then $2s+p/3=p$, so $s=p/3\Rightarrow s=b\Rightarrow$ the triangle is equilateral.



48.

See the figure. The area is given by

$$A(x)=\frac{1}{2}\left(2\sqrt{a^2-x^2}\right)x+\frac{1}{2}\left(2\sqrt{a^2-x^2}\right)\left(\sqrt{x^2+b^2-a^2}\right)=\sqrt{a^2-x^2}\left(x+\sqrt{x^2+b^2-a^2}\right)$$

for $0\le x\le a$. Now

$$A'(x)=\sqrt{a^2-x^2}\left(1+\frac{x}{\sqrt{x^2+b^2-a^2}}\right)+\left(x+\sqrt{x^2+b^2-a^2}\right)\frac{-x}{\sqrt{a^2-x^2}}=0\Leftrightarrow$$

$$\frac{x}{\sqrt{a^2-x^2}}\left(x+\sqrt{x^2+b^2-a^2}\right)=\sqrt{a^2-x^2}\left(\frac{x+\sqrt{x^2+b^2-a^2}}{\sqrt{x^2+b^2-a^2}}\right).$$

Except for the trivial case where $x=0$, $a=b$ and $A(x)=0$, we have

$x+\sqrt{x^2+b^2-a^2}>0$. Hence, cancelling this factor gives $\dfrac{x}{\sqrt{a^2-x^2}}=\dfrac{\sqrt{a^2-x^2}}{\sqrt{x^2+b^2-a^2}} \Rightarrow x\sqrt{x^2+b^2-a^2}=a^2-x^2$

$\Rightarrow x^2\left(x^2+b^2-a^2\right)=a^4-2a^2x^2+x^4 \Rightarrow x^2\left(b^2-a^2\right)=a^4-2a^2x^2 \Rightarrow x^2\left(b^2+a^2\right)=a^4 \Rightarrow x=\dfrac{a^2}{\sqrt{a^2+b^2}}$.

Now we must check the value of A at this point as well as at the endpoints of the domain to see which gives the maximum value. $A(0)=a\sqrt{b^2-a^2}$, $A(a)=0$ and

$$A\left(\frac{a^2}{\sqrt{a^2+b^2}}\right)=\sqrt{a^2-\left(\frac{a^2}{\sqrt{a^2+b^2}}\right)^2}\left[\frac{a^2}{\sqrt{a^2+b^2}}+\sqrt{\left(\frac{a^2}{\sqrt{a^2+b^2}}\right)^2+b^2-a^2}\right]=$$

$$\frac{ab}{\sqrt{a^2+b^2}}\left[\frac{a^2}{\sqrt{a^2+b^2}}+\frac{b^2}{\sqrt{a^2+b^2}}\right]=\frac{ab\left(a^2+b^2\right)}{a^2+b^2}=ab=0$$

Since $b\geq\sqrt{b^2-a^2}$, $A\left(a^2/\sqrt{a^2+b^2}\right)\geq A(0)$. So there is an absolute maximum when $x=\dfrac{a^2}{\sqrt{a^2+b^2}}$.

In this case the horizontal piece should be $\dfrac{2ab}{\sqrt{a^2+b^2}}$ and the vertical piece should be

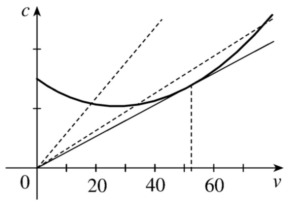
$\dfrac{a^2+b^2}{\sqrt{a^2+b^2}}=\sqrt{a^2+b^2}$.

49. Note that $|AD|=|AP|+|PD| \Rightarrow 5=x+|PD| \Rightarrow |PD|=5-x$. Using the Pythagorean Theorem for $\triangle PDB$ and $\triangle PDC$ gives us

$$\begin{aligned} L(x) &= |AP|+|BP|+|CP|=x+\sqrt{(5-x)^2+2^2}+\sqrt{(5-x)^2+3^2} \\ &= x+\sqrt{x^2-10x+29}+\sqrt{x^2-10x+34} \end{aligned}$$

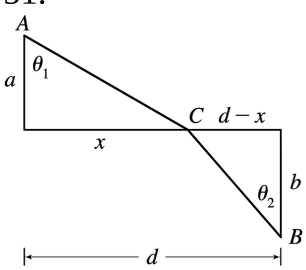
$\Rightarrow L'(x)=1+\dfrac{x-5}{\sqrt{x^2-10x+29}}+\dfrac{x-5}{\sqrt{x^2-10x+34}}$ From the graphs of L and L' , it seems that the minimum value of L is about $L(3.59)=9.35$ m.

50.



We note that since c is the consumption in gallons per hour, and v is the velocity in miles per hour, then $\frac{c}{v} = \frac{\text{gallons/hour}}{\text{miles/hour}} = \frac{\text{gallons}}{\text{mile}}$ gives us the consumption in gallons per mile, that is, the quantity G. To find the minimum, we calculate $\frac{dG}{dv} = \frac{d}{dv}\left(\frac{c}{v}\right) = \frac{v\frac{dc}{dv} - c\frac{dv}{dv}}{v^2} = \frac{v\frac{dc}{dv} - c}{v^2}$. This is 0 when $v\frac{dc}{dv} - c = 0 \Leftrightarrow \frac{dc}{dv} = \frac{c}{v}$. This implies that the tangent line of $c(v)$ passes through the origin, and this occurs when $v \approx 53$ mi / h. Note that the slope of the secant line through the origin and a point $(v, c(v))$ on the graph is equal to $G(v)$, and it is intuitively clear that G is minimized in the case where the secant is in fact a tangent.

51.



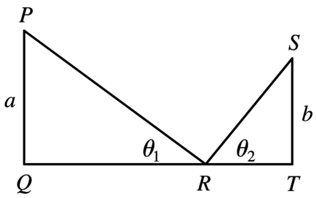
The total time is

$$T(x) </ = (\text{time from } A \text{ to } C) + (\text{time from } C \text{ to } B) = \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(d-x)^2}}{v_2},\ 0<x<d$$

$$T^{/}(x) = \frac{x}{v_1\sqrt{a^2+x^2}} - \frac{d-x}{v_2\sqrt{b^2+(d-x)^2}} = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}$$

The minimum occurs when $T^{/}(x) = 0 \Rightarrow \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$.

52.



If $d=|QT|$, we minimize $f\left(\theta_1\right)=|PR|+|RS|=a\ \theta_1+b\ \theta_2$. Differentiating with respect to θ_1 , and setting $\dfrac{df}{d\theta_1}$ equal to 0 , we get $\dfrac{df}{d\theta_1}=0=-a\ \theta_1\cot\theta_1-b\ \theta_2\cot\theta_2\dfrac{d\theta_2}{d\theta_1}$.

So we need to find an expression for $\dfrac{d\theta_2}{d\theta_1}$. We can do this by observing that $|QT|$ = constant $=a\cot\theta_1+b\cot\theta_2$.

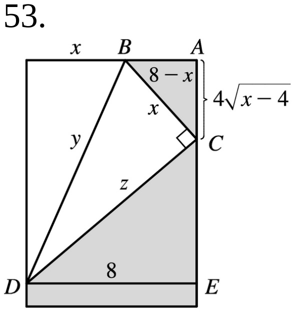
Differentiating this equation implicitly with respect to θ_1 , we get $-a^2\theta_1-b^2\theta_2\dfrac{d\theta_2}{d\theta_1}=0\Rightarrow$

$\dfrac{d\theta_2}{d\theta_1}=-\dfrac{a^2\theta_1}{b^2\theta_2}$. We substitute this into the expression for $\dfrac{df}{d\theta_1}$ to get

$$-a\ \theta_1\cot\theta_1-b\ \theta_2\cot\theta_2\left(-\frac{a^2\theta_1}{b^2\theta_2}\right)=0\Leftrightarrow-a\csc\theta_1\cot\theta_1+a\frac{\csc^2\theta_1\cot\theta_2}{\csc\theta_2}=0\Leftrightarrow$$

$\cot\theta_1\csc\theta_2=\csc\theta_1\cot\theta_2\Leftrightarrow\dfrac{\cot\theta_1}{\csc\theta_1}=\dfrac{\cot\theta_2}{\csc\theta_2}\Leftrightarrow\cos\theta_1=\cos\theta_2$. Since θ_1 and θ_2 are both acute, we have $\theta_1=\theta_2$.

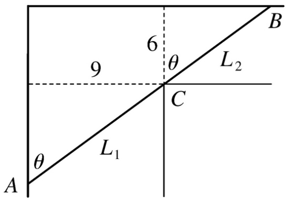
53.



$y^2=x^2+z^2$, but triangles CDE and BCA are similar, so $z/8=x\big/\left(4\sqrt{x-4}\right)\ \Rightarrow z=2x\big/\sqrt{x-4}$. Thus, we minimize $f(x)=y^2=x^2+4x^2/(x-4)=x^3/(x-4)$, $4<x\le 8$.

$f^{/}(x)=\dfrac{(x-4)\left(3x^2\right)-x^3}{(x-4)^2}=\dfrac{x^2[3(x-4)-x]}{(x-4)^2}=\dfrac{2x^2(x-6)}{(x-4)^2}=0$ when $x=6$. $f^{/}(x)<0$ when $x<6$, $f^{/}(x)>0$ when $x>6$, so the minimum occurs when $x=6$ in.

54.



Paradoxically, we solve this maximum problem by solving a minimum problem. Let L be the length of the line ACB going from wall to wall touching the inner corner C . As $\theta \to 0$ or $\theta \to \dfrac{\pi}{2}$, we have $L\to\infty$ and there will be an angle that makes L a minimum. A pipe of this length will just fit around the corner.

From the diagram, $L=L_1+L_2=9\csc\theta+6\sec\theta \Rightarrow dL/d\theta=-9\csc\theta\ \cot\theta+6\sec\theta\ \tan\theta=0$ when

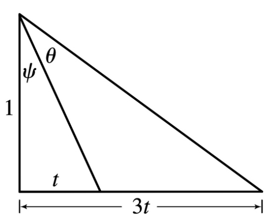
$6\sec\theta\ \tan\theta=9\csc\theta\ \cot\theta \Leftrightarrow \tan^3\theta=\dfrac{9}{6}=1.5 \Leftrightarrow \tan\theta=\sqrt{[3]1.5}$. Then $\sec^2\theta=1+\left(\dfrac{3}{2}\right)^{2/3}$ and

$\csc^2\theta=1+\left(\dfrac{3}{2}\right)^{-2/3}$, so the longest pipe has length

$L=9\left[1+\left(\dfrac{3}{2}\right)^{-2/3}\right]^{1/2}+6\left[1+\left(\dfrac{3}{2}\right)^{2/3}\right]^{1/2}\approx 21.07$ ft.

Or, use $\theta=\tan^{-1}\left(\sqrt[3]{1.5}\right)\approx 0.852 \Rightarrow L=9\ \theta+6\sec\theta\approx 21.07$ ft.

55.



It suffices to maximize $\tan\theta$. Now $\dfrac{3t}{1}=\tan\left(\psi+\theta\right)=\dfrac{\tan\psi+\tan\theta}{1-\tan\psi\tan\theta}=\dfrac{t+\tan\theta}{1-t\tan\theta}$. So

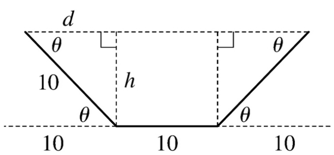
$3t\left(1-t\tan\theta\right)=t+\tan\theta \Rightarrow 2t=\left(1+3t^2\right)\tan\theta \Rightarrow \tan\theta=\dfrac{2t}{1+3t^2}$. Let $f(t)=\tan\theta=\dfrac{2t}{1+3t^2} \Rightarrow$

$f^{/}(t)=\dfrac{2\left(1+3t^2\right)-2t(6t)}{\left(1+3t^2\right)^2}=\dfrac{2\left(1-3t^2\right)}{\left(1+3t^2\right)^2}=0 \Leftrightarrow 1-3t^2=0 \Leftrightarrow t=\dfrac{1}{\sqrt{3}}$ since $t\geq 0$.

Now

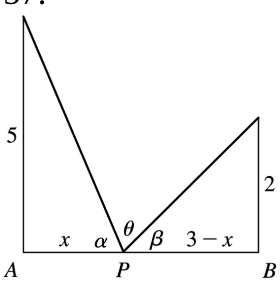
$f'(t)>0$ for $0\le t<\dfrac{1}{\sqrt{3}}$ and $f'(t)<0$ for $t>\dfrac{1}{\sqrt{3}}$, so f has an absolute maximum when $t=\dfrac{1}{\sqrt{3}}$ and $\tan\theta=\dfrac{2\left(1/\sqrt{3}\right)}{1+3\left(1/\sqrt{3}\right)^2}=\dfrac{1}{\sqrt{3}}\Rightarrow\theta=\dfrac{\pi}{6}$. Substituting for t and θ in $3t=\tan\left(\psi+\theta\right)$ gives us $\sqrt{3}=\tan\left(\psi+\dfrac{\pi}{6}\right)\Rightarrow\psi=\dfrac{\pi}{6}$.

56.



We maximize the cross–sectional area $A(\theta)=$
$10h+2\left(\dfrac{1}{2}dh\right)=10h+dh=10(10\sin\theta)+(10\cos\theta)(10\sin\theta) > =100(\sin\theta+\sin\theta\cos\theta),\ 0\le\theta\le\dfrac{\pi}{2}$
$A'(\theta)=100\left(\cos\theta+\cos^2\theta-\sin^2\theta\right)=100\left(\cos\theta+2\cos^2\theta-1\right)=100(2\cos\theta-1)(\cos\theta+1)=0$ when $\cos\theta=\dfrac{1}{2}\Leftrightarrow\theta=\dfrac{\pi}{3}$. ($\cos\theta\ne-1$ since $0\le\theta\le\dfrac{\pi}{2}$.)
Now $A(0)=0$, $A\left(\dfrac{\pi}{2}\right)=100$ and $A\left(\dfrac{\pi}{3}\right)=75\sqrt{3}\approx129.9$, so the maximum occurs when $\theta=\dfrac{\pi}{3}$.

57.



From the figure, $\tan\alpha=\dfrac{5}{x}$ and $\tan\beta=\dfrac{2}{3-x}$. Since $\alpha+\beta+\theta=180^\circ=\pi$,
$\theta=\pi-\tan^{-1}\left(\dfrac{5}{x}\right)-\tan^{-1}\left(\dfrac{2}{3-x}\right)\Rightarrow$

$$\frac{d\theta}{dx}=-\frac{1}{1+\left(\frac{5}{x}\right)^2}\left(-\frac{5}{x^2}\right)-\frac{1}{1+\left(\frac{2}{3-x}\right)^2}\left[\frac{2}{(3-x)^2}\right]$$

$$=\frac{x^2}{x^2+25}\cdot\frac{5}{x^2}-\frac{(3-x)^2}{(3-x)^2+4}\cdot\frac{2}{(3-x)^2}.$$

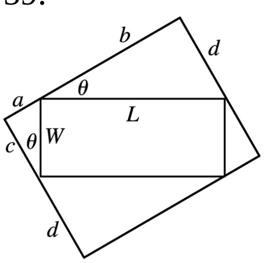
Now $\frac{d\theta}{dx}=0 \Rightarrow \frac{5}{x^2+25}=\frac{2}{x^2-6x+13} \Rightarrow 2x^2+50=5x^2-30x+65 \Rightarrow$ $3x^2-30x+15=0 \Rightarrow x^2-10x+5=0 \Rightarrow x=5\pm 2\sqrt{5}$. We reject the root with the + sign, since it is larger than 3 . $d\theta/dx>0$ for $x<5-2\sqrt{5}$ and $d\theta/dx<0$ for $x>5-2\sqrt{5}$, so θ is maximized when $|AP|=x=5-2\sqrt{5}\approx 0.53$.

58. Let x be the distance from the observer to the wall. Then, from the given figure, <
$\theta=\tan^{-1}\left(\frac{h+d}{x}\right)-\tan^{-1}\left(\frac{d}{x}\right)$, $x>0 \Rightarrow$

$$\begin{aligned}
\frac{d\theta}{dx} &= \frac{1}{1+[(h+d)/x]^2}\left[-\frac{h+d}{x^2}\right]-\frac{1}{1+(d/x)^2}\left[-\frac{d}{x^2}\right]=-\frac{h+d}{x^2+(h+d)^2}+\frac{d}{x^2+d^2} \\
&= \frac{d\left[x^2+(h+d)^2\right]-(h+d)\left(x^2+d^2\right)}{\left[x^2+(h+d)^2\right]\left(x^2+d^2\right)}=\frac{h^2d+hd^2-hx^2}{\left[x^2+(h+d)^2\right]\left(x^2+d^2\right)}=0 \Leftrightarrow
\end{aligned}$$

$hx^2=h^2d+hd^2 \Leftrightarrow x^2=hd+d^2 \Leftrightarrow x=\sqrt{d(h+d)}$. Since $d\theta/dx>0$ for all $x<\sqrt{d(h+d)}$ and $d\theta/dx<0$ for all $x>\sqrt{d(h+d)}$, the absolute maximum occurs when $x=\sqrt{d(h+d)}$.

59.



In the small triangle with sides a and c and hypotenuse W , $\sin\theta=\frac{a}{W}$ and $\cos\theta=\frac{c}{W}$. In the triangle with sides b and d and hypotenuse L , $\sin\theta=\frac{d}{L}$ and $\cos\theta=\frac{b}{L}$. Thus, $a=W\sin\theta$, $c=W\cos\theta$, $d=L\sin\theta$, and $b=L\cos\theta$, so the area of the circumscribed rectangle is

$$\begin{aligned}
A(\theta) &= (a+b)(c+d)=(W\sin\theta+L\cos\theta)(W\cos\theta+L\sin\theta)=1-12\text{pt} \\
&= W^2\sin\theta\cos\theta+WL\sin^2\theta+LW\cos^2\theta+L^2\sin\theta\cos\theta=1-12\text{pt} \\
&= LW\sin^2\theta+LW\cos^2\theta+\left(L^2+W^2\right)\sin\theta\cos\theta=1-12\text{pt} \\
&= LW\left(\sin^2\theta+\cos^2\theta\right)+\left(L^2+W^2\right)\cdot\frac{1}{2}\cdot 2\sin\theta\cos\theta=1-12\text{pt} \\
&= LW+\frac{1}{2}\left(L^2+W^2\right)\sin 2\theta,\ 0\le\theta\le\frac{\pi}{2}
\end{aligned}$$

This expression shows, without calculus, that the maximum value of $A(\theta)$ occurs when $\sin 2\theta = 1 \Leftrightarrow 2\theta = \dfrac{\pi}{2}$

$\Rightarrow \theta = \dfrac{\pi}{4}$. So the maximum area is $A\left(\dfrac{\pi}{4}\right) = LW + \dfrac{1}{2}\left(L^2 + W^2\right) = \dfrac{1}{2}\left(L^2 + 2LW + W^2\right) = \dfrac{1}{2}(L+W)^2$.

60. **(a)** Let D be the point such that $a = |AD|$. From the figure, $\sin\theta = \dfrac{b}{|BC|} \Rightarrow |BC| = b\ \theta$ and $\cos\theta = \dfrac{|BD|}{|BC|} = \dfrac{a - |AB|}{|BC|} \Rightarrow |BC| = (a - |AB|)\sec\theta$. Eliminating $|BC|$ gives $(a - |AB|)\sec\theta = b\ \theta \Rightarrow b\cot\theta = a - |AB| \Rightarrow |AB| = a - b\cot\theta$. The total resistance is

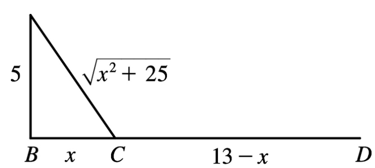
$$R(\theta) = C\frac{|AB|}{r_1^4} + C\frac{|BC|}{r_2^4} = C\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right).$$

(b) $R'(\theta) = C\left(\dfrac{b\csc^2\theta}{r_1^4} - \dfrac{b\csc\theta\cot\theta}{r_2^4}\right) = bC\csc\theta\left(\dfrac{\csc\theta}{r_1^4} - \dfrac{\cot\theta}{r_2^4}\right)$.

$R'(\theta) = 0 \Leftrightarrow \dfrac{\csc\theta}{r_1^4} = \dfrac{\cot\theta}{r_2^4} \Leftrightarrow \dfrac{r_2^4}{r_1^4} = \dfrac{\cot\theta}{\csc\theta} = \cos\theta$.

$R'(\theta) > 0 \Leftrightarrow \dfrac{\csc\theta}{r_1^4} > \dfrac{\cot\theta}{r_2^4} \Rightarrow \cos\theta < \dfrac{r_2^4}{r_1^4}$ and $R'(\theta) < 0$ when $\cos\theta > \dfrac{r_2^4}{r_1^4}$, so there is an absolute minimum when $\cos\theta = r_2^4 / r_1^4$.

(c) When $r_2 = \dfrac{2}{3}r_1$, we have $\cos\theta = \left(\dfrac{2}{3}\right)^4$, so $\theta = \cos^{-1}\left(\dfrac{2}{3}\right)^4 \approx 79^\circ$.

61. **(a)** 

If k= energy / km over land, then energy / km over water $= 1.4k$. So the total energy is $E = 1.4k\sqrt{25 + x^2} + k(13 - x)$, $0 \le x \le 13$, and so $\dfrac{dE}{dx} = \dfrac{1.4kx}{\left(25 + x^2\right)^{1/2}} - k$.

Set $\dfrac{dE}{dx} = 0$: $1.4kx = k\left(25 + x^2\right)^{1/2} \Rightarrow 1.96x^2 = x^2 + 25 \Rightarrow 0.96x^2 = 25 \Rightarrow x = \dfrac{5}{\sqrt{0.96}} \approx 5.1$. Testing against the value of E at the endpoints: $E(0)=1.4k(5)+13k=20k$, $E(5.1)\approx 17.9k$, $E(13)\approx 19.5k$. Thus, to minimize energy, the bird should fly to a point about 5.1 km from B .

(b) If W / L is large, the bird would fly to a point C that is closer to B than to D to minimize the energy used flying over water. If W / L is small, the bird would fly to a point C that is closer to D than to B to minimize the distance of the flight. $E=W\sqrt{25+x^2}+L(13-x) \Rightarrow \dfrac{dE}{dx}=\dfrac{Wx}{\sqrt{25+x^2}}-L=0$ when $\dfrac{W}{L}=\dfrac{\sqrt{25+x^2}}{x}$. By the same sort of argument as in part (a), this ratio will give the minimal expenditure of energy if the bird heads for the point x km from B .

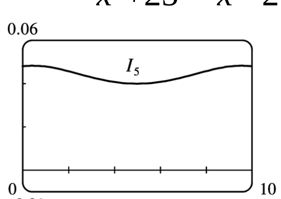
(c) For flight direct to D , $x=13$, so from part (b), $W / L=\dfrac{\sqrt{25+13^2}}{13}\approx 1.07$. There is no value of W / L for which the bird should fly directly to B . But note that $\lim_{x\to 0^+}(W / L)=\infty$, so if the point at which E is a minimum is close to B , then W / L is large.

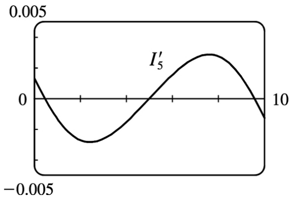
(d) Assuming that the birds instinctively choose the path that minimizes the energy expenditure, we can use the equation for $dE/dx=0$ from part (a) with $1.4k=c$, $x=4$, and $k=1$: $(c)(4)=1\cdot\left(25+4^2\right)^{1/2} \Rightarrow c=\sqrt{41}/4\approx 1.6$.

62. **(a)** $I(x) \propto \dfrac{\text{strength of source}}{(\text{distance from source})^2}$. Adding the intensities from the left and right lightbulbs,
$I(x)=\dfrac{k}{x^2+d^2}+\dfrac{k}{(10-x)^2+d^2}=\dfrac{k}{x^2+d^2}+\dfrac{k}{x^2-20x+100+d^2}$.

(b) The magnitude of the constant k won’t affect the location of the point of maximum intensity, so for convenience we take $k=1$. $I'(x)=-\dfrac{2x}{\left(x^2+d^2\right)^2}-\dfrac{2(x-10)}{\left(x^2-20x+100+d^2\right)^2}$.

Substituting $d=5$ into the equations for $I(x)$ and $I'(x)$, we get
$I_5(x)=\dfrac{1}{x^2+25}+\dfrac{1}{x^2-20x+125}$ and $I_5'(x)=-\dfrac{2x}{\left(x^2+25\right)^2}-\dfrac{2(x-10)}{\left(x^2-20x+125\right)^2}$

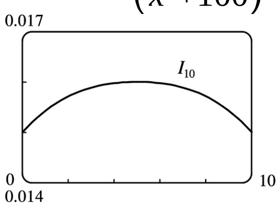


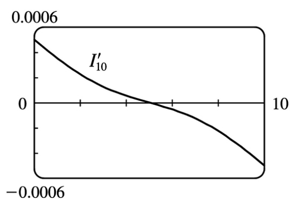


From the graphs, it appears that $I_5(x)$ has a minimum at $x=5$ m.

(c) Substituting $d=10$ into the equations for $I(x)$ and $I'(x)$ gives $I_{10}(x)=\dfrac{1}{x^2+100}+\dfrac{1}{x^2-20x+200}$ and

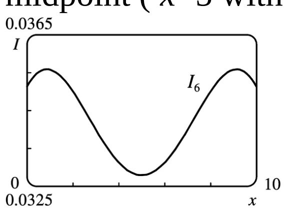
$$I_{10}'(x)=-\frac{2x}{\left(x^2+100\right)^2}-\frac{2(x-10)}{\left(x^2-20x+200\right)^2}\ .$$

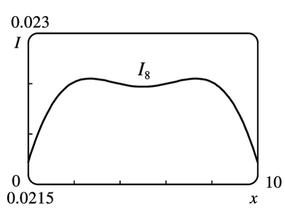


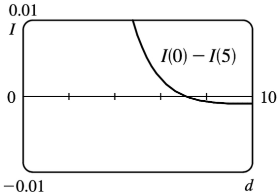


From the graphs, it seems that for $d=10$, the intensity is minimized at the endpoints, that is, $x=0$ and $x=10$. The midpoint is now the most brightly lit point!

(d) From the first figures in parts (b) and (c), we see that the minimal illumination changes from the midpoint ($x=5$ with $d=5$) to the endpoints ($x=0$ and $x=10$ with $d=10$).







So we try $d=6$ (see the first figure) and we see that the minimum value still occurs at $x=5$. Next, we let $d=8$ (see the second figure) and we see that the minimum value occurs at the endpoints. It appears that for some value of d between 6 and 8 , we must have minima at both the midpoint and the endpoints, that is, $I(5)$ must equal $I(0)$. To find this value of d , we solve $I(0)=I(5)$ (with $k=1$):
$\dfrac{1}{d^2}+\dfrac{1}{100+d^2}=\dfrac{1}{25+d^2}+\dfrac{1}{25+d^2}=\dfrac{2}{25+d^2}\Rightarrow \left(25+d^2\right)\left(100+d^2\right)+d^2\left(25+d^2\right)=2d^2\left(100+d^2\right)\Rightarrow$
$2500+125d^2+d^4+25d^2+d^4=200d^2+2d^4\Rightarrow 2500=50d^2\Rightarrow d^2=50\Rightarrow d=5\sqrt{2}\approx 7.071$ (for $0\le d\le 10$). The third figure, a graph of $I(0)–I(5)$ with d independent, confirms that $I(0)–I(5)=0$, that is, $I(0)=I(5)$, when $d=5\sqrt{2}$. Thus, the point of minimal illumination changes abruptly from the midpoint to the endpoints when $d=5\sqrt{2}$.