Cálculo A

Funções trigonométricas e trigonométricas inversas

1. Encontre o valor numérico das expressões a seguir

   $a)$ $\arcsin\frac{1}{2}$ $b)$ $\arccos 1$ $c)$ $\operatorname{arccsc}(-1)$ $d)$ $\arctan 0$ $e)$ $\operatorname{arccot}(-\sqrt{3})$
   $f)$ $\arctan(-\sqrt{3})$ $g)$ $\operatorname{arccsc}\sqrt{2}$ $h)$ $\operatorname{arcsec} 2$ $i)$ $\operatorname{arccsc} 2\sqrt{3}/3$ $j)$ $\operatorname{arcsec}(-2)$
   $k)$ $\operatorname{arcsec}(-2\sqrt{3}/3)$ $l)$ $\arcsin 0$ $m)$ $\arcsin -\frac{1}{2}$ $n)$ $\arccos(-\sqrt{3}/2)$ $o)$ $\arctan 1$

   Respostas:

   $a)\ \pi/6$ $b)\ 0$ $c)\ -\frac{\pi}{2}$ $d)\ 0$ $e)\ \frac{5\pi}{6}$ $f)\ -\frac{\pi}{3}$ $g)\ \frac{\pi}{4}$ $h)\ \frac{\pi}{3}$ $i)\ \frac{\pi}{3}$ $j)\ \frac{4\pi}{3}$ $k)\ \frac{7\pi}{6}$ $l)\ 0$
   $m)\ -\frac{\pi}{6}$ $n)\ \frac{5\pi}{6}$ $o)\ \frac{\pi}{4}$

2. Encontre o valor numérico das expressões a seguir

   $a)$ $\sin(\arccos\frac{1}{2})$ $b)$ $\tan(\arcsin\sqrt{3}/2)$ $c)$ $\sec(\arccos\sqrt{3}/2)$ $d)$ $\csc(\arctan(-1))$
   $e)$ $\sin(\arcsin(-\frac{1}{2}))$ $f)$ $\csc(\operatorname{arccot}(-\sqrt{3}))$ $g)$ $\csc(\operatorname{arcsec}\sqrt{2})$ $h)$ $\arcsin(\cos\pi/6)$
   $i)$ $\operatorname{arccot}(\tan\pi/3)$ $j)$ $\arctan(\tan 0)$

   Respostas:

   $a)\ \frac{\sqrt{3}}{2}$ $b)\ \sqrt{3}$ $c)\ \frac{2}{\sqrt{3}}$ $d)\ -\sqrt{2}$ $e)\ -\frac{1}{2}$ $f)\ 2$ $g)\ \sqrt{2}$ $h)\ \frac{\pi}{3}$ $i)\ \frac{\pi}{6}$ $j)\ 0$

3. Determinar o domínio das funções

   (a) $f(x) = \frac{\cot 2x}{\sin\frac{x}{3}}$ $\Big[$ Resp.: $\{x \in \mathbb{R} : x \neq n\pi/2; n \in \mathbb{Z}\}$ $\Big]$
   (b) $f(x) = \sqrt{\cos x}$ $\Big[$Resp.: $\cup_{n\in\mathbb{Z}}[-\frac{\pi}{2} + 2\pi n, \frac{\pi}{2} + 2\pi n]$$\Big]$
   (c) $f(x) = \left(\sin x - 2\sin^2 x\right)^{-3/4}$ $\Big[$ Resp.: $\cup_{n\in\mathbb{Z}}(2\pi n, \frac{\pi}{6} + 2\pi n) \cup (\frac{5\pi}{6} + 2\pi n, \pi + 2\pi n)$ $\Big]$
   (d) $f(x) = \arccos(3 - x)$ $\Big[$Resp.: $[2, 4]$$\Big]$
   (e) $f(x) = \arcsin\left(\frac{1}{2}x - 1\right) + \arccos\left(1 - \frac{1}{2}x\right)$ $\Big[$Resp.: $[0, 4]$$\Big]$
   (f) $f(x) = 3\arcsin\sqrt{\frac{3x-1}{2}}$ $\Big[$Resp.: $[\frac{1}{3}, 1]$$\Big]$
   (g) $f(x) = \arccos\frac{1}{x-1}$ $\Big[$Resp.: $(-\infty, 0] \cup [2, \infty)$$\Big]$
   (h) $f(x) = \frac{\tan x}{\cos 2x}$ $\Big[$ Resp.: $\mathbb{R} - \cup_{n\in\mathbb{Z}}\{\frac{\pi}{2} + n\pi, \frac{\pi}{4} + \frac{\pi}{2}n\}_{n\in\mathbb{Z}}$$\Big]$
   (i) $f(x) = \sqrt{\frac{\sin x + \cos x}{\sin x - \cos x}}$ $\Big[$Resp.: $\cup_{n\in Z}(\frac{\pi}{4} + n\pi, \frac{3\pi}{4} + n\pi]$$\Big]$
   (j) $f(x) = \arccos x - \arcsin(3 - x)$ $\Big[$Resp.: $\emptyset$$\Big]$
   (k) $f(x) = \arctan\frac{x}{x^2-9}$ $\Big[$Resp.: $\mathbb{R} - \{\pm 3\}$$\Big]$
   (l) $f(x) = \arcsin\frac{x^2-1}{x}$ $\Big[$Resp.: $[-\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}] \cup [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}+1}{2}$$\Big]$
   (m) $f(x) = \sqrt{\arcsin x - \arccos x}$ $\Big[$Resp.: $[\frac{1}{\sqrt{2}}, 1]$$\Big]$

(n) $f(x) = \arcsin(2\cos x)$ $\Big[$Resp.: $\cup_{n\in\mathbb{Z}}[\frac{\pi}{3} + n\pi, \frac{2\pi}{3} + n\pi]\Big]$

(o) $f(x) = \tan(2\arccos x)$ $\Big[$Resp.: $[-1, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, 1]\Big]$

(p) $f(x) = \frac{\arcsin(\frac{1}{2}x-1)}{\sqrt{x^2-3x+1}}$ $\Big[$Resp.: $[0, \frac{3-\sqrt{5}}{2}) \cup (\frac{3+\sqrt{5}}{3}, 4]\Big]$

(q) $f(x) = \frac{\sqrt{4-x^2}}{\arcsin(2-x)}$ $\Big[$Resp.: $(1, 2)\Big]$

(r) $f(x) = \frac{\sqrt{\cos x - \frac{1}{2}}}{\sqrt{6-35x-6x^2}}$ $\Big[$Resp.: $(-6, -\frac{5\pi}{3}] \cup [-\frac{\pi}{3}, \frac{1}{6})\Big]$

4. Seja $f : [-\pi/4, \pi/4] \to \mathbb{R}$ tal que $f(\sin 2x) = \sin x + \cos x$. Determine $f(x)$. $\Big[$Resp.: $f(x) = \sqrt{1+x}\Big]$

5. Seja $f : A \to [0, 1]$ com $f(x) = \sin^2 2x$. Determine $A \subset [0, 2\pi]$ de modo que f admita inversa. (Há mais de uma possibilidade) $\Big[$Resp.: $[\frac{\pi}{4}, \frac{\pi}{2}]$, etc.$\Big]$

6. Mostre que

   a) $\sec(\arctan x) = \sqrt{1+x^2}$
   b) $\sin(\operatorname{arccsc} x) = \frac{1}{x}$
   c) $\cos(2\arcsin x) = 1 - 2x^2$
   d) $\sin(2\arcsin x) = 2x\sqrt{1-x^2}$
   e) $\tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}, \ -1 < x < 1$
   f) $\sin(\operatorname{arccot} x) = \frac{1}{\sqrt{1+x^2}}$
   g) $\cot(\arcsin x) = \frac{\sqrt{1-x^2}}{x}$
   h) $\cos(2\arccos x) = 2x^2 - 1, \ -1 \le x \le 1$
   i) $\sin(3\arcsin x) = 3x - 4x^3, \ -1 \le x \le 1$
   j) $\tan(3\arctan x) = \frac{x(3-x^2)}{1-3x^2}, \ x \ne \pm 1/3$
   k) $3\arccos x - \arccos(3x - 4x^3) = \pi, \ -1/2 \le x \le 1/2$
   l) $\arccos \frac{1-x^2}{1+x^2} = 2|\arctan x|$
   m) $\arctan(-x) = -\arctan x$
   o) $\operatorname{arccot}(-x) = \pi - \operatorname{arccot} x$
   n) $\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$

7. Encontre os valores de x para o qual as equações a seguir são verdadeiras

   a) $\arccos\sqrt{1-x^2} = \arcsin x$ $\Big[$Resp.: $0 \le x \le 1\Big]$

   b) $\arccos\sqrt{1-x^2} = -\arcsin x$ $\Big[$Resp.: $-1 \le x \le 0\Big]$

   c) $\operatorname{arccot} x = \arctan \frac{1}{x}$ $\Big[$Resp.: $x > 0\Big]$

   d) $\arctan x = \operatorname{arccot}(\frac{1}{x})$ $\Big[$Resp.: $x > 0\Big]$

e) $\arctan x = \text{arccot}(\frac{1}{x}) - \pi$ $\left[\text{Resp.: } x < 0\right]$

f) $\arctan \frac{1+x}{1-x} = \arctan x + \frac{\pi}{4}$ $\left[\text{Resp.: } x < 1\right]$

g)$\arctan \frac{1+x}{1-x} = \arctan x - \frac{3\pi}{4}$ $\left[\text{Resp.: } x > 1\right]$

8. Resolva as equações

a) $\sin\left(\frac{1}{5} \arccos x\right) = 1$ $\left[\text{Resp.: } \emptyset\right]$

b) $\arcsin \frac{1}{\sqrt{x}} - \arcsin \sqrt{1-x} = \frac{\pi}{2}$ $\left[\text{Resp.: } x = 1\right]$

c) $\text{arccot}x = \arccos x$ $\left[\text{Resp.: } x = 0\right]$

d) $\arcsin x - \arccos x = \arccos \frac{\sqrt{3}}{2}$ $\left[\text{Resp.: } x = \frac{\sqrt{3}}{2}\right]$

9. Mostre que

$$\arcsin\, x + \arcsin y = \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\,\right)$$

desde que o valor da expressão do lado esquerdo esteja entre $[-\frac{\pi}{2}, \frac{\pi}{2}]$. [1]

10. Mostre que

$$\arctan \frac{x}{\sqrt{1-x^2}} = \arcsin\, x, \quad \text{com} \quad -1 < x < 1$$

11. Mostre que

$$\arctan\, x + \arctan\, y = \arctan\, \frac{x+y}{1-xy} \quad \text{para} \quad xy \neq 1$$

considerando que o valor da expressão do lado esquerdo esteja entre $(-\frac{\pi}{2}, \frac{\pi}{2})$.

12. Sejam a, b, c números satisfazendo $bc = 1 + a^2$. Mostre que

$$\arctan \frac{1}{a+b} + \arctan \frac{1}{a+c} = \arctan \frac{1}{a}$$

desde que a expressão do lado esquerdo esteja entre $(-\frac{\pi}{2}, \frac{\pi}{2})$, e que tenhamos $a+b \neq 0, \quad a+c \neq 0, \quad a \neq 0$.

13. Mostre que

$$\begin{aligned} a)\ \arcsin\left(\frac{x}{3} - 1\right) &= \frac{\pi}{2} - 2\arcsin\sqrt{1 - \frac{x}{6}} \\ b)\ \arcsin\left(\frac{x}{3} - 1\right) &= 2\left(\arcsin \frac{\sqrt{x}}{\sqrt{6}}\right) - \frac{\pi}{2} \end{aligned}$$

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[1] Obs.: Tal condição é usada apenas para garantir que pode-se escrever o lado esquerdo como o arco seno da expressão dada do lado direito.

14. Mostre que existe uma constante c tal que se tem

$$\arcsin x + \operatorname{arccos} x = c, \quad \text{com} \quad -1 \leq x \leq 1$$

15. Seja $f(x) = \arctan x + \arctan \frac{1}{x}$. Mostre que $f(x)$ é constante em cada um dos intervalos $(-\infty, 0)$ e $(0, \infty)$. Encontre as constantes.

16. Mostre que $\arcsin \frac{m-1}{m+1} = \arccos \frac{2\sqrt{m}}{m+1}$ $(m > 0)$

17. Determine $x \in (0,1)$ tal que $\arcsin x + \arcsin 2x = \frac{\pi}{2}$ $\Big[$Resp.: $\frac{\sqrt{5}}{5}\Big]$

18. Se $a \in \mathbb{R}, a > 0$ e $0 \leq \arcsin \frac{a-1}{a+1} < \frac{\pi}{2}$ mostre que $\tan\big(\arcsin \frac{a-1}{a+1} + \arctan \frac{1}{2\sqrt{a}}\big) = \frac{2a\sqrt{a}}{3a+1}$

19. Determine a solução de $\arctan x + \arctan \frac{x}{x+1} = \frac{\pi}{4}$ $(x \neq -1)$ $\Big[$Resp.: $\frac{1}{2}\Big]$

20. Determine a solução de

$$\sec\Big(\arctan \frac{1}{1+e^x} - \arctan(1-e^x)\Big) = \frac{\sqrt{5}}{2}$$

$\Big[$Resp.: $0\Big]$

21. Determine os valores de $a \in (-\frac{\pi}{2}, \frac{\pi}{2})$ para os quais existe $x \in \mathbb{R}$ solução de

$$\arctan\Big(\sqrt{2} - 1 + \frac{e^x}{2}\Big) + \arctan\Big(\sqrt{2} - 1 - \frac{e^x}{2}\Big) = a$$

$\Big[$Resp.: $(0, \frac{\pi}{4})\Big]$

22. Seja $x = \arcsin \frac{b}{a}$ com $|a| > |b|$, $0 \leq x \leq \frac{\pi}{4}$. Mostre que $x = \frac{1}{2} \arcsin \frac{2b\sqrt{a^2-b^2}}{a|a|}$

23. Determine um intervalo I que contêm todas as soluções de

$$\arctan \frac{1+x}{2} + \arctan \frac{1-x}{2} \geq \frac{\pi}{4}$$

$\Big[$Resp.: $[-1, 1]\Big]$

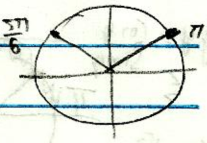
24. Mostre que

$$\sin\Big(2 \operatorname{arccot} \frac{4}{3}\Big) + \cos\Big(2 \operatorname{arccsc} \frac{5}{4}\Big) = \frac{17}{25}$$

Lista A

1.

a) $y = \text{arc sin } \frac{1}{2} \Leftrightarrow \sin y = \frac{1}{2}$

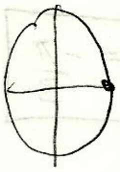


$\Downarrow$

Im arc sin $= [-\frac{\pi}{2}, \frac{\pi}{2}]$;

$y = \frac{\pi}{6}, \frac{5\pi}{6}$

Mas $y \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow \boxed{y = \frac{\pi}{6}}$

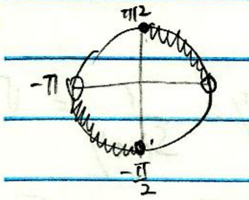


b) $y = \text{arc cos } 1 \Leftrightarrow \cos y = 1 \Rightarrow y = 0, 2\pi$

Im arc cos $= [0, \pi]$; Mas $y \in [0, \pi] \Rightarrow \boxed{y = 0}$

c) $y = \text{arc csc}(-1) \Leftrightarrow \csc y = -1 \Rightarrow y = \frac{3\pi}{2}$ ou $-\frac{\pi}{2}$

Im arc csc $= (-\pi, -\frac{\pi}{2}] \cup (0, \frac{\pi}{2}]$; Mas $y \in (-\pi, -\frac{\pi}{2}] \cup (0, \frac{\pi}{2}]$

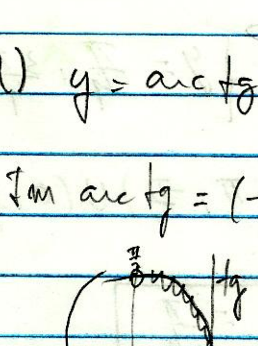


$\csc y = \frac{1}{\sin y}$

$\Rightarrow \boxed{y = -\frac{\pi}{2}}$

d) $y = \text{arc tg } 0 \Leftrightarrow \text{tg } y = 0 \Rightarrow y = 0, \pi$

Im arc tg $= (-\frac{\pi}{2}, \frac{\pi}{2})$; Mas $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$



$\Rightarrow \boxed{y = 0}$

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e) $y = \text{arc cotg}(-\sqrt{3}) \Leftrightarrow \text{cotg}\, y = -\sqrt{3} \Rightarrow y = \frac{-\pi}{6}, \frac{5\pi}{6}$

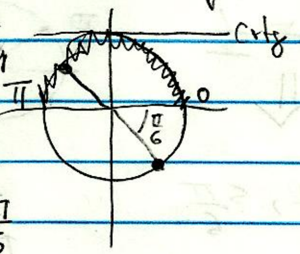
$\text{Im arc cotg} = (0, \pi)$; Mas $y \in (0, \pi) \Rightarrow \boxed{y = \frac{5\pi}{6}}$

$\text{cotg}\, y = \frac{\cos y}{\sin y}$

$\Leftrightarrow$

$\text{cotg}\, y = -\sqrt{3}$

$\therefore$ $y = \frac{-\pi}{6}$

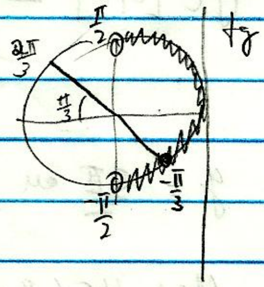


f) $y = \text{arc tg}(-\sqrt{3}) \Leftrightarrow \text{tg}\, y = -\sqrt{3} \Rightarrow y = \frac{-\pi}{3}, \frac{2\pi}{3}$

$\text{Im arc tg} = (-\frac{\pi}{2}, \frac{\pi}{2})$; Mas $y \in (\frac{-\pi}{2}, \frac{\pi}{2}) \Rightarrow \boxed{y = -\frac{\pi}{3}}$

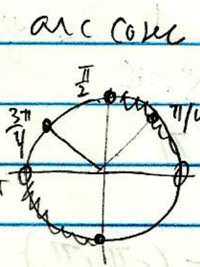
$\text{tg}\, y = -\sqrt{2}$

$y = \frac{\pi}{3}$



g) $y = \text{arc csc}\sqrt{2} \Leftrightarrow \csc y = \sqrt{2}$, $\text{cosec}\, y = \frac{1}{\sin y} = \sqrt{2}$

$\text{Im arc cosec} = (-\pi, -\frac{\pi}{2}] \cup (0, \frac{\pi}{2}]$; $\sin y = \frac{\sqrt{2}}{2}$



$\Rightarrow \Big\} y = \frac{\pi}{4}, \frac{3\pi}{4}$

Mas $y \in (-\pi, -\frac{\pi}{2}) \cup (0, \frac{\pi}{2}]$

$\Rightarrow \boxed{y = \frac{\pi}{4}}$

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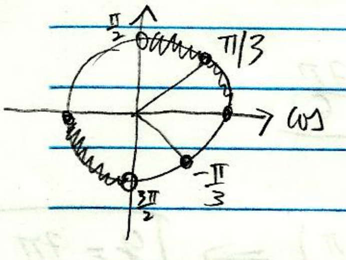
h) $y = \text{arcsec } 2 \Leftrightarrow \sec y = 2 \Leftrightarrow \sec y = \frac{1}{\cos y} = 2$

$\text{Im sec} = [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2}]$; $\therefore \cos y = \frac{1}{2}$



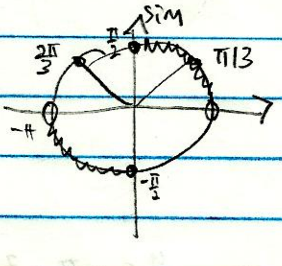
$\Rightarrow \underbrace{y = -\frac{\pi}{3}, \frac{\pi}{3}}$

Mas $y \in [0, \frac{\pi}{2}) \cup [\pi, 3\frac{\pi}{2})$

$\Rightarrow \boxed{y = \frac{\pi}{3}}$

i) $y = \text{arc cosec } \frac{2\sqrt{3}}{3} \rightarrow \underbrace{\text{cosec } y} = \frac{2\sqrt{3}}{3}$

$\text{Im arc cosec} = (-\pi, -\frac{\pi}{2}] \cup (0, \frac{\pi}{2}]$; $\sin y = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$



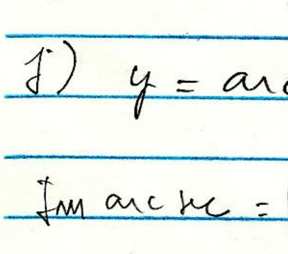
$\Rightarrow \underbrace{y = \frac{\pi}{3}, \frac{2\pi}{3}}$

Mas $y \in (-\pi, -\frac{\pi}{2}] \cup (0, \frac{\pi}{2}]$

$\Rightarrow \boxed{y = \frac{\pi}{3}}$

j) $y = \text{arc sec}(-2) \Leftrightarrow \sec y = -2 \Leftrightarrow \frac{1}{\cos y} = -2$

$\text{Im arc sec} = [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$; $\cos y = -\frac{1}{2}$



$\Rightarrow \underbrace{y = +\frac{2\pi}{3}, \frac{4\pi}{3}}$

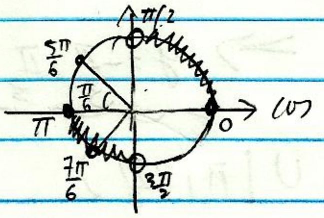
Mas

$y \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2}) \Rightarrow \boxed{y = \frac{4\pi}{3}}$

k) $y = \text{arc sec}\left(-\frac{2\sqrt{3}}{3}\right) \Leftrightarrow \sec y = -\frac{2\sqrt{3}}{3}$

$\therefore$

$\text{Im arc sec} = [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$; $\cos y = -\frac{3}{2\sqrt{3}} = -\frac{\sqrt{3}}{2}$

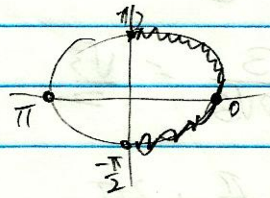


$\Rightarrow y = \frac{5\pi}{6}, \frac{7\pi}{6}$

Mas, $y \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2}) \Rightarrow \boxed{y = \frac{7\pi}{6}}$

l) $y = \text{arc sin } 0 \Leftrightarrow \sin y = 0 \Rightarrow y = 0, \pi$

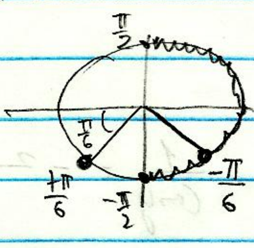
$\text{Im arc sin} = [-\frac{\pi}{2}, \frac{\pi}{2}]$; $y \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow \boxed{y = 0}$



m) $y = \text{arc sin } -\frac{1}{2} \Leftrightarrow \sin y = -\frac{1}{2} \Rightarrow y = -\frac{\pi}{6}, \frac{7\pi}{6}$

$\text{Im arc sin} = [-\frac{\pi}{2}, \frac{\pi}{2}]$

$y \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow \boxed{y = -\frac{\pi}{6}}$



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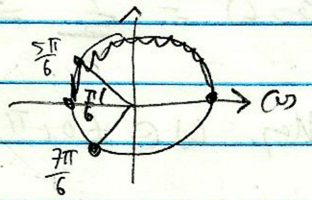
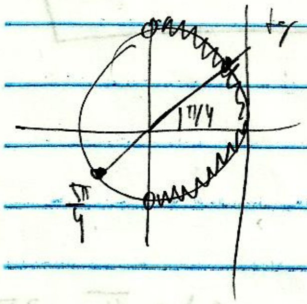
n) $y = \text{arc cos} \frac{-\sqrt{3}}{2} \Leftrightarrow \cos y = \frac{-\sqrt{3}}{2} \Rightarrow \underbrace{y = \frac{5\pi}{6}, \frac{7\pi}{6}}$

Im arc cos $= [0, \pi]$; Mas $y \in [0, \pi] \Rightarrow \boxed{y = \frac{5\pi}{6}}$



o) $y = \text{arc tg } 1 \Leftrightarrow \text{tg } y = 1 \Rightarrow \underbrace{y = \frac{\pi}{4}, \frac{5\pi}{4}}$

Im arc tg $= (-\frac{\pi}{2}, \frac{\pi}{2})$; Mas $y \in (-\frac{\pi}{2}, \frac{\pi}{2}) \Rightarrow \boxed{y = \frac{\pi}{4}}$



2)

a) $y = \sin(\arccos \frac{1}{2})$

Seja, $\omega = \arccos \frac{1}{2} \Leftrightarrow \cos\omega = \frac{1}{2} \Rightarrow \omega = -\frac{\pi}{3}, \frac{\pi}{3}$

$\text{Im} \arccos = [0, \pi]$; Mas $\omega \in [0, \pi]$

$\Rightarrow \omega = \frac{\pi}{3}$

∴

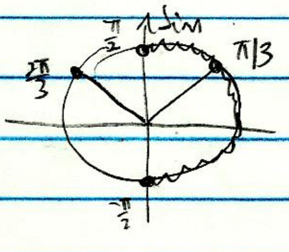
$y = \sin(\underbrace{\arccos \frac{1}{2}}_{\omega}) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

∴ $\boxed{\sin(\arccos \frac{1}{2}) = \frac{\sqrt{3}}{2}}$

b) $tg(\arcsin \frac{\sqrt{3}}{2})$

$\omega = \arcsin \frac{\sqrt{3}}{2} \Leftrightarrow \sin\omega = \frac{\sqrt{3}}{2} \Rightarrow \omega = \frac{\pi}{3}, \frac{2\pi}{3}$

$\text{Im} \arcsin = [-\frac{\pi}{2}, \frac{\pi}{2}]$; Mas $\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow \omega = \frac{\pi}{3}$



∴

$tg(\underbrace{\arcsin \frac{\sqrt{3}}{2}}_{\omega}) = tg \frac{\pi}{3} = \sqrt{3}$

∴ $\boxed{tg(\arcsin \frac{\sqrt{3}}{2}) = \sqrt{3}}$

c) $\sec\left(\text{arc}\cos\frac{\sqrt{3}}{2}\right)$

$\omega = \text{arc}\cos\frac{\sqrt{3}}{2} \Leftrightarrow \cos\omega = \frac{\sqrt{3}}{2} \Rightarrow \omega = -\frac{\pi}{6}, \frac{\pi}{6}$

$\text{Im arc}\cos = [0, \pi]$; Mas $\omega \in [0, \pi] \Longrightarrow \omega = \frac{\pi}{6}$

$\therefore$

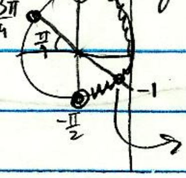
$\sec\left(\underbrace{\text{arc}\cos\frac{\sqrt{3}}{2}}_{\omega}\right) = \sec\frac{\pi}{6} = \frac{1}{\cos\frac{\pi}{6}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$

$$\boxed{\sec\left(\text{arc}\cos\frac{\sqrt{3}}{2}\right) = \frac{2}{\sqrt{3}}}$$

d) $\text{cosec}(\text{arc tg}(-1))$

$\omega = \text{arc tg}(-1) \Leftrightarrow \text{tg}\,\omega = -1 \Longrightarrow \omega = -\frac{\pi}{4}, \frac{3\pi}{4}$

$\text{Im arctg} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$\omega \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\Downarrow$
$\omega = -\frac{\pi}{4}$

$\therefore$

$\text{Cosec}\left(\underbrace{\text{arc tg}(-1)}_{\omega}\right) = \text{Cosec}\left(-\frac{\pi}{4}\right) = \frac{1}{\sin\left(-\frac{\pi}{4}\right)}$

$= \frac{1}{-\frac{\sqrt{2}}{2}}$

$= -\frac{2}{\sqrt{2}} = -\sqrt{2}$

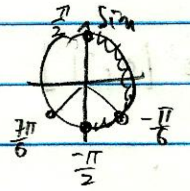
$\therefore$

$$\boxed{\text{Cosec}(\text{arc tg}(-1)) = -\sqrt{2}}$$

e) $\sin(\arcsin(-\frac{1}{2}))$

$\omega = \arcsin(-\frac{1}{2}) \Longrightarrow \sin\omega = -\frac{1}{2} \Rightarrow \omega = -\frac{\pi}{6}, \frac{7\pi}{6}$

Im arc sin $= [-\frac{\pi}{2}, \frac{\pi}{2}]$



$\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\Downarrow$

$\omega = -\frac{\pi}{6}$

∴

$\sin(\underbrace{\arcsin(-\frac{1}{2})}_{\omega}) = \sin(-\frac{\pi}{6}) = -\sin\frac{\pi}{6} = -\frac{1}{2}$

∴ $\boxed{\sin(\arcsin(-\frac{1}{2})) = -\frac{1}{2}}$

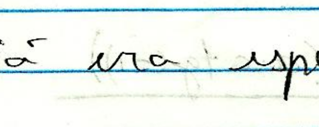
Obs.: Tal resultado já era esperado pois

$\sin(\arcsin x) = x \Longrightarrow \sin(\arcsin -\frac{1}{2}) = -\frac{1}{2}$

f) $\operatorname{cosec}(\text{arc cotg}(-\sqrt{3}))$

$\omega = \text{arc cotg}(-\sqrt{3}))$ $\Leftrightarrow$ $\operatorname{cotg}\omega = -\sqrt{3} \Longrightarrow \omega = -\frac{\pi}{6}, \frac{5\pi}{6}$

Im cotg $= (0, \pi)$



$\omega \in (0, \pi)$

$\Downarrow$

$\omega = \frac{5\pi}{6}$

∴

$\operatorname{cosec}(\underbrace{\text{arc cotg}(-\sqrt{3})}_{\omega}) = \operatorname{cosec}(\overset{150°}{\frac{5\pi}{6}}) = \frac{1}{\sin\frac{5\pi}{6}} = \frac{1}{\frac{1}{2}} = 2$

∴ $\boxed{\operatorname{cosec}(\text{arc cotg}(-\sqrt{3})) = \cancel{1/2}}$ 2

1 / 15

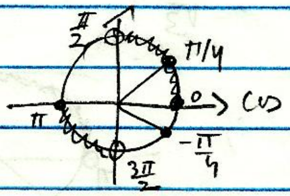
g) $\underline{\text{cosec}\,(\text{arc sec}\,\sqrt{2})}$

$\omega = \text{arc sec}\,\sqrt{2} \iff \sec\omega = \sqrt{2} \iff \cos\omega = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\Downarrow$

$\omega = -\frac{\pi}{4}, \frac{\pi}{4}$

$\text{Im sec} = [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$



$\omega \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$

$\Longrightarrow \omega = \frac{\pi}{4}$

$\therefore$

$\text{cosec}\,(\underbrace{\text{arc sec}\,\sqrt{2}}_{\omega}) = \text{cosec}\,\frac{\pi}{4} = \frac{1}{\sin\frac{\pi}{4}} = \frac{1}{\frac{\sqrt{2}}{2}}$

$= \frac{2}{\sqrt{2}} = \sqrt{2}$

$\therefore$

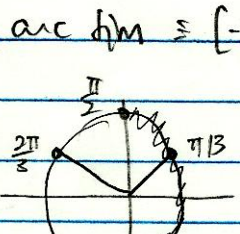
$$\boxed{\text{cosec}\,(\text{arc sec}\,\sqrt{2}) = \sqrt{2}}$$

h) $\underline{\text{arc sin}\,(\cos\frac{\pi}{6})}$

$y = \text{arc sin}\,(\underbrace{\cos\frac{\pi}{6}}) = \text{arc sin}\,(\frac{\sqrt{3}}{2}) = y$

$\Updownarrow$

$\text{Im arc sin} = [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\quad \sin y = \frac{\sqrt{3}}{2} \Longrightarrow y = \frac{\pi}{3}, \frac{2\pi}{3}$



Mas $y \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Longrightarrow y = \frac{\pi}{3}$

$\therefore$

$$\boxed{\text{arc sin}\,(\cos\frac{\pi}{6}) = \frac{\pi}{3}}$$

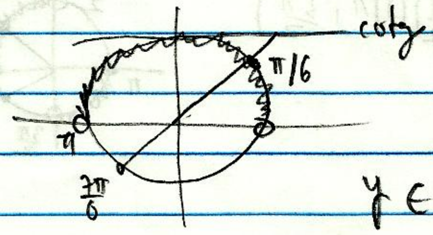
i) $\text{arc cotg}\left(\text{tg}\frac{\pi}{3}\right)$

$$\underline{y} = \text{arc cotg}\left(\underbrace{\text{tg}\frac{\pi}{3}}_{\sqrt{3}}\right) = \underline{\text{arc cotg}\sqrt{3}} \Longrightarrow \text{cotg}\, y = \sqrt{3}$$

$$\Downarrow$$

$$y = \frac{\pi}{6}, \frac{7\pi}{6}$$

$\text{Im cotg} = (0, \pi)$;



$$y \in (0, \pi) \Longrightarrow \underline{y = \frac{\pi}{6}}$$

$$\therefore \boxed{\text{arc cotg}\left(\text{tg}\frac{\pi}{3}\right) = \frac{\pi}{6}}$$

j) $\text{arc tan}(\tan 0)$

$$\text{arc tan}(\tan 0) = 0$$

$$f^{-1}(f(0)) = 0$$

$$\therefore \boxed{\text{arc tan}(\tan 0) = 0}$$

3a)

$$f(x) = \frac{\cotg 2x}{\sin \frac{x}{3}}$$

Dom f

$\cotg 2x$: $2x \neq n\pi \; ; \; n \in \mathbb{Z}$

$\therefore \; x \neq \frac{n\pi}{2} \; ; \; n \in \mathbb{Z} \quad (*)$

$\frac{1}{\sin \frac{x}{3}}$: $\frac{x}{3} \neq n\pi \; ; \; n \in \mathbb{Z}$

$x \neq 3n\pi \; ; \; n \in \mathbb{Z} \quad (**)$

De (*) e (**) devemos ter

$x \neq \frac{n\pi}{2}$

$x \neq 3n\pi$

Notemos que a forma $x \neq \frac{n\pi}{2}$ quando $n = 6k$ $(k \in \mathbb{Z})$ inclui a condição (**) :

$$x \neq \overset{\curvearrowright}{\frac{n\pi}{2} \neq \frac{6k\pi}{2}} = 3k\pi$$

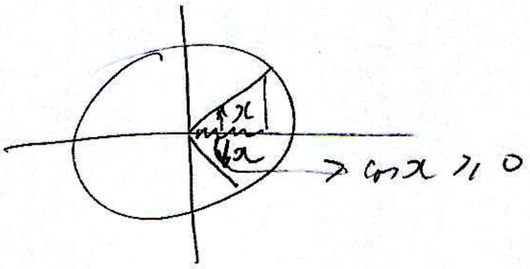
$x \neq 3k\pi \; ; \; k \in \mathbb{Z}.$

Assim $\| \mathrm{Dom}\, f = \mathbb{R} - \{ \frac{n\pi}{2} \; ; \; n \in \mathbb{Z} \} \|$

3b)

$$f(x) = \sqrt{\cos x}$$

$\cos x \geq 0 \implies -\frac{\pi}{2} + 2n\pi \leq x \leq \frac{\pi}{2} + 2n\pi \;, n \in \mathbb{Z}$



$\therefore$ // $\mathrm{Dom}\, f = \bigcup_{n \in \mathbb{Z}} \left[-\frac{\pi}{2} + 2n\pi \,,\; \frac{\pi}{2} + 2n\pi \right]$ //

3e)

$$f(x) = (\sin x - 2\sin^2 x)^{-3/4}$$

$$= \frac{1}{\sqrt[4]{(\sin x - 2\sin^2 x)^3}}$$

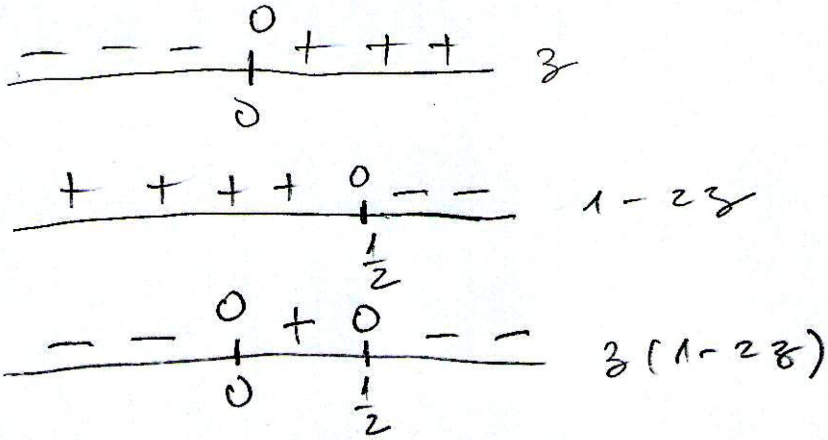
Devemos ter então,

$$\sin x - 2\sin^2 x > 0$$

$$\sin x\,(1 - 2\sin x) > 0$$

Seja $z = \sin x$, daí

$$z(1-2z) > 0$$

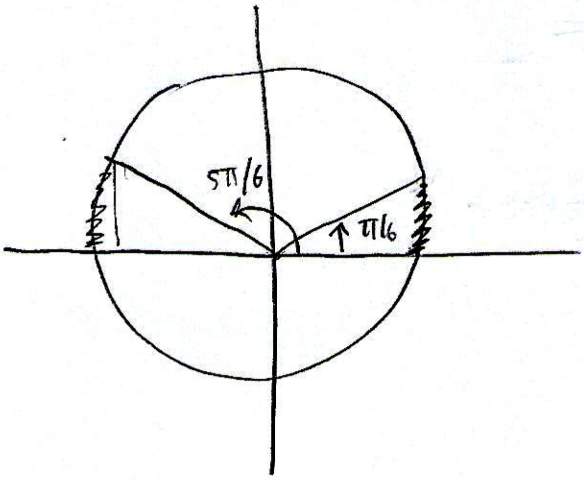


Daí, $z(1-2z) > 0 \Rightarrow 0 < \underset{\sim}{z} < \frac{1}{2}$

$$0 < \sin x < \frac{1}{2}$$

→

$$0 < \text{sen}\, x < \frac{1}{2}$$



Vemos do ciclo trigonométrico que se

$\to 0 < \omega < \frac{\pi}{6}$ então

$0 < \text{sen}\, \omega < \frac{1}{2}$

$\to \frac{5\pi}{6} < \omega < \pi$ então

$0 < \text{sen}\, \omega < \frac{1}{2}$

Daí $x = \omega + 2n\pi$ resolve

$$0 < \text{sen}\, x < \frac{1}{2}$$

$$\therefore \quad 0 + 2n\pi < x < \frac{\pi}{6} + 2n\pi \ , \ n \in \mathbb{Z}$$

ou

$$\frac{5\pi}{6} + 2n\pi < x < \pi + 2n\pi \ ; \ n \in \mathbb{Z}$$

$$\therefore \quad // \ \text{Dom}\, f = \bigcup_{n \in \mathbb{Z}} \left(\left(2n\pi, \frac{\pi}{6} + 2n\pi\right) \cup \left(\frac{5\pi}{6} + 2n\pi, \pi + 2n\pi\right) \right) //$$

3d)

$$f(x) = \arccos(3-x)$$

$3 - x \in [-1, 1]$

$\therefore \quad -1 \leq 3 - x \leq 1$

$$-1-3 \leq -x \leq 1-3$$

$$-4 \leq -x \leq -2$$

$$4 \geqslant x \geqslant 2$$

$\therefore$

$//$ Dom $f = [2, 4]$ $//$

3e)

$$f(x) = \arcsin\left(\tfrac{1}{2}x - 1\right) + \arccos\left(1 - \tfrac{1}{2}x\right)$$

$\arcsin\left(\tfrac{1}{2}x - 1\right)$: $\quad \tfrac{1}{2}x - 1 \in [-1,1]$

$$\therefore$$

$$-1 \leq \tfrac{1}{2}x - 1 \leq 1$$

$$-1+1 \leq \tfrac{1}{2}x \leq 1+1$$

$$0 \leq \tfrac{1}{2}x \leq 2$$

$$0 \leq x \leq 4 \qquad (*)$$

$\arccos\left(1 - \tfrac{1}{2}x\right)$: $\quad 1 - \tfrac{1}{2}x \in [-1,1]$

$$\therefore$$

$$-1 \leq 1 - \tfrac{1}{2}x \leq 1$$

$$-1-1 \leq -\tfrac{1}{2}x \leq 1-1$$

$$-2 \leq -\tfrac{1}{2}x \leq 0$$

$$\downarrow \times(-2)$$

$$4 \geq x \geq 0$$

$$\therefore \quad 0 \leq x \leq 4 \qquad (**)$$

De $(*)$ $\underline{\underline{e}}$ $(**)$ temos : $\quad 0 \leq x \leq 4$

$$\therefore \quad // \operatorname{Dom} f = [0,4] //$$

3f)

$$f(x) = 3 \operatorname{arc\,sin} \sqrt{\frac{3x-1}{2}}$$

$\sqrt{\frac{3x-1}{2}}$; $\frac{3x-1}{2} \geq 0$ (*)

$\operatorname{arc\,sin} \sqrt{\frac{3x-1}{2}}$ ∴ $\sqrt{\frac{3x-1}{2}} \in [-1,1]$ (**)

De (*) ; $\frac{3x-1}{2} \geq 0$

$3x - 1 \geq 0$

$3x \geq 1$

∴ $x \geq \frac{1}{3}$ (4*)

De (**) : $-1 \leq \underbrace{\sqrt{\frac{3x-1}{2}}}_{\text{positivo}} \leq 1$

∴ $0 \leq \sqrt{\frac{3x-1}{2}} \leq 1$

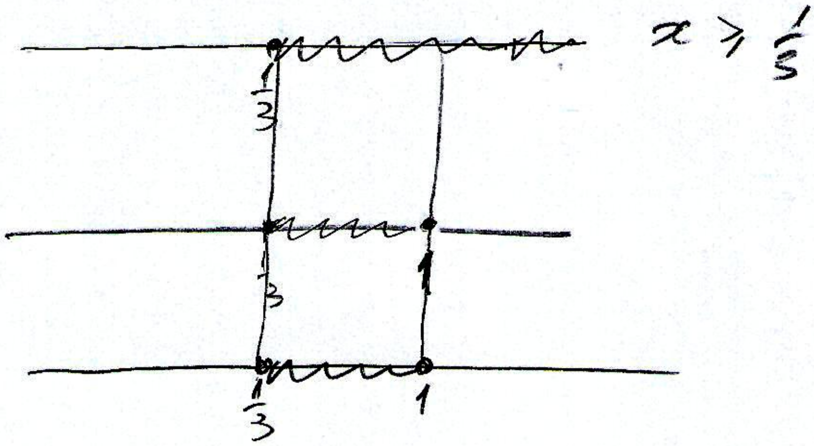
$0 \leq \frac{3x-1}{2} \leq 1$

$0 \leq 3x - 1 \leq 2$

$1 \leq 3x \leq 3$

$\frac{1}{3} \leq x \leq 1$ (5*)

De (4*) e (5*) :



$\frac{1}{3} \leq x \leq 1$

$\therefore$ $\| \text{Dom} f = [\frac{1}{3}, 1] \|$

38)

$$f(x) = \arc\cos \frac{1}{x-1}$$

$\frac{1}{x-1}$: $x \neq 1$

$\arc\cos \frac{1}{x-1}$: $\frac{1}{x-1} \in [-1,1]$

$$\therefore \quad -1 \leq \frac{1}{x-1} \leq 1$$

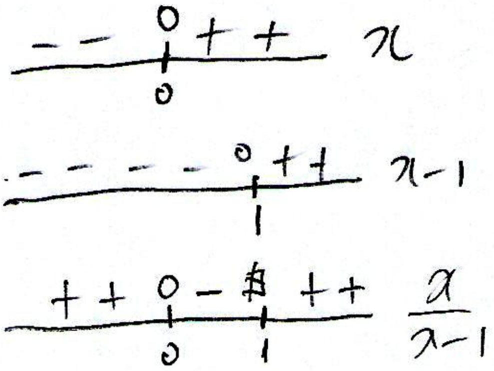
Esta inequação é equivalente à :

$$-1 \leq \frac{1}{x-1} \quad \underline{\underline{e}} \quad \frac{1}{x-1} \leq 1$$

$$\therefore \quad 0 \leq \frac{1}{x-1} + 1$$

$$0 \leq \frac{1+x-1}{x-1}$$

$$0 \leq \frac{x}{x-1}$$

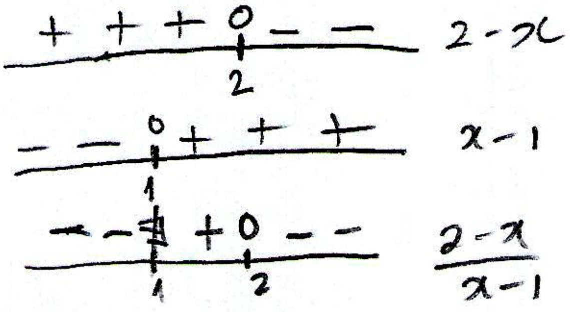


$$\frac{1}{x-1} - 1 \leq 0$$

$$\frac{1-(x-1)}{x-1} \leq 0$$

$$\frac{1-x+1}{x-1} \leq 0$$

$$\frac{2-x}{x-1} \leq 0$$

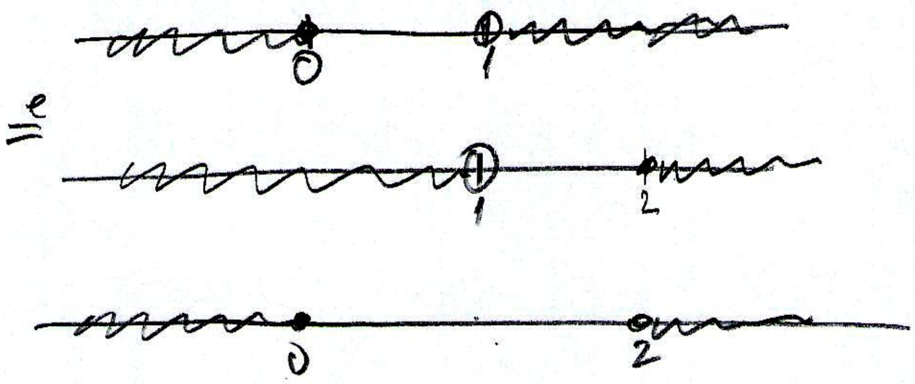


Destes diagramas temos que

$\frac{x}{x-1} \geqslant 0 \Longrightarrow x \leq 0$ ou $x > 1$

e

$\frac{2-x}{x-1} \leq 0 \Longrightarrow x < 1$ ou $x \geqslant 2$



$\| \text{Dom } f = (-\infty, 0] \cup [2, +\infty) \|$

3h) $f(x) = \dfrac{\operatorname{tg} x}{\cos 2x}$

$\operatorname{tg} x$: $x \neq \frac{\pi}{2} + n\pi$; $n \in \mathbb{Z}$ $(*)$

$\cos 2x$: $2x \neq \frac{\pi}{2} + n\pi$; $n \in \mathbb{Z}$

$x \neq \frac{\pi}{4} + \frac{n\pi}{2}$ $(**)$

De $(*)$ e $(**)$ temos :

$$\operatorname{Dom} f = \mathbb{R} - \left\{ \frac{\pi}{2} + n\pi , \frac{\pi}{4} + \frac{n\pi}{2} : n \in \mathbb{Z} \right\}$$

3i)

$$f(x) = \sqrt{\frac{\sin x + \cos x}{\sin x - \cos x}}$$

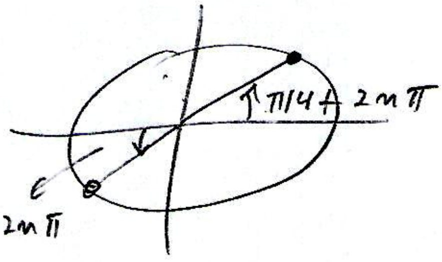
Dom f

$$\begin{cases} \dfrac{\sin x + \cos x}{\sin x - \cos x} \geq 0 & (*) \\ \sin x - \cos x \neq 0 & (2*) \end{cases}$$

(2*) :

$$\sin x - \cos x = 0$$

$$\sin x = \cos x$$



$$\Downarrow$$

$x = \frac{\pi}{4} + n\pi$; $n \in \mathbb{Z}$ (inclui ambas as possibilidades: $x = \frac{\pi}{4} + 2n\pi$ e $x = \frac{5\pi}{4} + 2n\pi$)

Daí devemos ter,

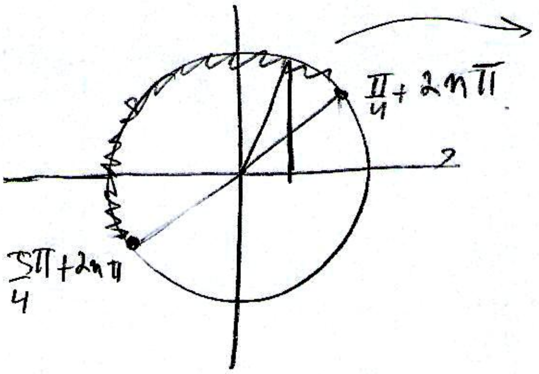
$$\sin x - \cos x \neq 0$$

$$\therefore \quad \| x \neq \frac{\pi}{4} + n\pi \|$$

(*)

$$\frac{\sin x + \cos x}{\sin x - \cos x} \geq 0 .$$

Análise do sinal de $\sin x - \cos x$:



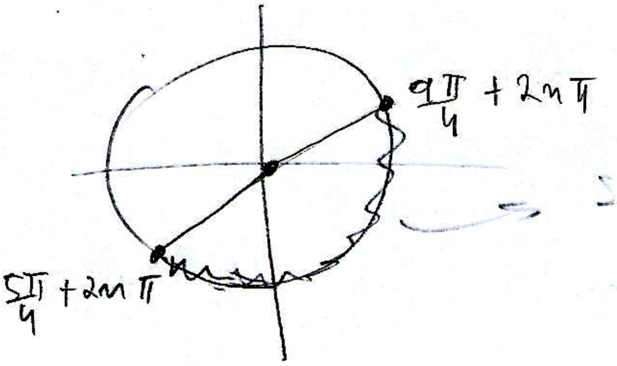
Se $\frac{\pi}{4} + 2n\pi \leq x \leq \frac{5\pi}{4} + 2n\pi$, $n \in \mathbb{Z}$

Vemos que

$\sin x \geq \cos x$

isto é :

$\sin x - \cos x \geq 0$.



Se $\frac{5\pi}{4} + 2n\pi \leq x \leq \frac{9\pi}{4} + 2n\pi$

Vemos que

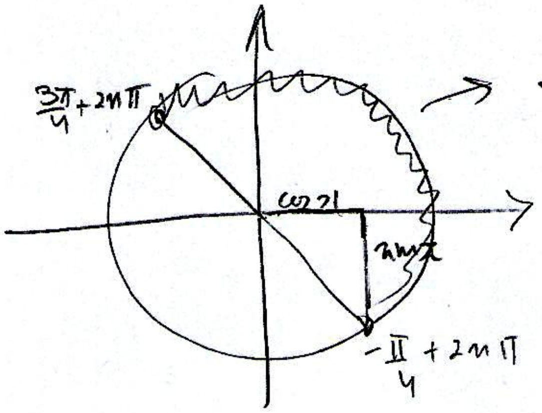
$\sin x \leq \cos x$

$\therefore \sin x - \cos x \leq 0$

O que nos dá então a seguinte representação simbólica para o sinal de $\sin x - \cos x$:

	$\frac{\pi}{4} + 2n\pi$		$\frac{5\pi}{4} + 2n\pi$		$\frac{9\pi}{4} + 2n\pi$
$\sin x - \cos x$	0	+ + +	0	− − −	0

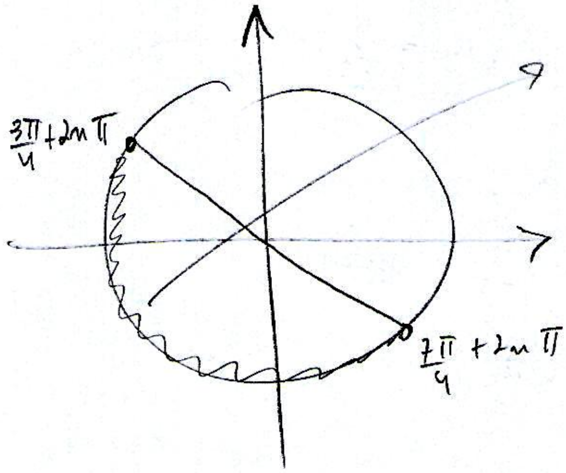
Análise do sinal de $\sin x + \cos x$;



Se $-\frac{\pi}{4}+2n\pi \leq x \leq \frac{3\pi}{4}+2n\pi$; $n \in \mathbb{Z}$

temos

$\sin x + \cos x \geq 0$

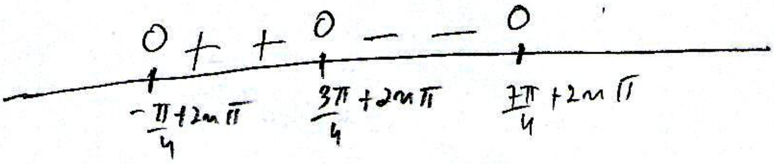


Se $\frac{3\pi}{4}+2n\pi \leq x \leq \frac{7\pi}{4}+2n\pi$, $n \in \mathbb{Z}$

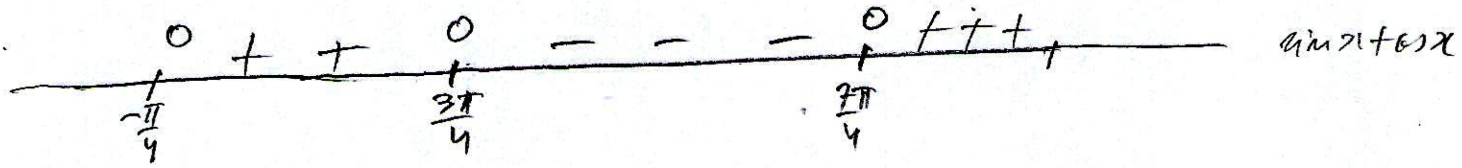
temos

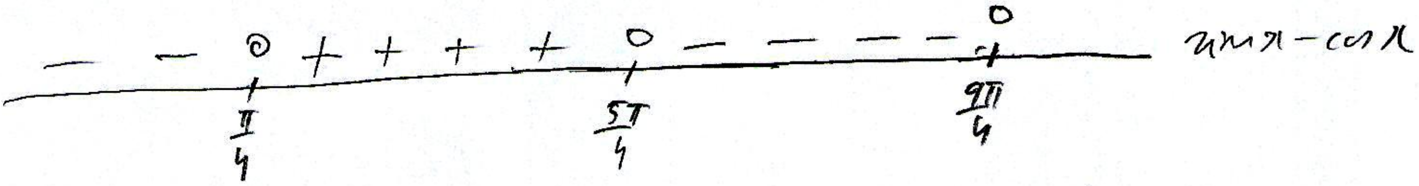
$\sin x + \cos x \leq 0$

o que nos dá o seguinte diagrama representando o sinal de $\sin x + \cos x$

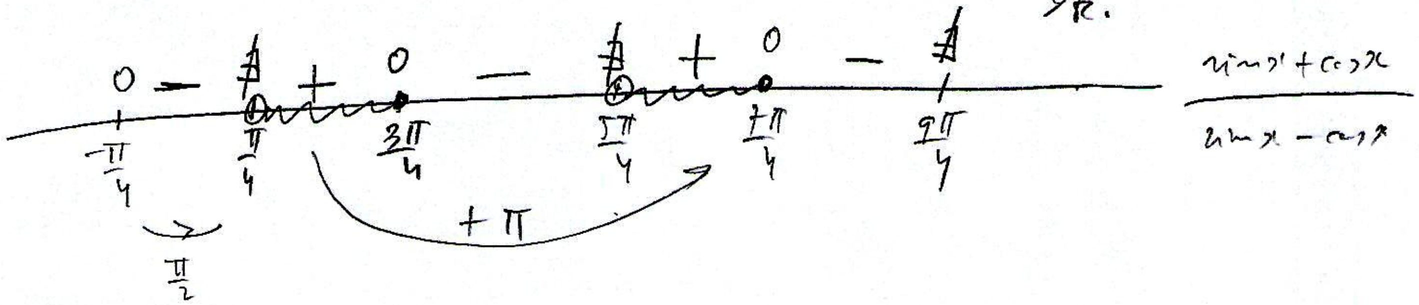


Temos então





ok.



$$\frac{\sin x + \cos x}{\sin x - \cos x} \geqslant 0 \implies \left(\frac{\pi}{4} + n\pi, \frac{3\pi}{4} + n\pi\right]$$

$$\therefore \; \Big\| \; \mathrm{Dom}\, f = \bigcup_{n \in \mathbb{Z}} \left(\frac{\pi}{4} + n\pi, \frac{3\pi}{4} + n\pi\right] \; \Big\|$$

3j) $f(x) = \arccos x - \arcsin(3-x)$

Dom f

(*) $x \in [-1,1]$ (Domínio da arc cos)

e

(**) $3-x \in [-1,1]$ (Domínio da arc sin)

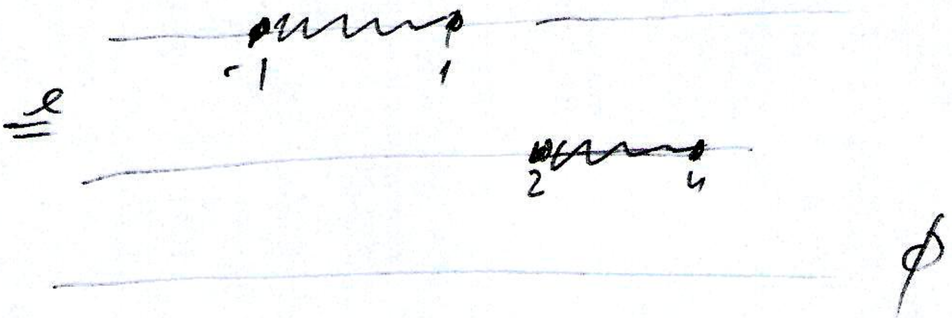
(*): $-1 \leq x \leq 1$

(**): $-1 \leq 3-x \leq 1$

$-4 \leq -x \leq -2$

$\therefore 4 \geq x \geq 2$

Daí



// Dom f = ∅ // (Não é função)

3K)

$f(x) = \operatorname{arc\,tg} \dfrac{x}{x^2-9}$

Dom f

$\dfrac{x}{x^2-9} \in \mathbb{R}$ (Dominio da arc tg)

$\therefore \quad x^2 - 9 \neq 0$

$\therefore \quad x \neq \pm 3$

$$\| \text{Dom } f = \mathbb{R} - \{-3, +3\} \|$$

3l)

$$f(x) = \text{arc sin } \frac{x^2-1}{x}$$

Dom f

$$\frac{x^2-1}{x} \in [-1,1]$$ (Domínio da arc sin)

$$\therefore \quad -1 \le \frac{x^2-1}{x} \le 1$$

$$-1 \le \frac{x^2-1}{x} \quad \text{e} \quad \frac{x^2-1}{x} \le 1$$

$$\therefore \quad 0 \le \frac{x^2-1}{x} + 1 \quad \text{e} \quad \frac{x^2-1}{x} - 1 \le 0$$

$$0 \le \frac{x^2+x-1}{x}$$

+	0	−		−	0	+	x^2+x-1
	$\frac{-1-\sqrt{5}}{2}$				$\frac{-1+\sqrt{5}}{2}$		
− −		−	0	+		+ +	x
			0				
−	0	+	∄	−	0	+	$\frac{x^2+x-1}{x}$
	$\frac{-1-\sqrt{5}}{2}$		0		$\frac{-1+\sqrt{5}}{2}$		

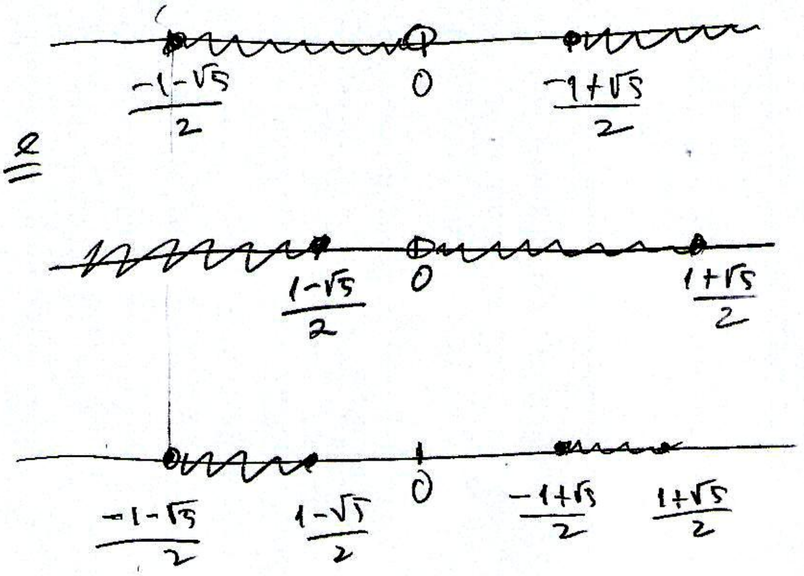
$$\frac{x-x-1}{x} \le 0$$

+	0	−		−	0	+	x^2-x-1
	$\frac{1-\sqrt{5}}{2}$				$\frac{1+\sqrt{5}}{2}$		
− −			0	+		+ +	x
			0				
−	0	+	∄	−	0	+	$\frac{x^2-x-1}{x}$
	$\frac{1-\sqrt{5}}{2}$		0		$\frac{1+\sqrt{5}}{2}$		

$$0 \le \frac{x^2+x-1}{x} \Rightarrow$$

$$\frac{x^2-x-1}{x} \le 0 \Rightarrow$$

$$\Rightarrow \left(\frac{-1-\sqrt{5}}{2} \le x < 0 \text{ ou } x \geq \frac{-1+\sqrt{5}}{2}\right) \quad \text{e} \quad \Rightarrow \left(x \le \frac{1-\sqrt{5}}{2} \text{ ou } 0 < x \le \frac{1+\sqrt{5}}{2}\right)$$



$\therefore$ $\frac{-1-\sqrt5}{2} \leq x \leq \frac{1-\sqrt5}{2}$ ou $\frac{-1+\sqrt5}{2} \leq x \leq \frac{1+\sqrt5}{2}$

$\therefore$ $\boxed{\text{Dom} f = \left[\frac{-1-\sqrt5}{2}, \frac{1-\sqrt5}{2}\right] \cup \left[\frac{-1+\sqrt5}{2}, \frac{1+\sqrt5}{2}\right]}$

3m)

$$f(x) = \sqrt{\arcsin x - \arccos x}$$

Dom f

$$\begin{cases} \arcsin x - \arccos x \geq 0 & (*) \\ x \in [-1,1] & \text{(para termos definido } \arcsin x \text{ e } \arccos x) \quad (**) \end{cases}$$

Seja

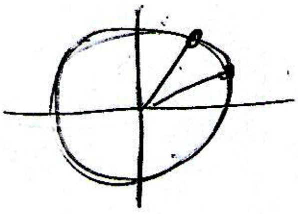
$w = \arcsin x$	$z = \arccos x$
$\therefore \ \sin w = x$	$\therefore \ \cos z = x$
$w \in [-\frac{\pi}{2}, \frac{\pi}{2}]$	$z \in [0, \pi]$

Queremos então resolver a desigualdade

$$w - z \geq 0$$

Mas $\sin w = \cos z \implies w + z = \frac{\pi}{2}$

com $\begin{cases} w \in [-\frac{\pi}{2}, \pi] \\ z \in [0, \pi] \end{cases}$



Com $0 \leq w \leq \frac{\pi}{2}$

$0 \leq z \leq \frac{\pi}{2}$

Daí, $w - z \geq 0 \implies w - (\frac{\pi}{2} - w) \geq 0$

$$2w - \frac{\pi}{2} \geq 0$$

$$2w \geq \frac{\pi}{2}$$

$$w \geq \frac{\pi}{4}$$ isto é

$$\frac{\pi}{4} \leq \omega \leq \frac{\pi}{2}$$

$$\therefore \quad \frac{\sqrt{2}}{2} \leq \sin \omega \leq 1.$$

$$\frac{\sqrt{2}}{2} \leq x \leq 1 \qquad \left(\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \right)$$

$$\therefore \quad /\!/ \; \text{Dom } f = \left[\frac{1}{\sqrt{2}}, 1 \right] \; /\!/$$

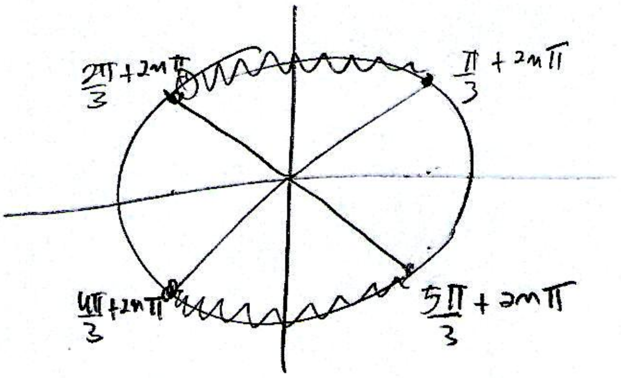
3n)

$f(x) = \arcsin(2\cos x)$

Dom f

$2\cos x \in [-1,1]$ (Domínio do arc sin)

$\therefore \cos x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$



Mas $\cos x = -\frac{1}{2}$

$$\therefore \begin{cases} x = \frac{2\pi}{3} + 2n\pi \\ \text{ou} \\ x = \frac{4\pi}{3} + 2n\pi \end{cases}$$

e $\cos x = \frac{1}{2}$

$$\therefore \begin{cases} x = \frac{\pi}{3} + 2n\pi \\ \text{ou} \\ x = \frac{5\pi}{3} + 2n\pi \end{cases}$$

$$\therefore \text{ou} \begin{cases} \frac{4\pi}{3} + 2n\pi \leq x \leq \frac{5\pi}{3} + 2n\pi \\ \frac{\pi}{3} + 2n\pi \leq x \leq \frac{2\pi}{3} + 2n\pi \end{cases}$$

que podem ser escritos compactamente na forma

$$\frac{\pi}{3} + n\pi \leq x \leq \frac{2\pi}{3} + n\pi \quad ; n \in \mathbb{Z}$$

$$\text{Dom } f = \bigcup_{n \in \mathbb{Z}} \left[\frac{\pi}{3} + n\pi, \frac{2\pi}{3} + n\pi\right]$$

30)

$$f(x) = tg\,(2 \arccos x)$$

Dom f

(*) $2 \arccos x \neq \frac{\pi}{2} + n\pi$ (domínio da tangente)

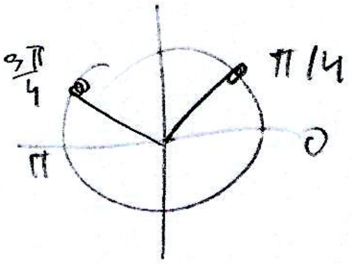
(**) $x \in [-1, 1]$ (domínio da arccos)

(*) : $2 \arccos x = \frac{\pi}{2} + n\pi \; ; \; n \in \mathbb{Z}$

$\therefore \; \arccos x = \frac{\pi}{4} + \frac{n\pi}{2} \; ; \; n \in \mathbb{Z}$

Mas $\arccos x \in [0, \pi]$, daí temos duas possibilidades :

$\arccos x = \frac{\pi}{4}$ ou $\arccos x = \frac{3\pi}{4}$



$\therefore \; \cos \frac{\pi}{4} = x$

$\frac{\sqrt{2}}{2} = x$

$\cos \frac{3\pi}{4} = x$

$-\frac{\sqrt{2}}{2} = x$

$\therefore \; 2 \arccos x \neq \frac{\pi}{2} + n\pi \Longrightarrow x \neq \pm\frac{\sqrt{2}}{2}$.

De (**) : $x \in [-1,1]$, daí uma vez que $x \neq \pm \frac{\sqrt{2}}{2}$ temos que $x \in [-1, -\frac{\sqrt{2}}{2}) \cup (\frac{\sqrt{2}}{2}, 1]$

// $\text{Dom } f = [-1, -\frac{\sqrt{2}}{2}) \cup (\frac{\sqrt{2}}{2}, 1]$ //

3 p)

$$f(x) = \frac{\text{arc sin}\left(\frac{1}{2}x - 1\right)}{\sqrt{x^2 - 3x + 1}}$$

Dom f

$(*)\ \frac{1}{2}x - 1 \in [-1, 1]$ (Domínio da arc sin)

e

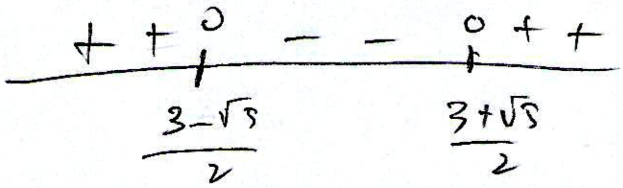
$(**)\ x^2 - 3x + 1 > 0$ (Vem do termo em raiz quadrada no denominador)

$(*)$: $-1 \leq \frac{1}{2}x - 1 \leq 1$

$\therefore\ -2 \leq x - 2 \leq 2$

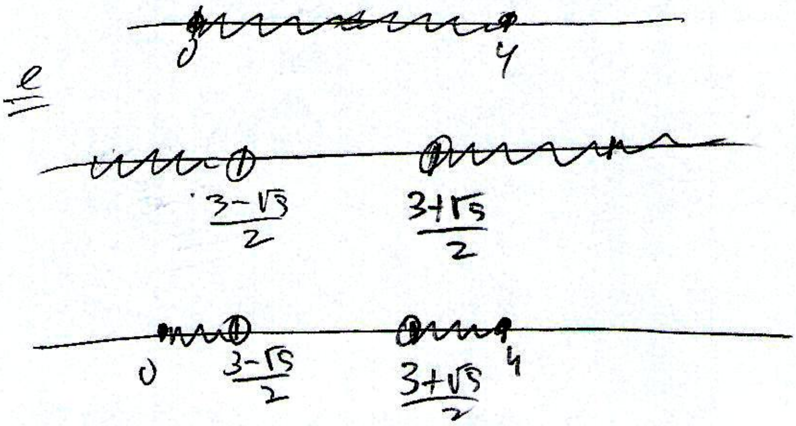
$0 \leq x \leq 4$

$(**)$: $x^2 - 3x + 1 > 0$



$x^2 - 3x + 1 > 0 \implies x < \frac{3-\sqrt{5}}{2}$ ou $x > \frac{3+\sqrt{5}}{2}$

e



$$\text{Dom } f = \left[0, \frac{3-\sqrt{5}}{2}\right) \cup \left(\frac{3+\sqrt{5}}{2}, 4\right]$$

39)

$$f(x) = \frac{\sqrt{4-x^2}}{\arcsin(2-x)}$$

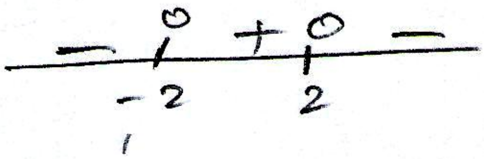
Dom f

(*) $4-x^2 \geq 0$ (oriundo da raiz quadrada)

e

(**) $\begin{cases} 2-x \in [-1,1] \\ x \neq 2 \end{cases}$ (Domínio da arc sin e do fato de arc sin está no denominador)

(*) $4-x^2 \geq 0$



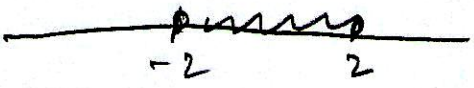
$4-x^2 \geq 0 \Longrightarrow -2 \leq x \leq 2$

(**) $-1 \leq 2-x \leq 1$, $x \neq 2$

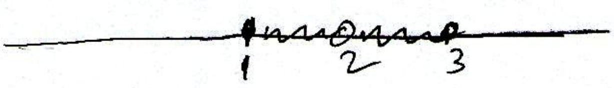
$\therefore -3 \leq -x \leq -1$ e $x \neq 2$

$\therefore 3 \geq x \geq 1$ e $x \neq 2$

Daí :



e



$$\boxed{\text{Dom } f = [1, 2)}$$

3η)

$$f(x) = \frac{\sqrt{\cos x - \frac{1}{2}}}{\sqrt{6 - 35x - 6x^2}}$$

Dom f

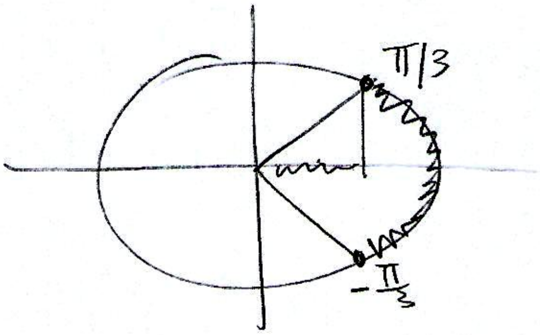
(*) $\cos x - \frac{1}{2} \geq 0$

&

(**) $6 - 35x - 6x^2 > 0$

(*) $\cos x \geq \frac{1}{2} \Longrightarrow -\frac{\pi}{3} + 2n\pi \leq x \leq \frac{\pi}{3} + 2n\pi$

$(n \in \mathbb{Z})$



(**) $6 - 35x - 6x^2 > 0$

$-6x^2 - 35x + 6 = 0$

$x = \frac{35 \pm \sqrt{1225 + 144}}{-12}$

$= \frac{35 \pm 37}{-12} = \begin{cases} \nearrow -6 \\ \searrow \frac{1}{6} \end{cases}$

$$\begin{array}{r} 35 \\ 35 \\ \hline 175 \\ 105 \\ \hline 1225 \\ 144 \\ \hline 1369 \end{array} \qquad \begin{array}{r} 36 \\ 4 \\ \hline 144 \end{array}$$

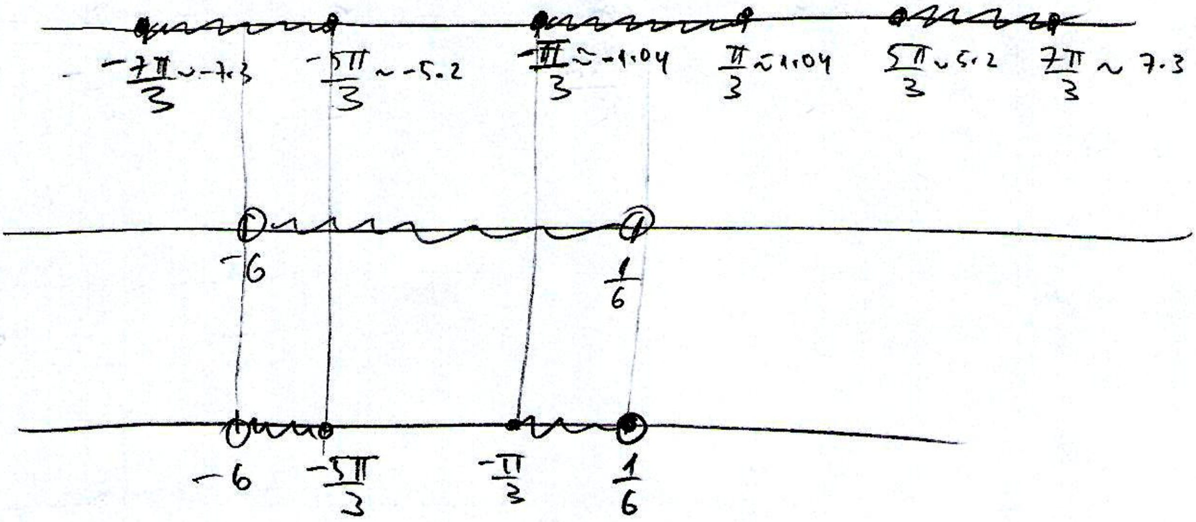
$$\begin{array}{ccccc} - & 0 & + & 0 & - \\ & -6 & & \frac{1}{6} & \end{array}$$

$6 - 35x - 6x^2 > 0 \Rightarrow$

$\Rightarrow -6 < x < \frac{1}{6}$

Analisando as duas condições

$$\left(-\frac{\pi}{3}+2n\pi \le x \le \frac{\pi}{3}+2n\pi \;;\; n\in\mathbb{Z}\right)$$



$$-6 < x \le -\frac{5\pi}{3} \quad \text{ou} \quad -\frac{\pi}{3} \le x < \frac{1}{6}$$

$$\therefore \; \Big\| \; \text{Dom } f = \left(-6, -\frac{5\pi}{3}\right] \cup \left[-\frac{\pi}{3}, \frac{1}{6}\right) \; \Big\|$$

4.

$$\begin{cases} I = [-\frac{\pi}{4}, \frac{\pi}{4}] \\ f : I \to \mathbb{R} \\ f(\sin 2x) = \sin x + \cos x \end{cases}$$

$$f^2(\sin 2x) = \sin^2 x + 2\underline{\sin x \cos x} + \cos^2 x$$

$$= 1 + \sin 2x$$

$\therefore$

$$f^2(z) = 1 + z$$

$$f(z) = \pm\sqrt{1+z} \qquad (*)$$

Mas $\quad -\frac{\pi}{4} \le x \le \frac{\pi}{4}$

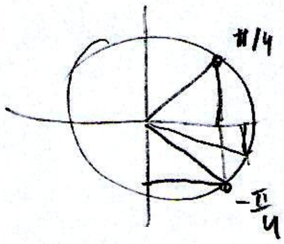
$\therefore \quad -\frac{\pi}{2} \le 2x \le \frac{\pi}{2}$

$\therefore \quad -1 \le \sin 2x \le 1$

$-1 \le z \le 1$

Mas $\quad f(\sin 2x) = \sin x + \cos x$

Se $\quad -\frac{\pi}{4} \le x \le \frac{\pi}{4} \Rightarrow \sin x \le \cos x$



$\therefore \quad \sin x + \cos x \geq 0$

$\therefore \quad f(\sin 2x) \geq 0$

$\therefore \quad // \; f(z) = \sqrt{1+z} \; //$

5:

$$f : A \longrightarrow [0,1]$$

$$f(x) = \sin^2 2x \quad , \quad A \subset [0, 2\pi]$$

Solução

$\left\{ \begin{array}{l} \text{Im} f = [0,1] \implies 0 \leq \sin^2 2x \leq 1 \\ f \text{ deve ser bijetiva} \end{array} \right.$

$\therefore$

$$0 \leq \sin 2x \leq 1$$

ou

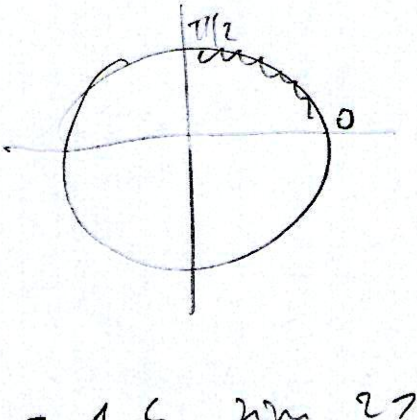
$$-1 \leq \sin 2x \leq 0$$

Mas $0 \leq \underbrace{\sin 2x}_{\text{injetiva}} \leq 1 \implies 0 \leq 2x \leq \frac{\pi}{2}$

ou

$$\frac{\pi}{2} \leq 2x \leq \pi$$

$\therefore$



i.e. $\underline{0 \leq x \leq \frac{\pi}{4}}$

ou

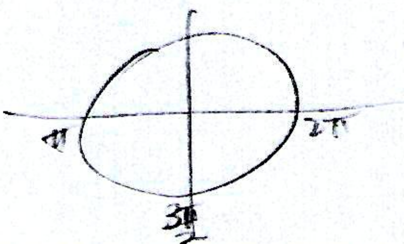
$$\underline{\frac{\pi}{4} \leq x \leq \frac{\pi}{2}}$$

$-1 \leq \underbrace{\sin 2x}_{\text{injetiva}} \leq 0 \implies \pi \leq 2x \leq \frac{3\pi}{2}$

ou

$$\frac{3\pi}{2} \leq 2x \leq 2\pi$$

i.e.



$$\frac{\pi}{2} \leq x \leq \frac{3\pi}{4}$$

ou

$$\frac{3\pi}{4} \leq x \leq \pi$$

Temos assim as possibilidades

$$A = \left[0, \frac{\pi}{4}\right] \text{ ou } \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \text{ ou } \left[\frac{\pi}{2}, \frac{3\pi}{4}\right] \text{ ou } \left[\frac{3\pi}{4}, \pi\right]$$

6a)

$\sec(\operatorname{arctg} x) = \sqrt{1+x^2}$

Seja $\omega = \operatorname{arctg} x$

$\therefore \begin{cases} \operatorname{tg}\omega = x & (*) \\ \omega \in (-\frac{\pi}{2}, \frac{\pi}{2}) \end{cases}$

Mas $\sec^2\omega = 1+\operatorname{tg}^2\omega$ (relação trigonométrica)

$\sec\omega = \pm\sqrt{1+\operatorname{tg}^2\omega}$ (**)

Mas, se $\omega \in (-\frac{\pi}{2}, \frac{\pi}{2})$ temos que

$\cos\omega > 0 \quad \therefore \quad \underbrace{\frac{1}{\cos\omega}} > 0 \quad \therefore \quad \underbrace{\sec\omega} > 0$

Assim, em (**) devemos tomar

$$\sec\underset{\sim}{\omega} = +\sqrt{1+\underset{\sim}{\operatorname{tg}^2\omega}}$$

$$\sec(\operatorname{arctg} x) = \sqrt{1+\underset{\sim}{x^2}} \quad \Big\downarrow \text{ de } (*)$$

$$\therefore \quad \Big/\!\!\Big/ \sec(\operatorname{arctg} x) = \sqrt{1+x^2} \Big/\!\!\Big/$$

6b)

$$\sin(\text{arc cosec } x) = \frac{1}{x}$$

Seja

$$w = \text{arc cosec } x \quad (*)$$

$$\therefore \begin{cases} \text{cosec } w = x & (**) \\ w \in (-\pi, -\frac{\pi}{2}] \cup (0, \frac{\pi}{2}] \end{cases}$$

Daí

$$\sin(\underset{\sim\sim\sim}{\text{arccosec}}\ x) \overset{\text{De } (*)}{=} \sin \underset{\sim}{w} = \frac{1}{\text{cosec } w} \overset{\text{De } (**)}{=} \frac{1}{x}$$

$$\therefore \Big/\Big/ \sin(\text{arc cosec } x) = \frac{1}{x} \Big/\Big/$$

6e)

$$\cos(2\arcsin x) = 1 - 2x^2$$

Seja

$$\omega = \arcsin x$$

$$\therefore \quad \sin\omega = x \qquad (*)$$

$$\omega \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Daí,

$$\begin{aligned}
\cos(\underline{2\arcsin x}) &= \cos \underline{2\omega} \\
&= \cos^2\omega - \sin^2\omega \\
&= (1 - \sin^2\omega) - \sin^2\omega \\
&= 1 - 2\,\underline{\sin^2\omega} \\
&= 1 - 2x^2 \qquad \downarrow \text{de } (*)
\end{aligned}$$

$$\therefore \quad /\!/ \cos(2\arcsin x) = 1 - 2x^2 /\!/$$

6d)

$$\sin(2\arcsin x) = 2x\sqrt{1-x^2}$$

Seja

$$\omega = \arcsin x$$

$$\therefore \begin{cases} \sin\omega = x & (*) \\ \omega \in [-\frac{\pi}{2}, \frac{\pi}{2}] \end{cases}$$

Daí

$$\sin(2\underbrace{\arcsin x}) = \sin 2\underline{\omega}$$

$$= 2\sin\omega\cos\omega \quad (**)$$

Mas

$$\sin^2\omega + \cos^2\omega = 1$$

$$\therefore \cos^2\omega = 1 - \sin^2\omega$$

$$\cos\omega = \pm\sqrt{1-\sin^2\omega}$$

uma vez que $\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ tem-se que $\cos\omega \geq 0$, daí devemos tomar

$$\cos\omega = +\sqrt{1-\sin^2\omega}.$$

Voltando a $(**)$ temos:

$$\sin(2\arcsin x) = 2\underline{\sin\omega}\sqrt{1-\sin^2\omega}$$

$$= 2x\sqrt{1-x^2} \quad \Big\} \text{De } (*)$$

$$\therefore \; // \sin(2\arcsin x) = 2x\sqrt{1-x^2} //$$

6l)

$$tg(\arcsin x) = \frac{x}{\sqrt{1-x^2}} \quad , \quad -1 < x < 1$$

Seja $w = \arcsin x$

$$\therefore \begin{cases} \sin w = x \\ w \in [-\frac{\pi}{2}, \frac{\pi}{2}] \end{cases} \quad (*)$$

Daí, $tg(\arcsin x) = tg\, w$

$$= \frac{\sin w}{\cos w}$$

Mas, sendo $w \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ temos que

$\cos w = +\sqrt{1-\sin^2 w}$

Daí

$$tg(\arcsin x) = \frac{\sin w}{\sqrt{1-\sin^2 w}} \Big\downarrow \text{de } (*)$$

$$= \frac{x}{\sqrt{1-x^2}}$$

$$\therefore \quad // \; tg(\arcsin x) = \frac{x}{\sqrt{1-x^2}} \; //$$

6f)

$$\sin(\text{arc cotg}\, x) = \frac{1}{\sqrt{1+x^2}}$$

Seja

$$\omega = \text{arc cotg}\, x$$

$$\therefore \begin{cases} \text{cotg}\,\omega = x & (*) \\ \omega \in (0,\pi) \end{cases}$$

Daí

$$\sin(\text{arc cotg}\, x) = \sin\omega = \frac{1}{\text{cosec}\,\omega}$$

Mas

$$\text{cosec}^2\omega = 1 + \text{cotg}^2\omega$$

$$\text{cosec}\,\omega = \pm\sqrt{1+\text{cotg}^2\omega}$$

Sendo $\omega \in (0,\pi)$ temos que $\underline{\sin\omega} > 0$,

$\underline{\frac{1}{\text{cosec}\,\omega}} > 0 \;\therefore\; \text{cosec}\,\omega > 0$, daí

devemos tomar $\text{cosec}\,\omega = +\sqrt{1+\text{cotg}^2\omega}$.

Daí

$$\sin(\text{arc cotg}\,\omega) = \frac{1}{\sqrt{1+\text{cotg}^2\omega}} \underset{\text{de } (*)}{=} \frac{1}{\sqrt{1+x^2}}$$

$$\therefore \;//\; \sin(\text{arc cotg}\,\omega) = \frac{1}{\sqrt{1+x^2}} \;//$$

6g)

$$\text{cotg}\,(\text{arc}\sin x) = \frac{\sqrt{1-x^2}}{x}$$

Seja

$$\omega = \text{arc}\sin x$$

$$\therefore \begin{cases} \sin\omega = x & (*) \\ \omega \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \end{cases}$$

Daí

$$\text{cotg}\,(\text{arc}\sin x) = \text{cotg}\,\omega$$
$$= \frac{\cos\omega}{\sin\omega}$$

Mas, sendo $\omega \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ temos que $\cos\omega = +\sqrt{1-\sin^2\omega}$, daí,

$$\text{cotg}\,(\text{arc}\sin x) = \frac{\sqrt{1-\sin^2\omega}}{\sin\omega} \quad \Big\downarrow (*)$$
$$= \frac{\sqrt{1-x^2}}{x}$$

$$\therefore \quad \Big\| \; \text{cotg}\,(\text{arc}\sin x) = \frac{\sqrt{1-x^2}}{x} \; \Big\|$$

61)

$\cos(2\arccos x) = 2x^2 - 1, \quad -1 \leq x \leq 1$

Seja $\omega = \arccos x$

$\therefore \begin{cases} \cos\omega = x & (*) \\ \omega \in [0, \pi] \end{cases}$

Daí

$$\begin{aligned} \cos(2\arccos x) &= \cos(2\omega) \\ &= \cos^2\omega - \sin^2\omega \\ &= \cos^2\omega - (1-\cos^2\omega) \\ &= 2\cos^2\omega - 1 \\ &= 2x^2 - 1 \end{aligned} \quad \Big\downarrow \text{De } (*)$$

$\therefore \| \cos(2\arccos x) = 2x^2 - 1 \|$

6i)

$$\sin(3\arcsin x) = 3x - 4x^3$$

Seja

$$\omega = \arcsin x$$

$$\therefore \begin{cases} \sin\omega = x \\ \omega \in [-\frac{\pi}{2}, \frac{\pi}{2}] \end{cases}$$

Daí

$$\begin{aligned}
\sin(3\arcsin x) &= \sin 3\omega \\
&= \sin(\omega + 2\omega) \\
&= \sin\omega\cos 2\omega + \sin 2\omega\cos\omega \\
&= \sin\omega(\cos^2\omega - \sin^2\omega) + 2\sin\omega\cos\omega\cos\omega \\
&= \underbrace{\sin\omega\cos^2\omega} - \sin^3\omega + \underbrace{2\sin\omega\cos^2\omega} \\
&= \underbrace{3\sin\omega\,\underbrace{\cos^2\omega}} - \sin^3\omega \\
&= 3\sin\omega(1 - \sin^2\omega) - \sin^3\omega \\
&= 3\sin\omega - \underbrace{3\sin^3\omega - \sin^3\omega} \\
&= 3\underbrace{\sin\omega} - 4\sin^3\omega \quad \swarrow \text{ De } (*) \\
&= 3x - 4x^3
\end{aligned}$$

$$\therefore \quad \| \sin(3\arcsin x) = 3x - 4x^3 \|$$

6j)

$$tg(3 \operatorname{arctg} x) = \frac{x(3-x^2)}{1-3x^2}$$

Seja $\omega = \operatorname{arc\,tg} x$

$\therefore \ tg\,\omega = x$ (*)

$\omega \in (-\frac{\pi}{2}, \frac{\pi}{2})$

Daí

$$tg(3 \operatorname{arc\,tg} x) = tg(3\omega)$$

$$= tg(\omega + 2\omega)$$

$$= \frac{tg\,\omega + \boxed{tg\,2\omega}}{1 - tg\,\omega\, tg\,2\omega}$$

$$= \frac{tg\,\omega + \boxed{\dfrac{tg\,\omega + tg\,\omega}{1 - tg\,\omega\, tg\,\omega}}}{1 - tg\,\omega\left(\dfrac{tg\,\omega + tg\,\omega}{1 - tg\,\omega\, tg\,\omega}\right)}$$

$$= \frac{tg\,\omega + \dfrac{2\,tg\,\omega}{1-tg^2\omega}}{1 - tg\,\omega\,\dfrac{2\,tg\,\omega}{(1-tg^2\omega)}} =$$

$$= \frac{\dfrac{\operatorname{tg}\omega(1-\operatorname{tg}^2\omega) + 2\operatorname{tg}\omega}{\cancel{(1-\operatorname{tg}^2\omega)}}}{\dfrac{1-\operatorname{tg}^2\omega - 2\operatorname{tg}^2\omega}{\cancel{(1-\operatorname{tg}^2\omega)}}}$$

$$= \frac{\operatorname{tg}\omega - \operatorname{tg}^3\omega + 2\operatorname{tg}\omega}{1-3\operatorname{tg}^2\omega}$$

$$= \frac{3\operatorname{tg}\omega - \operatorname{tg}^3\omega}{1-3\operatorname{tg}^2\omega} \Bigg\downarrow \text{se } (*)$$

$$= \frac{3x - x^3}{1-3x^2}$$

$$\therefore \quad \Bigg\| \operatorname{tg}(3\operatorname{arctg} x) = \frac{x(3-x^2)}{1-3x^2} \Bigg\|$$

6K)

$3 \arccos x - \arccos(3x - 4x^3) = \pi$

O resultado que queremos mostrar é equivalente a mostrar que

$$3 \arccos x - \pi = \arccos(3x - 4x^3)$$

Seja então

$$\omega = \arccos x$$

$$\therefore \begin{cases} \cos \omega = x \\ \omega \in [0, \pi] \end{cases}$$

Consideremos então

$$\cos(3 \arccos x - \pi) = \cos(3\omega - \pi)$$

$$= \cos 3\omega \underbrace{\cos \pi}_{-1} + \sin 3\omega \cancelto{0}{\sin \pi}$$

$$= -\cos 3\omega$$

$$= -(\cos \omega \underline{\cos 2\omega} - \sin \omega \sin 2\omega)$$

$$= -\cos \omega (2\cos^2 \omega - 1) + \sin \omega \, 2 \sin \omega \cos \omega$$

$$= -2\cos^3 \omega + \cos \omega + 2 \sin^2 \omega \cos \omega$$

$$= -2\cos^3 \omega + \cos \omega + 2(1 - \cos^2 \omega) \cos \omega$$

$$= -\underline{2\cos^3 \omega} + \cos \omega + 2\cos \omega - \underline{2\cos^3 \omega}$$

$$= -4\cos^3\omega + 3\cos\omega$$

isto é ;

$$\cos\left(3\arccos x - \pi\right) = -4\cos^3\underset{\sim}{\omega} + 3\underset{\sim}{\cos\omega}$$

$$= -4x^3 + 3x$$

$$\therefore \quad \Big\| \, 3\arccos x - \pi = \arccos(-4x^3+3x) \, \Big\|$$

6l)

$$\arccos \frac{1-x^2}{1+x^2} = 2|\operatorname{arctg} x|$$

Solução

Uma vez que

$\operatorname{arctg} x \geq 0$ se $x \geq 0$

e $\operatorname{arctg} x < 0$ se $x < 0$

temos que a equação que queremos mostrar se escreve como

$$\begin{cases} \arccos \frac{1-x^2}{1+x^2} = 2 \operatorname{arctg} x & \text{se } x \geq 0 \\ \arccos \frac{1-x^2}{1+x^2} = -2 \operatorname{arctg} x & \text{se } x < 0 \end{cases}$$

→ Seja então $x \geq 0$.

Seja $\omega = \operatorname{arctg} x$

$$\therefore \begin{cases} \operatorname{tg} \omega = x \\ \omega \in (-\frac{\pi}{2}, \frac{\pi}{2}) \end{cases}$$

Mas, sendo $x \geq 0$ isso nos restringe

$$\omega \in [0, \tfrac{\pi}{2})$$

Mas

$$\frac{1-x^2}{1+x^2} = \frac{1-tg^2\omega}{1+tg^2\omega}$$

$$= \frac{1-\frac{\sin^2\omega}{\cos^2\omega}}{\sec^2\omega}$$

$$= \frac{\frac{\cos^2\omega-\sin^2\omega}{\cos^2\omega}}{\frac{1}{\cos^2\omega}}$$

$$= \cos^2\omega - \sin^2\omega$$

$\therefore$

$$\frac{1-x^2}{1+x^2} = \cos 2\omega$$

Aqui, estamos tratando com $\omega \in [0, \frac{\pi}{2})$ daí $2\omega \in [0, \pi)$ e da definição de arc cos podemos escrever que

$$\underset{\sim}{2\omega} = \text{arc cos}\,\frac{1-x^2}{1+x^2}$$

$$\underline{(*)} \quad 2\,\text{arctg}\,x = \text{arc cos}\,\frac{1-x^2}{1+x^2}\ ; \quad x \geqslant 0$$

→ Seja então $x < 0$.

Seja
$$\omega = \text{arc tg } x$$
$$\therefore \begin{cases} \text{tg}\,\omega = x \\ \omega \in (-\frac{\pi}{2}, \frac{\pi}{2}) \end{cases}$$

Mas, sendo $x < 0$ isso nos restringe
$$\omega \in (-\tfrac{\pi}{2}, 0).$$

Mas, $\frac{1-x^2}{1+x^2} = \cos 2\omega$ (já feito)

Aqui, sendo $\omega \in (-\frac{\pi}{2}, 0)$ temos
$$\underline{2\omega \in (-\pi, 0)}$$

Contudo, nesta faixa <u>não</u> podemos de $\frac{1-x^2}{1+x^2} = \cos 2\omega$ afirmar que
$$2\omega = \text{arc cos}\, \frac{1-x^2}{1+x^2}.$$

Contudo escrevendo $\cos 2\omega = \cos -2\omega$ (pois cosseno é função par) temos que
$$\frac{1-x^2}{1+x^2} = \cos(-2\omega)$$ e agora

sendo $\omega \in (-\frac{\pi}{2}, 0)$

tem-se $2\omega \in (-\pi, 0)$

e $\underbrace{-2\omega \in (0, \pi)}$.

Assim, da definição de arcocoseno temos que

$$\cos -2\omega = \frac{1-x^2}{1+x^2} \Longrightarrow \underset{\sim}{-2\omega} = \arccos \frac{1-x^2}{1+x^2}$$

$$-2 \operatorname{arctg} x = \arccos \frac{1-x^2}{1+x^2}$$

i.e. obtivemos

$$(**) \quad \arccos \frac{1-x^2}{1+x^2} = -2 \operatorname{arctg} x \quad , x < 0$$

De $(*)$ e $(**)$ temos

$$\left\| \arccos \frac{1-x^2}{1+x^2} = 2 |\operatorname{arctg} x| \right\| ; x \in \mathbb{R}$$

6m)

$$\operatorname{arc\,tg}(-x) = -\operatorname{arc\,tg} x$$

Solução

Seja

$$\omega = \operatorname{arc\,tg}(-x)$$

$$\therefore \quad \operatorname{tg}\omega = -x$$

$$\omega \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$x = -\operatorname{tg}\omega$$

$$x = \operatorname{tg}(-\omega)$$

$\downarrow$ tangente é ímpar

uma vez que $-\omega \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ podemos aplicar a função arc tg em x e em $\operatorname{tg}(-\omega)$;

$$\operatorname{arc\,tg} x = \underbrace{\operatorname{arc\,tg}(\operatorname{tg}}(-\omega)) = -\omega$$

$$\therefore \quad \underline{\omega} = -\operatorname{arc\,tg} x$$

$$\| \operatorname{arc\,tg}(-x) = -\operatorname{arc\,tg} x \|$$

6n)

$$\text{arc tg } x + \text{arc cotg } x = \frac{\pi}{2}$$

Solução

Seja

$w = \text{arc tg } x$ | $z = \text{arc cotg } x$

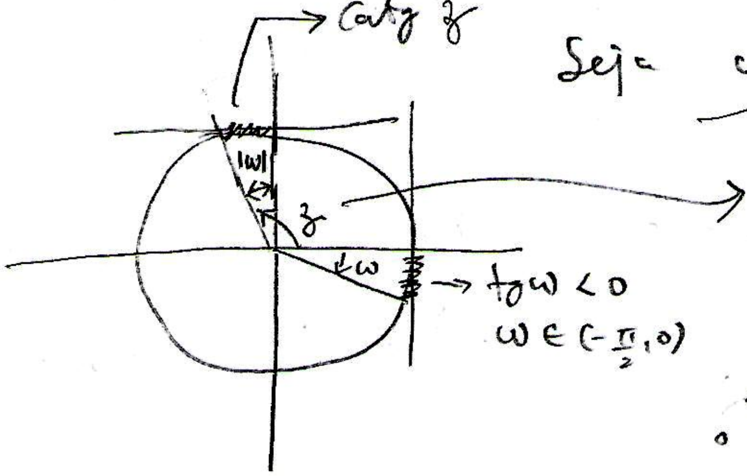
$\therefore$ $\text{tg } w = x$ (*) | $\therefore$ $\text{cotg } z = x$ (**)

$w \in (-\frac{\pi}{2}, \frac{\pi}{2})$ | $z \in (0, \pi)$

De (*) e (**):

$$\text{tg } w = \text{cotg } z$$

Vamos analisar a solução desta equação usando o ciclo trigonométrico:

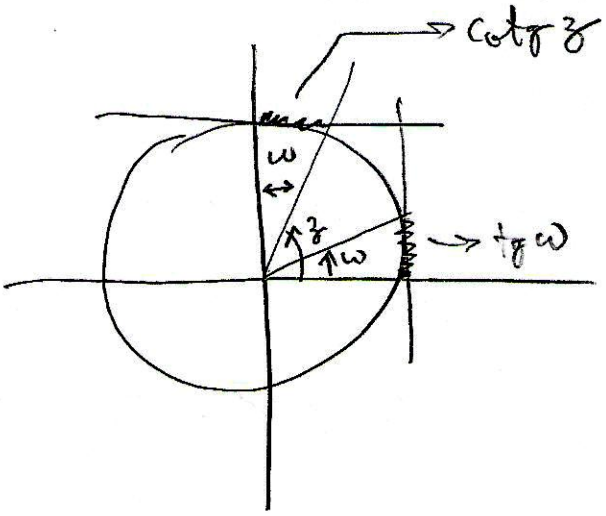


Seja $w \in (-\frac{\pi}{2}, 0)$.

$$z = \frac{\pi}{2} + |w| = \frac{\pi}{2} - w$$ pois $w < 0$

$$\therefore \quad z + w = \frac{\pi}{2} \qquad (3*)$$

Seja $\omega \in (0, \frac{\pi}{2})$.



$z = \frac{\pi}{2} - \omega$

$\therefore$

$z + \omega = \frac{\pi}{2}$ (4*)

Da (3*) e (4*) temos que

$\underset{\sim}{z} + \underbrace{\omega} = \frac{\pi}{2}$

$\text{arccotg } x + \underbrace{\text{arctg } x} = \frac{\pi}{2}$

$\therefore$ $\| \text{arctg } x + \text{arccotg } x = \frac{\pi}{2} \|$

6a)

$$\text{arccotg}(-x) = \pi - \text{arccotg}\, x$$

Solução

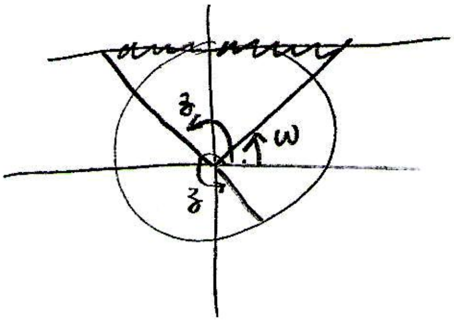
Seja

$\omega = \text{arccotg}(-x)$ $\qquad$ $z = \text{arccotg}\, x$

$\therefore \quad \text{cotg}\,\omega = -x \quad (*)$ $\qquad$ $\therefore \quad \text{cotg}\, z = x \quad (**)$

$\omega \in (0,\pi)$ $\qquad$ $z \in (0,\pi)$

De $(*)$ e $(**)$: $\quad \text{cotg}\,\omega = -\,\text{cotg}\, z \Longrightarrow$



$$\Longrightarrow \begin{cases} z = \pi - \omega + 2n\pi \;;\; n \in \mathbb{Z} \\ \text{ou} \\ z = 2\pi - \omega + 2n\pi \;;\; n \in \mathbb{Z} \end{cases}$$

Mas $z \in (0,\pi)$ e $\omega \in (0,\pi)$ nos obriga a tomar por solução $z = \pi - \omega$

$$\therefore \quad \underset{\sim}{\omega} = \pi - z$$

$$// \; \operatorname{arc\,cotg}(-x) = \pi - \operatorname{arc\,cotg} x \; //$$

7a) $\operatorname{arc\,cos}\sqrt{1-x^2} = \operatorname{arc\,sin} x$

Solução

$\operatorname{arc\,sin} x$: $x \in [-1,1]$ (*)

$\operatorname{arc\,cos}\sqrt{1-x^2}$: $1-x^2 \geq 0 \Rightarrow -1 \leq x \leq 1$ (**)

e

$\sqrt{1-x^2} \in [-1,1]$

$\therefore\ -1 \leq \sqrt{1-x^2} \leq 1$

$\therefore\ 0 \leq \sqrt{1-x^2} \leq 1$

$0 \leq 1-x^2 \leq 1$

$-1 \leq -x^2 \leq 0$

$1 \geq x^2 \geq 0$

$\therefore\ -1 \leq x \leq 1$ (***)

De (*), (**) e (***) tem-se : $\underline{\underline{x \in [-1,1]}}$

Seja $w = \operatorname{arc\,cos}\sqrt{1-x^2}$

$\therefore\ \cos w = \sqrt{1-x^2}$ (4*)

$w \in [0,\pi]$

$z = \operatorname{arc\,sin} x$

$\therefore\ \sin z = x$ (5*)

$z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

De (5*) e (4*) tem-se

$$
\begin{aligned}
\cos w &= \sqrt{1 - \sin^2 z} \\
&= \sqrt{\cos^2 z} \\
&= |\cos z| \\
&= \cos z
\end{aligned}
$$

Se $z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ então $\cos z \geq 0$

$$\therefore \quad \cos w = \cos z \qquad (6*) \qquad \left(\Rightarrow \begin{array}{l} w = z + 2n\pi \ ; \ n \in \mathbb{Z} \\ \text{ou} \\ w = -z + 2n\pi \ ; \ n \in \mathbb{Z} \end{array} \right)$$

A equação original se escreve na forma

$$\underbrace{\arccos \sqrt{1-x^2}}_{w} = \underbrace{\arcsin x}_{z} \qquad (7*)$$

e vemos que (7*) é solução de (6*).

Sendo $w \in [0, \pi]$ e $z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ temos que (6*) restringe w e z à

$$w, z \in [0, \frac{\pi}{2}].$$

De (5*) :

$$\sin z = x$$

Daí, sendo $0 \leq z \leq \frac{\pi}{2} \Rightarrow 0 \leq \underbrace{\sin z}_{} \leq 1$

$$0 \leq x \leq 1$$

$$\therefore \quad // x \in [0, 1] //$$

7b) $\arccos \sqrt{1-x^2} = -\arcsin x$

Solução

$$\arccos \sqrt{1-x^2} : \begin{cases} 1-x^2 \geqslant 0 \Rightarrow -1 \leq x \leq 1 \quad (*) \\ \text{e} \\ \sqrt{1-x^2} \in [-1,1] \end{cases}$$

$$\therefore \quad -1 \leq \sqrt{1-x^2} \leq 1$$

$$0 \leq \sqrt{1-x^2} \leq 1$$

$$0 \leq 1-x^2 \leq 1$$

$$-1 \leq -x^2 \leq 0$$

$$1 \geqslant x^2 \geqslant 0$$

$$\therefore \quad -1 \leq x \leq 1 \quad (**)$$

$\arcsin x$: $x \in [-1,1]$ (3*)

De (*), (**) e (***) tem-se : $\underline{x \in [-1,1]}$.

Seja $\omega = \arccos \sqrt{1-x^2}$

$\therefore \cos \omega = \sqrt{1-x^2}$ (3*)

$\omega \in [0,\pi]$

$z = \arcsin x$

$\therefore \sin z = x$ (4*)

$z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

De (4*) em (3*) :

$$\cos\omega = \sqrt{1-\sin^2 z}$$
$$= \sqrt{\cos^2 z}$$
$$= |\cos z|$$

$\Big\downarrow$ se $z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ temos $\cos z \geq 0$

$$\cos\omega = \cos z \qquad (5*)$$

Em termos de ω e z temos a equação original na forma

$$\underbrace{\arccos\sqrt{1-x^2}}_{\omega} = -\underline{\arcsin x}$$
$$= -z \qquad (6*)$$

Vemos que (6*) é solução de (5*) compatível com $\omega \in [0, \frac{\pi}{2}]$ e $\underline{z \in [-\frac{\pi}{2}, 0]}$.

Mas, de (4*) : $\sin z = x$

e sendo $-\frac{\pi}{2} \leq z \leq 0$ isso nos dá $-1 \leq \sin z \leq 0$

$$-1 \leq x \leq 0$$

$$\therefore \quad \| x \in [-1, 0] \|$$

7c) $\text{arccot}\, x = \text{arc tg}\, \frac{1}{x}$

<u>Solução</u>

$\text{arccatg}\, x \; : \; x \in \mathbb{R}$

e

$\text{arc tg}\, \frac{1}{x} \; : \; x \in \mathbb{R}$ e $x \neq 0$

Daí, dessas duas condições temos que $\underline{\underline{x \neq 0}}$. (*)

Seja

$\omega = \text{arc catg}\, x$

$\therefore \begin{cases} \text{catg}\, \omega = x & (2*) \\ \omega \in (0, \pi) \end{cases}$

$z = \text{arc tg}\, \frac{1}{x}$

$\therefore \begin{cases} \text{tg}\, z = \frac{1}{x} \\ z \in (-\frac{\pi}{2}, \frac{\pi}{2}) \end{cases}$

$\therefore \; x = \frac{1}{\text{tg}\, z} = \text{catg}\, z$ (3*)

Substituindo (3*) em (2*):

$\text{catg}\, \omega = \text{catg}\, z$ (4*).

Em termos de ω e z vemos que a equação que temos de resolver assume a forma

$$\underbrace{\text{arc cotg } x} = \underbrace{\text{arc tg } \tfrac{1}{x}}$$

$$w = z \qquad (5*)$$

que satisfaz a equação (4*).

Mas, sendo $w \in (0, \pi)$ e $z \in (-\frac{\pi}{2}, \frac{\pi}{2})$ vemos que (5*) restringe

$$w, z \in (0, \tfrac{\pi}{2}) .$$

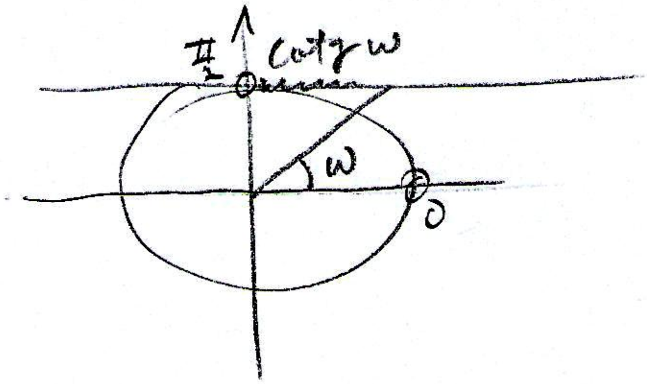
De (2*):

$$\text{cotg } w = x$$

se $0 < w < \frac{\pi}{2}$ então $0 < \underbrace{\text{cotg } w} < +\infty$

$$0 < x < +\infty$$

$$\therefore \quad \| x \in (0, +\infty) \|$$



7d) $\operatorname{arc\,tg} x = \operatorname{arc\,cotg} \frac{1}{x}$

$\operatorname{arctg} x \; : \; x \in \mathbb{R}$

$\operatorname{arccotg} \frac{1}{x} \; : \; \frac{1}{x} \in \mathbb{R} \implies x \neq 0$

$\rightarrow \underline{\underline{x \neq 0}}$

Seja

$w = \operatorname{arctg} x$ $\qquad$ $z = \operatorname{arc\,cotg} \frac{1}{x}$

$\therefore \; \operatorname{tg} w = x \quad (*)$ $\qquad$ $\operatorname{cotg} z = \frac{1}{x} \quad (**)$

$w \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\qquad$ $z \in (0, \pi)$

De $(*)$ e $(**)$: $\operatorname{tg} w = \dfrac{1}{\operatorname{cotg} z}$

$$(***) \quad \operatorname{tg} w = \operatorname{tg} z \implies \left| \begin{array}{l} w = z + 2n\pi \\ \text{ou} \\ w = z + \pi + 2n\pi \end{array} \right.$$

Em termos de w e z a equação original se escreve como:

$$w = z$$

que satisfaz $(***)$.

Mas, sendo $w \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ e $z \in (0, \pi)$ temos que $w = z$ obriga a ter $\underline{w, z \in \left(0, \frac{\pi}{2}\right)}$.

Mas, de (*)

se $0 < \omega < \frac{\pi}{2}$ então $0 < \underset{\sim}{\operatorname{tg}\omega} < +\infty$

$0 < x < +\infty$

$\therefore \quad \| x \in (0, +\infty) \|$

7e) $\text{arc tg}\, x = \text{arc cotg}\, \frac{1}{x} - \pi$

$\text{arc tg}\, x \; : \; x \in \mathbb{R}$

$\text{arc cotg}\, \frac{1}{x} \; : \; \frac{1}{x} \in \mathbb{R} \; , \; x \neq 0 \; \rightarrow \; \underline{\underline{x \neq 0}}$

Seja

$\omega = \text{arc tg}\, x$	$z = \text{arc cotg}\, \frac{1}{x}$
$\therefore \; \text{tg}\, \omega = x \quad (*)$	$\therefore \; \text{cotg}\, z = \frac{1}{x} \quad (**)$
$\omega \in (-\frac{\pi}{2}, \frac{\pi}{2})$	$z \in (0, \pi)$

De $(*)$ e $(**)$: $\text{tg}\, \omega = \frac{1}{\text{cotg}\, z}$

$$(***) \quad \text{tg}\, \omega = \text{tg}\, z \Rightarrow \left\{ \begin{array}{l} \omega = z + 2n\pi \\ \text{ou} \\ \omega = z + \pi + 2n\pi \end{array} \right.$$

Em termos de ω e z a equação original se escreve como

$$\text{arc tg}\, x = \text{arc cotg}\, \frac{1}{x} - \pi$$

$$\omega = z - \pi$$

que satisfaz $(***)$.

Mas, sendo $0 < z < \pi$ tem-se $-\pi < \underbrace{z - \pi}_{\omega} < 0$

$\therefore$

$$-\pi < \omega < 0$$

e de (*) devemos ter

$$-\frac{\pi}{2} < \omega < \frac{\pi}{2}$$

Daí obtemos que essas duas condições delimitam ω ao intervalo

$$-\frac{\pi}{2} < \omega < 0 .$$

Daí,

se $-\frac{\pi}{2} < \omega < 0$ então $-\infty < \operatorname{tg} \omega < 0$

$$-\infty < x < 0$$

$$\therefore \; // \, x \in (-\infty, 0) \, //$$

7f) $\operatorname{arc\,tg} \frac{1+x}{1-x} = \operatorname{arc\,tg} x + \frac{\pi}{4}$

$\operatorname{arc\,tg} \frac{1+x}{1-x}$: $\frac{1+x}{1-x} \in \mathbb{R}$; $\underline{\underline{x \neq 1}} \to \underline{\underline{x \neq 1}}$

$\operatorname{arc\,tg} x$: $x \in \mathbb{R}$

Seja

$w = \operatorname{arc\,tg} \frac{1+x}{1-x}$

$\therefore \operatorname{tg} w = \frac{1+x}{1-x}$ (*)

$w \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$z = \operatorname{arc\,tg} x$

$\therefore \operatorname{tg} z = x$ (**)

$z \in (-\frac{\pi}{2}, \frac{\pi}{2})$

De (**) e (*) :

(***) $\operatorname{tg} w = \frac{1 + \operatorname{tg} z}{1 - \operatorname{tg} z}$

A eq. original assume a forma :

$$w = z + \frac{\pi}{4}$$

que satisfaz (***) .

Mas

$$-\frac{\pi}{2} < \omega < \frac{\pi}{2}$$

$$\therefore \quad -\frac{\pi}{2} - \frac{\pi}{4} < \omega - \frac{\pi}{4} < \frac{\pi}{2} - \frac{\pi}{4}$$

$$-\frac{3\pi}{4} < \underbrace{\omega - \frac{\pi}{4}} < \frac{\pi}{4}$$

$$-\frac{3\pi}{4} < z < \frac{\pi}{4}$$

Mas devemos ter também

$$-\frac{\pi}{2} < z < \frac{\pi}{2}$$

e essas duas condições delimita z ao intervalo

$$-\frac{\pi}{2} < z < \frac{\pi}{4}$$

Daí, se $-\frac{\pi}{2} < z < \frac{\pi}{4}$ então $-\infty < \operatorname{tg} z < 1$

$$-\infty < x < 1$$

$$\therefore \quad // \, x \in (-\infty, 1) \, //$$

7g)

$$\operatorname{arctg} \frac{1+x}{1-x} = \operatorname{arctg} x - \frac{3\pi}{4}$$

$$\begin{tikzcd} \operatorname{arctg} \frac{1+x}{1-x} : \quad x \neq 1 \arrow[dr] & \\ & \underline{x \neq 1} \\ \operatorname{arctg} x \; : \quad x \in \mathbb{R} \arrow[ur] & \end{tikzcd}$$

Seja

$$w = \operatorname{arctg} \frac{1+x}{1-x}$$

$$\operatorname{tg} w = \frac{1+x}{1-x} \qquad (*)$$

$$w \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$z = \operatorname{arctg} x$$

$$\operatorname{tg} z = x \qquad (**)$$

$$z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

De $(*)$ e $(**)$:

$$(***) \qquad \operatorname{tg} w = \frac{1 + \operatorname{tg} z}{1 - \operatorname{tg} z}$$

Em termos de w e z a equação original se escreve como

$$\operatorname{arctg} \frac{1+x}{1-x} = \operatorname{arctg} x - \frac{3\pi}{4}$$

$$(4*) \qquad w = z - \frac{3\pi}{4}$$

(e vemos que esta relação entre w e z é compatível com $(***)$.)

Mas,

$$\underbrace{\omega \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}, \quad z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$-\frac{\pi}{2} < \omega < \frac{\pi}{2} \qquad \Big\downarrow + \frac{3\pi}{4}$$

$$\therefore \quad -\frac{\pi}{2} + \frac{3\pi}{4} < \underbrace{\omega + \frac{3\pi}{4}}_{\text{De } (4*)} < \frac{\pi}{2} + \frac{3\pi}{4}$$

$$\underline{\underline{(5*)}} \quad \frac{\pi}{4} < z < \frac{5\pi}{4}$$

Mas, temos que $-\frac{\pi}{2} < z < \frac{\pi}{2}$

Daí, de (5*) devemos ter

$$\frac{\pi}{4} < z < \frac{\pi}{2} \Longrightarrow$$

$$\Longrightarrow \quad 1 < \underline{tg\, z} < +\infty$$

$$1 < x < +\infty$$

$$\therefore \quad \| x \in (1, +\infty) \|$$

8a)

$$\sin\left(\frac{1}{5}\arccos x\right) = 1$$

Solução

Para que a equação esteja definida devemos ter $x \in [-1,1]$.

Seja $$\omega = \arccos x$$

$$\therefore \begin{cases} \cos\omega = x \\ \omega \in [0,\pi] \quad (*) \end{cases}$$

A equação se escreve então na forma

$$\sin\left(\frac{1}{5}\omega\right) = 1 \quad (**)$$

$$\therefore \quad \frac{1}{5}\omega = \frac{\pi}{2} + 2n\pi \quad ; \quad n \in \mathbb{Z}$$

$$\therefore \quad \omega = \frac{5\pi}{2} + 10n\pi \quad ; \quad n \in \mathbb{Z} \quad (***)$$

De $(*)$: $$0 \leq \underset{\sim}{\omega} \leq \pi$$

$$\therefore \quad 0 \leq \frac{5\pi}{2} + 10n\pi \leq \pi \quad ; \quad n \in \mathbb{Z}$$

$$-\frac{5\pi}{2} \leq 10n\pi \leq \pi - \frac{5\pi}{2} = -\frac{3\pi}{2}$$

$$\therefore \quad -\frac{\pi}{4} \leq n\pi \leq -\frac{3\pi}{20}$$

$$\therefore \quad -\frac{1}{4} \leq n \leq -\frac{3}{20} \quad ; \quad n \in \mathbb{Z}$$

Mas, não existe nenhum inteiro $\underline{n}$ satisfazendo essa condição, logo, a condição dada em (***) que determina ω não admite solução, isto é,

$\nexists\ \omega$ solução de (**), ou ainda

$$\sin\left(\tfrac{1}{5}\arccos x\right) = 1 \Longrightarrow \underline{\text{não tem solução}}.$$

$$\underline{\underline{S = \phi}}$$

8b)

$$\arcsin \frac{1}{\sqrt{x}} - \arcsin \sqrt{1-x} = \frac{\pi}{2}$$

Solução

Inicialmente notamos que

$\arcsin \frac{1}{\sqrt{x}}$: $x > 0$ e $-1 < \sqrt{x} < 1$ (domínio de $\arcsin x$)

$$\underbrace{x > 0 \quad \therefore \quad 0 < \sqrt{x} < 1}$$

$$\therefore \quad \underline{0 < x < 1} \qquad (*)$$

$\arcsin \sqrt{1-x}$: $1 - x \geq 0$ e $-1 \leq \sqrt{1-x} \leq 1$

$$x \leq 1 \quad \text{e} \quad \begin{cases} 0 \leq \sqrt{1-x} \leq 1 \\ 0 \leq 1-x \leq 1 \\ -1 \leq -x \leq 0 \end{cases}$$

$$\underbrace{x \leq 1 \quad \text{e} \quad 1 \geq x \geq 0}$$

$$\therefore \quad \underline{0 \leq x \leq 1} \qquad (**)$$

Assim de $(*)$ e $(**)$ devemos ter $\underline{\underline{0 < x < 1}}$.

Seja

$$\omega = \arcsin \frac{1}{\sqrt{x}}$$

$$\therefore \begin{cases} \sin \omega = \dfrac{1}{\sqrt{x}} \\ \omega \in [-\frac{\pi}{2}, \frac{\pi}{2}] \end{cases}$$

$$z = \arcsin \sqrt{1-x}$$

$$\therefore \quad \sin z = \sqrt{1-x}$$

$$z \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

Daí :

$$\arcsin \frac{1}{\sqrt{x}} - \arcsin \sqrt{1-x} = \frac{\pi}{2}$$

$$\omega - z = \frac{\pi}{2}$$

$$\therefore \quad \sin(\omega - z) = \sin \frac{\pi}{2}$$

$$\sin\omega \cos z - \sin z \cos \omega = 1$$

$$\frac{1}{\sqrt{x}} \cos z - \sqrt{1-x} \cos \omega = 1 \qquad (**)$$

Mas,

$z \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow \cos z \geq 0$, daí podemos escrever que

$$\cos z = +\sqrt{1 - \sin^2 z}$$

$$= \sqrt{1-(1-x)} = \sqrt{x}$$

$$\omega \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Longrightarrow \cos\omega \geqslant 0$$

$$\therefore \quad \cos\omega = \sqrt{1-\sin^2\omega}$$

$$= \sqrt{1-\frac{1}{x}} = \sqrt{\frac{x-1}{x}}$$

Assim, voltando a (***) temos:

$$\frac{1}{\sqrt{x}}\sqrt{x} - \sqrt{1-x}\,\sqrt{\frac{x-1}{x}} = 1$$

$$\cancel{1} - \sqrt{\frac{(1-x)(x-1)}{x}} = \cancel{1}$$

$$\therefore \quad \sqrt{\frac{(1-x)(x-1)}{x}} = 0$$

$$\therefore \quad (1-x)(x-1) = 0$$

$$\therefore \quad \underline{x = 1}\,.$$

8c)
$\text{arc cot } x = \text{arc cos } x$

<u>Soluçaõ</u>

$\text{arc cotg } x : \quad x \in \mathbb{R}$
$\text{arc cos } x : \quad x \in [-1,1]$ $\Bigg\} \Longrightarrow \underline{x \in [-1,1]}$

Seja

$\omega = \text{arc cotg } x$

$\therefore \quad \text{cotg } \omega = x \quad (*)$

$\omega \in (0, \pi)$

$z = \text{arc cos } x$

$\therefore \quad \cos z = x \quad (**)$

$z \in [0, \pi]$

De $(*)$ e $(**)$ temos

$$\text{cotg } \omega = \cos z \qquad (3*)$$

Mas, em termos de ω e z a equação original fica na forma:

$$\underbrace{\text{arc cotg } x}_{\omega} = \underbrace{\text{arc cos } x}_{z} \qquad (4*)$$

De $(4*)$ em $(3*)$ temos

$$\text{cotg } \omega = \cos \omega \quad \Longrightarrow$$

$$\frac{\cos w}{\sin w} = \cos w$$

$$\therefore \quad \cos w = \cos w \sin w$$

$$\therefore \quad \cos w (1 - \sin w) = 0$$

$$\Rightarrow \quad \cos w = 0$$

ou

$$\sin w = 1$$

Mas $w \in (0, \pi)$, daí:

$$\left.\begin{array}{l} \cos w = 0 \Rightarrow w = \pi/2 \\ \quad\text{ou} \qquad\qquad\quad \text{ou} \\ \sin w = 1 \Rightarrow w = \pi/2 \end{array}\right\} \quad w = \frac{\pi}{2}.$$

De (*):

$$\operatorname{cotg} \frac{w}{2} = x$$

$$\underbrace{\operatorname{cotg} \frac{\pi}{2}}_{0} = x$$

$$0 = x$$

$$\therefore \quad \| x = 0 \|.$$

8d)

$$\arcsin x - \arccos x = \arccos \frac{\sqrt{3}}{2}$$

Solução

Para que a equação esteja definida devemos ter $\underline{x \in [-1,1]}$.

Seja

$w = \arcsin x$ | $z = \arccos x$

$\therefore \sin w = x \quad (*)$ | $\therefore \cos z = x \quad (**)$

$w \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ | $z \in [0, \pi]$

A equação que queremos resolver assume então a forma:

$$w - z = \arccos \frac{\sqrt{3}}{2}$$

$$\therefore$$

$$\cos(w - z) = \cos\left(\arccos \frac{\sqrt{3}}{2}\right)$$

$$\underline{\cos w}\, \underline{\cos z} + \underline{\sin w}\, \sin z = \frac{\sqrt{3}}{2}$$

$$\sqrt{1 - \sin^2 w}\; x + x\sqrt{1 - \cos^2 z} = \frac{\sqrt{3}}{2}$$

$$\underbrace{\sqrt{1 - x^2}\; x + x\sqrt{1 - x^2}} = \frac{\sqrt{3}}{2}$$

$$2x\sqrt{1 - x^2} = \frac{\sqrt{3}}{2}$$

$$\therefore \quad x\sqrt{1-x^2} = \frac{\sqrt{3}}{4} \qquad (3*) \quad (\Rightarrow x > 0)$$

$$\therefore \quad x^2(1-x^2) = \frac{3}{16}$$

$$x^2 - x^4 = \frac{3}{16}$$

$$\therefore \quad x^4 - x^2 + \frac{3}{16} = 0$$

$$\therefore \quad x^2 = \frac{+1 \pm \sqrt{1 - 4 \cdot \frac{3}{16}}}{2}$$

$$= \frac{1 \pm \sqrt{\frac{1}{4}}}{2} = \frac{1 \pm \frac{1}{2}}{2} \begin{array}{l} \nearrow \frac{3}{4} \\ \searrow \frac{1}{2} \end{array}$$

isto é:

$$x^2 = \frac{3}{4} \quad \text{ou} \quad x^2 = \frac{1}{2}$$

$$\therefore \quad x = \pm\frac{\sqrt{3}}{2} \quad \text{ou} \quad x = \pm\frac{1}{\sqrt{2}}$$

Devemos testar quais dessas possibilidades é, de fato, solução de (3*):

Inicialmente, descartamos $x = -\frac{\sqrt{3}}{2}$ e $x = -\frac{1}{\sqrt{2}}$ pois $x > 0$.

Se $x = \frac{1}{\sqrt{2}}$: $x\sqrt{1-x^2} = \frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{2}} =$

$$= \frac{1}{\sqrt{2}}\sqrt{\frac{1}{2}} = \frac{1}{2} \neq \frac{\sqrt{3}}{4}$$

Assim, descartamos também $x = \frac{1}{\sqrt{2}}$.

Se $x = \frac{\sqrt{3}}{2}$:

$$x\sqrt{1-x^2} = \frac{\sqrt{3}}{2}\sqrt{1-\frac{3}{4}}$$
$$= \frac{\sqrt{3}}{2}\sqrt{\frac{1}{4}}$$
$$= \frac{\sqrt{3}}{2}\,\frac{1}{2}$$
$$= \frac{\sqrt{3}}{4}$$

$\therefore$ $x = \frac{\sqrt{3}}{2}$ é a única solução de (3b).

Daí

$$\arcsin x - \arccos x = \arccos\frac{\sqrt{3}}{2}$$

$$\Longrightarrow \boxed{x = \frac{\sqrt{3}}{2}}$$

9)

$$\arcsin x + \arcsin y = \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

Solução

Seja $w = \arcsin x$ | $z = \arcsin y$

$\therefore \sin w = x \quad (*)$ | $\therefore \sin z = y \quad (**)$

$w \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ | $z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Suponhamos que $w + z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Temos

$$\sin(w+z) = \underset{\sim}{\sin w} \cos z + \underset{\sim}{\sin z} \cos w$$
$$= x \cos z + y \cos w \quad \underline{\underline{(3*)}}$$

Mas, se $z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ temos $\cos z \geq 0$,

daí, sendo $\sin^2 z + \cos^2 z = 1$

$$\therefore \cos^2 z = 1 + \sin^2 z$$
$$\cos z = +\sqrt{1 + \sin^2 z}$$
$$= \sqrt{1 + y^2} \quad \text{de } (**)$$

Também, sendo $w \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ temos que $\cos w \geqslant 0$ e podemos tomar

$$\cos w = +\sqrt{1 - \sin^2 w}$$
$$= +\sqrt{1 - x^2} \quad \downarrow \text{de } (*)$$

Voltando a (3*) temos:

$$\sin(w+z) = x\sqrt{1+y^2} + y\sqrt{1-x^2} \qquad (4*)$$

Mas, sendo $w+z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ temos de (4*) que:

$$\Big\| \; w + z = \arcsin\left(x\sqrt{1+y^2} + y\sqrt{1-x^2}\right) \Big\|$$

10.

$$\operatorname{arc\,tg} \frac{x}{\sqrt{1-x^2}} = \operatorname{arc\,sin} x \quad , \quad -1 < x < 1$$

Solução

Seja $w = \operatorname{arc\,sin} x \quad ; \quad -1 < x < 1$

$$\therefore \quad \sin w = x \quad (*)$$

$$w \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$(\text{pois } -1 < x < 1)$$

Mas $\operatorname{tg} w = \dfrac{\sin w}{\cos w} = \dfrac{x}{\cos w}$

Sendo $w \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ temos que $\cos w \geq 0$

daí, de $\sin^2 w + \cos^2 w = 1$

$$\therefore \quad \cos^2 w = 1 - \sin^2 w$$

$$\cos w = \underline{\pm}\sqrt{1-\sin^2 w}$$

$$\overset{(*)}{=} +\sqrt{1-x^2}$$

Daí $\operatorname{tg} w = \dfrac{\sin w}{\cos w} = \dfrac{x}{\sqrt{1-x^2}}$

Sendo $w \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ temos que

$$\operatorname{tg} w = \frac{x}{\sqrt{1-x^2}} \Rightarrow \underset{\sim}{w} = \operatorname{arc\,tg} \frac{x}{\sqrt{1-x^2}}$$

$$\left\| \arcsin x = \operatorname{arc\,tg} \frac{x}{\sqrt{1-x^2}} \right\|$$

$$(-1 < x < 1)$$

11.

$$\operatorname{arctg} x + \operatorname{arctg} y = \operatorname{arctg} \frac{x+y}{1-xy} \quad ; \quad xy \neq 1$$

Solução

Seja

$w = \operatorname{arctg} x$ $\qquad$ $z = \operatorname{arctg} y$

$\therefore$ $\operatorname{tg} w = x$ $\qquad$ $\operatorname{tg} z = y$

$w \in (-\frac{\pi}{2}, \frac{\pi}{2})$ $\qquad$ $z \in (-\frac{\pi}{2}, \frac{\pi}{2})$

O enunciado da questão nos diz que o lado esquerdo está na faixa $(-\frac{\pi}{2}, \frac{\pi}{2})$, isto é,

$$w + z \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

Seja

$$\operatorname{tg}(w+z) = \frac{\operatorname{tg} w + \operatorname{tg} z}{1 - \operatorname{tg} w \operatorname{tg} z}$$

$$= \frac{x+y}{1-xy}$$

Identificamos:

$\operatorname{tg} w = x$

$\operatorname{tg} z = y$

Sendo $(w+z) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ temos então que

$$\operatorname{tg}(w+z) = \frac{x+y}{1-xy} \Rightarrow w + z = \operatorname{arctg}\left(\frac{x+y}{1-xy}\right)$$

$$\underset{\sim}{\omega} + z = \operatorname{arc\,tg} \frac{x+y}{1-xy}$$

$$\Big\| \operatorname{arctg} x + \operatorname{arctg} y = \operatorname{arctg} \frac{x+y}{1-xy} \Big\|$$

12.

$$\operatorname{arc\,tg} \frac{1}{a+b} + \operatorname{arc\,tg} \frac{1}{a+c} = \operatorname{arc\,tg} \frac{1}{a}$$

$$bc = 1+a^2$$

Solução

Seja

$$w = \operatorname{arc\,tg} \frac{1}{a+b} \qquad z = \operatorname{arc\,tg} \frac{1}{a+c}$$

$$\therefore \qquad \therefore$$

$$\operatorname{tg} w = \frac{1}{a+b} \qquad \operatorname{tg} z = \frac{1}{a+c}$$

$$w \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \qquad z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

O enunciado diz que a soma do lado esquerdo está no intervalo $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, isto é,

$$w + z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Temos

$$\operatorname{tg}(w+z) = \frac{\operatorname{tg} w + \operatorname{tg} z}{1 - \operatorname{tg} w \operatorname{tg} z}$$

$$= \frac{\frac{1}{a+b} + \frac{1}{a+c}}{1 - \left(\frac{1}{a+b}\right) \cdot \left(\frac{1}{a+c}\right)}$$

$$=$$

$$= \frac{\dfrac{a+c+a+b}{\cancel{(a+c)(a+b)}}}{\dfrac{(a+b)(a+c)-1}{\cancel{(a+c)(a+b)}}}$$

$$= \frac{2a+c+b}{a^2+ac+ba+\underbrace{bc-1}}$$

temos

$bc-1=a^2$

$$= \frac{2a+c+b}{a^2+ac+ba+\underbrace{a^2}}$$

$$= \frac{2a+c+b}{2a^2+ac+ba}$$

$$= \frac{\cancel{(2a+c+b)}}{a\cancel{(2a+c+b)}}$$

$\therefore$

$$tg(w+z) = \frac{1}{a}$$

Sendo $w+z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ temos que

$$tg(w+z) = \frac{1}{a} \Rightarrow \underbrace{w}+z = arctg\,\frac{1}{a}$$

$$\| \; arctg\,\frac{1}{a+b} + arctg\,\frac{1}{a+c} = arctg\,\frac{1}{a} \; |$$

13a)

$$\arcsin\left(\frac{x}{3}-1\right) = \frac{\pi}{2} - 2\arcsin\sqrt{1-\frac{x}{6}}$$

<u>Solução</u>

Inicialmente notamos que

$\arcsin\left(\frac{x}{3}-1\right)$: $\frac{x}{3}-1 \in [-1,1]$

$$-1 \leq \frac{x}{3}-1 \leq 1$$

$$0 \leq \frac{x}{3} \leq 2$$

$$0 \leq x \leq 6$$

$\arcsin\sqrt{1-\frac{x}{6}}$: $1-\frac{x}{6} \geq 0 \Rightarrow \underline{x \leq 6}$ (*)

$$\sqrt{1-\frac{x}{6}} \in [-1,1]$$

$$-1 \leq \sqrt{1-\frac{x}{6}} \leq 1$$

$$\therefore 0 \leq \sqrt{1-\frac{x}{6}} \leq 1$$

$$\therefore 0 \leq 1-\frac{x}{6} \leq 1$$

$$-1 \leq -\frac{x}{6} \leq 0$$

$$1 \geq \frac{x}{6} \geq 0$$

$$\therefore \underline{6 \geq x \geq 0}$$ (**)

De (*) e (**) : $\underline{\underline{0 \leq x \leq 6}}$

Seja

$$y = \frac{\pi}{2} - 2 \arcsin \sqrt{1 - \frac{x}{6}} \quad (*)$$

Sendo $0 \leq x \leq 6$ temos

$$0 \leq \arcsin \sqrt{1 - \frac{x}{6}} \leq \frac{\pi}{2}$$

Daí

$$0 \geq -\arcsin \sqrt{1 - \frac{x}{6}} \geq -\frac{\pi}{2}$$

$$0 \geq -2\arcsin \sqrt{1 - \frac{x}{6}} \geq -\pi$$

$$\frac{\pi}{2} + 0 \geq \underbrace{\frac{\pi}{2} - 2\arcsin \sqrt{1 - \frac{x}{6}}} \geq \frac{\pi}{2} - \pi$$

$$\frac{\pi}{2} \geq y \geq -\frac{\pi}{2}$$

$$\therefore \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \quad (**)$$

Temos de (*)

$$\frac{y - \frac{\pi}{2}}{2} = -\arcsin \sqrt{1 - \frac{x}{6}}$$

$$\therefore \quad \frac{\pi}{4} - \frac{y}{2} = \arcsin \sqrt{1 - \frac{x}{6}}$$

Mas, $-\frac{\pi}{2} \leq y \leq \frac{3\pi}{2} \implies$

$$-\frac{\pi}{4} \leq \frac{y}{2} \leq \frac{3\pi}{4}$$

$$\frac{\pi}{4} \geqslant -\frac{y}{2} \geqslant -\frac{3\pi}{4}$$

$$\frac{\pi}{4}+\frac{\pi}{4} \geqslant \frac{\pi}{4}-\frac{y}{2} \geqslant \frac{\pi}{4}-\frac{3\pi}{4}$$

$$\frac{\pi}{2} \geqslant \frac{\pi}{4}-\frac{y}{2} \geqslant -\frac{\pi}{2} \qquad (***)$$

Daí, sendo

$$\frac{\pi}{4}-\frac{y}{2} = \arcsin\sqrt{1-\frac{x}{6}}$$

temos de (***) que

$$\sin\left(\frac{\pi}{4}-\frac{y}{2}\right) = \sqrt{1-\frac{x}{6}}$$

$$\sin\frac{\pi}{4}\cos\frac{y}{2} - \sin\frac{y}{2}\cos\frac{\pi}{4} = \sqrt{1-\frac{x}{6}}$$

$$\frac{\sqrt{2}}{2}\cos\frac{y}{2} - \frac{\sqrt{2}}{2}\sin\frac{y}{2} = \sqrt{1-\frac{x}{6}}$$

$$\frac{\sqrt{2}}{2}\left(\cos\frac{y}{2} - \sin\frac{y}{2}\right) = \sqrt{1-\frac{x}{6}}$$

$$\therefore \frac{2}{4}\left(\cos^2\frac{y}{2} + \underline{2\sin\frac{y}{2}\cos\frac{y}{2}} + \sin^2\frac{y}{2}\right) = 1-\frac{x}{6}$$

$$\frac{1}{2}\left(1 + \underline{\sin y}\right) = 1-\frac{x}{6}$$

$$1 + \sin y = 2 - \frac{x}{3}$$

$$\sin y = 1 - \frac{x}{3}$$

De (**) temos

$$y = \operatorname{arc\,sin}\left(1 - \frac{x}{3}\right)$$

$$\sim$$

$$\Big\| \; \frac{\pi}{2} - 2 \operatorname{arc\,sin} \sqrt{1 - \frac{x}{6}} = \operatorname{arc\,sin}\left(1 - \frac{x}{3}\right) \; \Big\|$$

13.b)

$$\text{arc sin}\left(\frac{x}{3}-1\right) = 2\left(\text{arc sin}\,\frac{\sqrt{x}}{\sqrt{6}}\right) - \frac{\pi}{2}$$

Solução

$\text{arc sin}\left(\frac{x}{3}-1\right)$: $0 \leq x \leq 6$ (*)

(Ver 13.a)

$\text{arc sin}\,\frac{\sqrt{x}}{\sqrt{6}}$: $x \geq 0$

$$\frac{\sqrt{x}}{\sqrt{6}} \in [-1,1]$$

$$-1 \leq \frac{\sqrt{x}}{\sqrt{6}} \leq 1$$

$$-\sqrt{6} \leq \sqrt{x} \leq \sqrt{6}$$

$$\therefore\ 0 \leq \sqrt{x} \leq \sqrt{6}$$

$$\therefore\ 0 \leq x \leq 6 \quad (**)$$

De (*) e (**) : $\underline{0 \leq x \leq 6}$.

Seja $y = 2\left(\text{arc sin}\,\frac{\sqrt{x}}{\sqrt{6}}\right) - \frac{\pi}{2}$ (3*)

Temos que se $0 \leq x \leq 6$ então

$$0 \leq \text{arc sin}\,\frac{\sqrt{x}}{\sqrt{6}} \leq \frac{\pi}{2} \quad \therefore$$

$$0 \le 2\arcsin\frac{\sqrt{x}}{\sqrt{6}} \le \pi$$

$$-\frac{\pi}{2} + 0 \le -\frac{\pi}{2} + 2\arcsin\frac{\sqrt{x}}{\sqrt{6}} \le -\frac{\pi}{2} + \pi$$

$$-\frac{\pi}{2} \le \underbrace{-\frac{\pi}{2} + 2\arcsin\frac{\sqrt{x}}{\sqrt{6}}}_{} \le \frac{\pi}{2}$$

$$-\frac{\pi}{2} \le y \le \frac{\pi}{2} \qquad (4*)$$

Dx (3*) :

$$\frac{y+\frac{\pi}{2}}{2} = \arcsin\frac{\sqrt{x}}{\sqrt{6}}$$

$$\frac{y}{2}+\frac{\pi}{4} = \arcsin\frac{\sqrt{x}}{\sqrt{6}}$$

De (4*) :

$$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$-\frac{\pi}{4} \le \frac{y}{2} \le \frac{\pi}{4}$$

$$0 \le \frac{y}{2}+\frac{\pi}{4} \le \frac{\pi}{2}$$

∴

$$\sin\left(\frac{y}{2}+\frac{\pi}{4}\right) = \sin\left(\arcsin\frac{\sqrt{x}}{\sqrt{6}}\right)$$

$$\sin\frac{y}{2}\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos\frac{y}{2} = \frac{\sqrt{x}}{\sqrt{6}}$$

$$\sin\frac{y}{2}\,\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cos\frac{y}{2} = \frac{\sqrt{x}}{\sqrt{6}}$$

$$\frac{\sqrt{2}}{2}\left(\sin\frac{y}{2}+\cos\frac{y}{2}\right) = \frac{\sqrt{x}}{\sqrt{6}}$$

$$\frac{\cancel{2}}{\cancel{4}\,2}\left(\sin^2\frac{y}{2} + +2\underbrace{\sin\frac{y}{2}\cos\frac{y}{2}}_{} + \cos^2\frac{y}{2}\right) = \frac{x}{6}$$

$$\frac{1}{2}\left(1+\operatorname{sen} y\right)=\frac{x}{6}$$

$$\therefore \quad 1+\operatorname{sen} y=\frac{x}{3}$$

$$\operatorname{sen} y=\frac{x}{3}-1$$

De (4*) temos que

$$\operatorname{sen} y=\frac{x}{3}-1 \Longrightarrow y=\operatorname{arcsen}\left(\frac{x}{3}-1\right)$$

$$\left\|\, 2\operatorname{arcsen}\frac{\sqrt{x}}{\sqrt{6}}-\frac{\pi}{2}=\operatorname{arcsen}\left(\frac{x}{3}-1\right)\right\|$$

14.

$\arcsin x + \arccos x = c \quad ; \quad -1 \leq x \leq 1$

<u>Solução</u>

Seja

$\omega = \arcsin x$

$\therefore \quad \sin \omega = x \quad (*)$

$\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$z = \arccos x$

$\therefore$

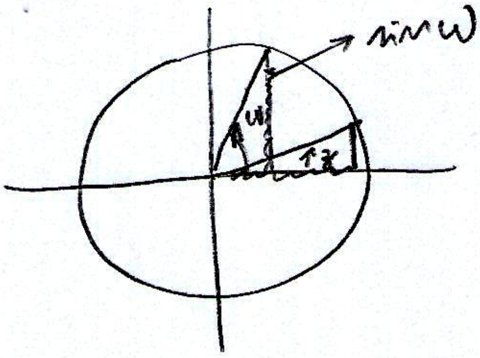
$\cos z = x \quad (**)$

$z \in [0, \pi]$

De $(*)$ e $(**)$:

$$\sin \omega = \cos z$$

A solução de $\sin \omega = \cos z$ com $\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ e $z \in [0, \pi]$ nos leva a seguinte situação



$\underline{\omega, z \in [0, \frac{\pi}{2}]}$

Neste caso, $\sin \omega = \cos z \Rightarrow$

$$\underline{\omega = \frac{\pi}{2} - z}$$

$$\therefore \quad \omega + z = \frac{\pi}{2} \quad (3*)$$

$$\underset{\sim}{w} + z = \frac{\pi}{2}$$

$$\left\| \operatorname{arc\,sin} x + \operatorname{arc\,cos} x = \frac{\pi}{2} \right\|$$

isto é: $c = \frac{\pi}{2}$.

15.

$$f(x) = \text{arc tg}\, x + \text{arc tg}\, \frac{1}{x} \,.$$

Seja $\omega = \text{arc tg}\, x$ | $z = \text{arc tg}\, \frac{1}{x}$

$\therefore\ \text{tg}\,\omega = x$ (*) | $\therefore\ \text{tg}\, z = \frac{1}{x}$ (**)

$\omega \in (-\frac{\pi}{2}, \frac{\pi}{2})$ | $z \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$(z \neq 0)$

De (*) e (**) :

$$\text{tg}\,\omega = \frac{1}{\text{tg}\, z} \quad \therefore \quad \text{tg}\,\omega = \text{cotg}\, z = x \quad (***)$$

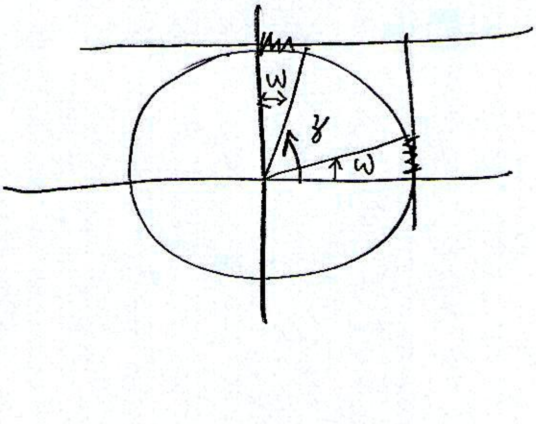
Seja $x > 0$.

Mas se $\omega \in (-\frac{\pi}{2}, \frac{\pi}{2})$ e $z \in (-\frac{\pi}{2}, \frac{\pi}{2})$

temos que $\text{tg}\,\omega = \text{cotg}\, z$. (positivo)

nos restringe $z, \omega \in (0, \frac{\pi}{2})$. Daí

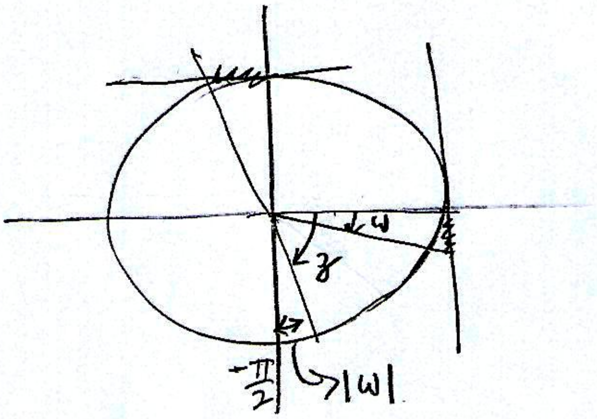
$\text{tg}\,\omega = \text{cotg}\,\omega \Longrightarrow$



$\Longrightarrow\ z + \omega = \frac{\pi}{2}$ (4*)

Seja $x < 0$.

Neste caso, temos que a equação dada em (***) nos obriga a restringir $w, z \in (-\frac{\pi}{2}, 0)$. Daí



$z < 0$ e temos

$$-\frac{\pi}{2} + |w| = z$$

$$\Downarrow w < 0$$

$$-\frac{\pi}{2} - w = z$$

$$\therefore \quad w + z = -\frac{\pi}{2} \quad (5*)$$

De (4*) e (5*) temos:

$$\left\| \; f(x) = \begin{cases} -\frac{\pi}{2} & \text{se } x < 0 \\ \frac{\pi}{2} & \text{se } x > 0 \end{cases} \; \right\|$$

16.

$$\arcsin \frac{m-1}{m+1} = \arccos \frac{2\sqrt{m}}{m+1} \; ; \; m > 0$$

Solução

Verifiquemos a consistência das quantidades que aparecem na expressão.

$\arcsin \frac{m-1}{m+1}$: $\rightarrow m \neq -1$ (é satisfeito por $m > 0$).

$m > 0$ $\rightarrow \frac{m-1}{m+1} \in [-1, 1]$

$$-1 \leq \frac{m-1}{m+1} \leq 1$$

Sendo $m > 0$ (enunciado da questão)

$\therefore \quad m+1 > 0$

daí $\quad -1 \leq \frac{m-1}{m+1} \leq 1$

$$\therefore -1(m+1) \leq m-1 \leq m+1$$

$$-m-1 \leq m-1 \leq m+1$$

$$-1 \leq 2m-1 \leq 2m+1$$

$$0 \leq 2m \leq 2m+2$$

$$0 \leq m \leq m+1$$

Como $m > 0$, tem-se verificado essa condição acima.

arc cos $\frac{2\sqrt{m}}{m+1}$: $\rightarrow m \geq 0$ (é satisfeita pois $m > 0$)

$m > 0$

$$\rightarrow \frac{2\sqrt{m}}{m+1} \in [-1,1]$$

$$-1 \leq \frac{2\sqrt{m}}{m+1} \leq 1$$

$$-1(m+1) \leq 2\sqrt{m} \leq m+1$$

$$-m-1 \leq 2\sqrt{m} \leq m+1$$

$$\underbrace{-m-1 \leq 2\sqrt{m}}_{\substack{\text{imediatamente}\\ \text{verificado pois}\\ m>0}} \quad \underline{\underline{e}} \quad 2\sqrt{m} \leq m+1$$

$$4m \leq (m+1)^2$$

$$4m \leq m^2+2m+1$$

$$0 \leq m^2-2m+1$$

$$\underbrace{0 \leq (m-1)^2}_{\substack{\text{imediatamente}\\ \text{verificado.}}}$$

Temos então que $\underline{m > 0}$ garante a boa definição de

$$\text{arc sin } \frac{m-1}{m+1} = \text{arc cos } \frac{2\sqrt{m}}{m+1}.$$

Seja

$$\omega = \text{arc sin } \frac{m-1}{m+1}$$

$\therefore$

$$\sin \omega = \frac{m-1}{m+1} \quad (*)$$

$$\omega \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$z = \text{arc cos } \frac{2\sqrt{m}}{m+1}$$

$\therefore$

$$\cos z = \frac{2\sqrt{m}}{m+1} \quad (**)$$

$$z \in [0, \pi]$$

Note que

se $\frac{m-1}{m+1} = -1 \Rightarrow m-1 = -m-1$

$2m = 0$

$m = 0$, mas $\underline{m > 0}$, logo

$$\underline{\underline{\frac{m-1}{m+1} \neq -1}}$$

se $\frac{m-1}{m+1} = 1 \Rightarrow m-1 = m+1$

$0 = 2$

$\therefore \frac{m-1}{m+1} \neq 1$

Daí, se $\underline{\frac{m-1}{m+1}} \neq \pm 1$ então $\omega \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

De (*) : $\operatorname{sen}\omega = \frac{m-1}{m+1}$

$$\therefore \quad (m+1)\operatorname{sen}\omega = m-1 \quad ($$

$$m\operatorname{sen}\omega + \operatorname{sen}\omega = m-1$$

$$m\operatorname{sen}\omega - m = -1-\operatorname{sen}\omega$$

$$m(\operatorname{sen}\omega - 1) = -1-\operatorname{sen}\omega$$

$$m = \frac{-1-\operatorname{sen}\omega}{\operatorname{sen}\omega - 1} \qquad \left\{ \begin{array}{c} \omega \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \Downarrow \\ \operatorname{sen}\omega \neq 1 \end{array} \right.$$

$$m = \frac{1+\operatorname{sen}\omega}{1-\operatorname{sen}\omega}$$

Substituindo em (**):

$$\cos z = \frac{2\sqrt{\frac{1+\operatorname{sen}\omega}{1-\operatorname{sen}\omega}}}{\frac{1+\operatorname{sen}\omega}{1-\operatorname{sen}\omega} + 1}$$

$$= \frac{2\sqrt{\frac{1+\operatorname{sen}\omega}{1-\operatorname{sen}\omega}}}{\frac{1+\cancel{\operatorname{sen}\omega}+1-\cancel{\operatorname{sen}\omega}}{1-\operatorname{sen}\omega}}$$

$$= \frac{\cancel{2}\sqrt{\frac{1+\operatorname{sen}\omega}{1-\operatorname{sen}\omega}}}{\frac{2}{1-\operatorname{sen}\omega}}$$

$$\cos z = \sqrt{\frac{1+\sin\omega}{1-\sin\omega}}\,(1-\sin\omega)$$

$$= \frac{\sqrt{1+\sin\omega}}{\sqrt{1-\sin\omega}}\,(1-\sin\omega)$$

$$= \sqrt{1+\sin\omega}\,\sqrt{1-\sin\omega}$$

$$= \sqrt{(1+\sin\omega)(1-\sin\omega)}$$

$$= \sqrt{1-\sin^2\omega}$$

$$= \sqrt{\cos^2\omega}$$

$$= |\cos\omega| \quad \Big\downarrow \ \omega \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

$$\cos z = \cos\omega$$

Sendo $z \in [0,\pi]$ e $\omega \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

tem que $\cos z = \cos\omega \Longrightarrow z = \omega$

$$\Downarrow$$

$$\left\| \ \arccos\frac{2\sqrt{m}}{m+1} = \arcsin\frac{m-1}{m+1} \ \right\|$$

17. $\arcsin x + \arcsin 2x = \frac{\pi}{2}$; $0 < x < 1$

Solução

Temos

$$\arcsin x + \arcsin 2x = \frac{\pi}{2}$$

$$\therefore \quad \sin\left(\arcsin x + \arcsin 2x\right) = \sin\frac{\pi}{2}$$

$$\therefore \quad \underline{\sin(\arcsin x)}\cos(\arcsin 2x) + \underline{\sin(\arcsin 2x)}\cos(\arcsin x) = 1$$

$$\therefore \quad \underline{x}\cos(\arcsin 2x) + \underline{2x}\cos(\arcsin x) = 1 \qquad (*)$$

Seja

$$w = \arcsin 2x$$

$$\therefore \quad \sin w = 2x$$

$$w \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Daí

$$\cos(\arcsin 2x) = \cos w = \sqrt{1 - \sin^2 w} = \sqrt{1 - 4x^2} \qquad (**)$$

$$z = \arcsin x$$

$$\therefore \quad \sin z = x$$

$$z \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Daí

$$\cos(\arcsin x) = \cos z = \sqrt{1 - \sin^2 z} = \sqrt{1 - x^2} \qquad (***)$$

Daí, inserindo (**) e (***) em (*) temos

$$x\sqrt{1-4x^2} + 2x\sqrt{1-x^2} = 1$$

$$\therefore \quad 2x\sqrt{1-x^2} = 1 - x\sqrt{1-4x^2}$$

$$\therefore \quad 4x^2(1-x^2) = 1 - 2x\sqrt{1-4x^2} + x^2(1-4x^2)$$

$$\underline{4x^2} - \cancel{4x^4} = 1 - 2x\sqrt{1-4x^2} + \underline{x^2} - \cancel{4x^4}$$

$$\therefore \quad 2x\sqrt{1-4x^2} = 1 - 3x^2$$

$$\therefore \quad 4x^2(1-4x^2) = 1 - 6x^2 + 9x^4$$

$$4x^2 - \underline{16x^4} = 1 - 6x^2 + \underline{9x^4}$$

$$0 = 25x^4 - 10x^2 + 1$$

$$x^2 = \frac{10 \pm \sqrt{100-100}}{50}$$

$$x^2 = \frac{1}{5} \quad \therefore \quad x = \pm\frac{1}{\sqrt{5}}$$

Mas $0 < x < 1 \Rightarrow /\!/ x = \frac{1}{\sqrt{5}} /\!/$

18.

$$\begin{cases} tg\left(\arcsin \frac{a-1}{a+1} + \operatorname{arctg} \frac{1}{2\sqrt{a}}\right) = \frac{2a\sqrt{a}}{3a+1} \\ a > 0 \\ 0 \leq \arcsin \frac{a-1}{a+1} < \frac{\pi}{2} \end{cases}$$

Solução

$$tg\left(\arcsin \frac{a-1}{a+1} + \operatorname{arc\,tg} \frac{1}{2\sqrt{a}}\right) =$$

$$= \frac{tg\left(\arcsin \frac{a-1}{a+1}\right) + tg\left(\operatorname{arc\,tg} \frac{1}{2\sqrt{a}}\right)}{1 - tg\left(\arcsin \frac{a-1}{a+1}\right) tg\left(\operatorname{arc\,tg} \frac{1}{2\sqrt{a}}\right)}$$

$$= \frac{tg\left(\arcsin \frac{a-1}{a+1}\right) + \frac{1}{2\sqrt{a}}}{1 - tg\left(\arcsin \frac{a-1}{a+1}\right) \frac{1}{2\sqrt{a}}} \quad (*)$$

Seja $\omega = \arcsin \frac{a-1}{a+1}$

$\therefore \sin \omega = \frac{a-1}{a+1}$

$\omega \in [0, \frac{\pi}{2})$ (pois é dado que $0 \leq \arcsin \frac{a-1}{a+1} < \frac{\pi}{2}$)

Daí

$$\operatorname{tg} \omega = \frac{\sin \omega}{\cos \omega}$$

$$= \frac{\sin \omega}{\sqrt{1 - \sin^2 \omega}}$$

$$= \frac{\frac{a-1}{a+1}}{\sqrt{1 - \left(\frac{a-1}{a+1}\right)^2}}$$

$$= \frac{\frac{a-1}{a+1}}{\sqrt{\frac{(a+1)^2 - (a-1)^2}{(a+1)^2}}} = \frac{\frac{a-1}{a+1}}{\sqrt{\frac{\cancel{a^2}+2a+\cancel{1}-\cancel{a^2}+2a-\cancel{1}}{(a+1)^2}}}$$

$$= \frac{\frac{a-1}{a+1}}{\sqrt{\frac{4a}{(a+1)^2}}} = \frac{\frac{a-1}{\cancel{(a+1)}}}{\frac{2\sqrt{a}}{\cancel{(a+1)}}}$$

$$\underset{\sim}{\operatorname{tg} \omega} = \frac{a-1}{2\sqrt{a}}$$

$$\therefore \quad \operatorname{tg}\left(\operatorname{arc} \sin \frac{a-1}{a+1}\right) = \frac{a-1}{2\sqrt{a}} \qquad (*1)$$

Substituindo (2*) em (*) temos:

$$tg\left(\arcsin\frac{a-1}{a+1}+\operatorname{arctg}\frac{1}{2\sqrt{a}}\right)=$$

$$=\frac{\frac{a-1}{2\sqrt{a}}+\frac{1}{2\sqrt{a}}}{1-\left(\frac{a-1}{2\sqrt{a}}\right)\cdot\frac{1}{2\sqrt{a}}}$$

$$=\frac{\frac{a}{2\sqrt{a}}}{1-\frac{a-1}{4a}}=\frac{\frac{a}{2\sqrt{a}}}{\frac{4a-a+1}{4a}}$$

$$=\frac{\frac{a}{2\sqrt{a}}}{\frac{3a+1}{4a}}=\frac{a}{\cancel{2}\sqrt{a}}\cdot\frac{\cancel{4}a}{3a+1}$$

$$=\frac{2a^2}{\sqrt{a}\,(3a+1)}$$

$$=\frac{2a\sqrt{a}}{3a+1}$$

$$\left\|\,tg\left(\arcsin\frac{a-1}{a+1}+\operatorname{arctg}\frac{1}{2\sqrt{a}}\right)=\frac{2a\sqrt{a}}{3a+1}\,\right\|$$

19. $\text{arc tg}\, x + \text{arc tg}\, \frac{x}{x+1} = \frac{\pi}{4} \quad (x \neq -1)$

Solução

Temos:

$$\text{tg}\left(\text{arc tg}\, x + \text{arc tg}\, \frac{x}{x+1}\right) = \text{tg}\, \frac{\pi}{4}$$

$$\therefore \frac{\text{tg}(\text{arc tg}\, x) + \text{tg}\left(\text{arc tg}\, \frac{x}{x+1}\right)}{1 - \text{tg}(\text{arc tg}\, x) \cdot \text{tg}\left(\text{arc tg}\, \frac{x}{x+1}\right)} = 1$$

$$\therefore \frac{x + \frac{x}{x+1}}{1 - x \cdot \frac{x}{x+1}} = 1$$

$$\therefore \frac{\frac{x(x+1) + x}{\cancel{(x+1)}}}{\frac{x+1-x^2}{\cancel{(x+1)}}} = 1$$

$$\therefore \frac{x^2 + x + x}{-x^2 + x + 1} = 1$$

$$\frac{x^2+2x}{-x^2+x+1} = 1 \qquad \left(-x^2+x+1 \neq 0 \quad x \neq \frac{-1 \pm \sqrt{1+4}}{-2} \quad x \neq \frac{1 \pm \sqrt{5}}{2}\right)$$

$$\therefore \quad x^2+2x = -x^2+x+1$$

$$\therefore \quad 2x^2+x-1=0$$

$$x = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} \begin{array}{l} \nearrow -1 \\ \searrow \frac{1}{2} \end{array}$$

Mas, devemos ter $x \neq -1$, daí obtém-se por solução apenas $\underline{\underline{x = \frac{1}{2}}}$.

$$// \; x = \frac{1}{2} \; //$$

20.

$$\sec\left(\operatorname{arctg}\frac{1}{1+e^x}-\operatorname{arctg}(1-e^x)\right)=\frac{\sqrt{5}}{2}$$

Solução

Seja

$$y=\operatorname{arctg}\frac{1}{1+e^x}$$

$\therefore$

$$\begin{cases}\operatorname{tg} y=\dfrac{1}{1+e^x} & (*)\\[2mm] y\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)\end{cases}$$

$$z=\operatorname{arctg}(1-e^x)$$

$\therefore$

$$\operatorname{tg} z=1-e^x \quad (**)$$

$$z\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

Em termos de y e z a equação fica

$$\sec(y-z)=\frac{\sqrt{5}}{2}$$

$$\therefore\quad \underbrace{\sec^2(y-z)}=\frac{5}{4}$$

$$1+\operatorname{tg}^2(y-z)=\frac{5}{4}$$

$$\therefore\quad \operatorname{tg}^2(y-z)=\frac{1}{4}$$

$$\left[\operatorname{tg}(y-z)\right]^2=\frac{1}{4}$$

$$\left(\frac{\operatorname{tg} y-\operatorname{tg} z}{1+\operatorname{tg} y\operatorname{tg} z}\right)^2=\frac{1}{4}\qquad (3*)$$

Substituindo (*) e (**) em (3*) temos:

$$\left(\frac{\frac{1}{1+e^x} - (1-e^x)}{1+\frac{1}{1+e^x}(1-e^x)}\right)^2 = \frac{1}{4}$$

$$\therefore \quad \left(\frac{\frac{1-(1-e^x)(1+e^x)}{1+e^x}}{\frac{1+\cancel{e^x}+1-\cancel{e^x}}{1+e^x}}\right)^2 = \frac{1}{4}$$

$$\therefore \quad \left(\frac{\frac{1-(1+\cancel{e^x}-\cancel{e^x}-e^{2x})}{\cancel{1+e^x}}}{\frac{2}{\cancel{1+e^x}}}\right)^2 = \frac{1}{4}$$

$$\therefore \quad \left(\frac{1-1+e^{2x}}{2}\right)^2 = \frac{1}{4}$$

$$\left(\frac{e^{2x}}{2}\right)^2 = \frac{1}{4}$$

$$\frac{e^{4x}}{4} = \frac{1}{4}$$

$$\therefore \quad e^{4x} = 1$$

$$\therefore \quad \underline{\underline{x = 0}} \, .$$

21. $\begin{cases} \operatorname{arctg}\left(\sqrt{2}-1+\frac{e^x}{2}\right) + \operatorname{arctg}\left(\sqrt{2}-1-\frac{e^x}{2}\right) = a \\ a \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{cases}$

Solução

Seja

$$y = \operatorname{arc\,tg}\left(\sqrt{2}-1+\frac{e^x}{2}\right)$$

$$\therefore \begin{cases} \operatorname{tg} y = \sqrt{2}-1+\frac{e^x}{2} & (*) \\ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{cases}$$

$$z = \operatorname{arc\,tg}\left(\sqrt{2}-1-\frac{e^x}{2}\right)$$

$$\therefore \begin{cases} \operatorname{tg} z = \sqrt{2}-1-\frac{e^x}{2} & (**) \\ z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{cases}$$

A equação que queremos resolver se escreve como

$$y + z = a$$

$$\therefore \operatorname{tg}(y+z) = \operatorname{tg} a$$

$$\frac{\operatorname{tg} y + \operatorname{tg} z}{1 - \operatorname{tg} y \operatorname{tg} z} = \operatorname{tg} a \qquad (***)$$

Substituindo (*) e (**) em (***) obtém-se:

$$\frac{\sqrt{2}-1+\cancel{\frac{e^x}{2}}+\sqrt{2}-1-\cancel{\frac{e^x}{2}}}{1-(\underbrace{\sqrt{2}-1}+\frac{e^x}{2})(\underbrace{\sqrt{2}-1}-\frac{e^x}{2})} = tg\,a$$

$$\therefore \quad \frac{2(\sqrt{2}-1)}{1-\left((\sqrt{2}-1)^2-\frac{e^{2x}}{4}\right)} = tg\,a$$

$$\therefore \quad \frac{2(\sqrt{2}-1)}{1-\left(2-2\sqrt{2}+1-\frac{e^{2x}}{4}\right)} = tg\,a$$

$$\therefore \quad \frac{2(\sqrt{2}-1)}{\underbrace{-2+2\sqrt{2}+\frac{e^{2x}}{4}}_{\neq 0}} = tg\,a \quad (\Rightarrow tg\,a \neq 0)$$

$$\therefore \quad \frac{2(\sqrt{2}-1)}{tg\,a} = -2+2\sqrt{2}+\frac{e^{2x}}{4}$$

$$\therefore \quad \frac{2(\sqrt{2}-1)}{tg\,a} - 2(\sqrt{2}-1) = \frac{e^{2x}}{4}$$

$$\therefore \quad 2(\sqrt{2}-1)\left(\frac{1}{tga}-1\right) = \frac{e^{2x}}{4}$$

$$\therefore \quad 8(\sqrt{2}-1)\left(\frac{1}{tga}-1\right) = e^{2x} \qquad (4*)$$

(forma equivalente da equação original)

Uma vez que $e^{2x} > 0$ vemos de (4*) que devemos ter

$$\frac{1}{tg a} - 1 > 0$$

$$\therefore \frac{1}{tg a} > 1$$

$$\therefore 0 < tg a < 1$$

$$\therefore 0 < a < \frac{\pi}{4}$$

$$\therefore \| a \in (0, \tfrac{\pi}{4}) \|$$

22.

$$x = \arcsin \frac{b}{a}$$

$$|a| > |b| \quad , \quad 0 \leq x \leq \frac{\pi}{4}.$$

Mostrar que $\quad x = \frac{1}{2} \arcsin \frac{2b\sqrt{a^2-b^2}}{a|a|}$

Solução

Seja $\quad x = \arcsin \frac{b}{a}$

$$\therefore \quad \sin x = \frac{b}{a}$$

É dado que $\quad 0 \leq x \leq \frac{\pi}{4}$.

Temos

$$\sin 2x = 2 \sin x \cos x$$

$$= 2 \frac{b}{a} \cdot \sqrt{1-\sin^2 x}$$

$$= 2 \frac{b}{a} \cdot \sqrt{1 - \frac{b^2}{a^2}}$$

$$= 2 \frac{b}{a} \sqrt{\frac{a^2-b^2}{a^2}}$$

$$= \frac{2b}{a} \frac{1}{|a|} \sqrt{a^2-b^2}$$

Mas: $\quad 0 \leq x \leq \frac{\pi}{4} \Rightarrow 0 \leq 2x \leq \frac{\pi}{2}$

Dan,

$$\sin 2x = 2\frac{b}{a} \cdot \frac{\sqrt{a^2-b^2}}{|a|}$$

$$\therefore \quad 2x = \arcsin \frac{2b}{a|a|}\sqrt{a^2-b^2}$$

$$\Big\| \; x = \frac{1}{2} \arcsin \frac{2b}{a|a|}\sqrt{a^2-b^2} \; \Big\|$$

23.

$$\operatorname{arctg} \frac{1+x}{2} + \operatorname{arctg} \frac{1-x}{2} \geq \frac{\pi}{4}$$

Solução

Seja

$$y = \operatorname{arctg} \frac{1+x}{2} \qquad \Big| \qquad z = \operatorname{arctg} \frac{1-x}{2}$$

$$\therefore \operatorname{tg} y = \frac{1+x}{2} \qquad \Big| \qquad \therefore \operatorname{tg} z = \frac{1-x}{2}$$

$$y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \qquad \Big| \qquad z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Assim:

$$\operatorname{tg}(y+z) = \frac{\operatorname{tg} y + \operatorname{tg} z}{1 - \operatorname{tg} y \operatorname{tg} z}$$

$$= \frac{\frac{1+x}{2} + \frac{1-x}{2}}{1 - \left(\frac{1+x}{2}\right)\left(\frac{1-x}{2}\right)}$$

$$= \frac{1}{1 - \frac{1-x^2}{4}} = \frac{1}{\frac{4-1+x^2}{4}}$$

$$= \frac{4}{x^2+3} > 0$$

Mas, em termos de y e z a inequação se escreve $y+z \geq \frac{\pi}{4} \Rightarrow \operatorname{tg}(y+z) \geq 1$

isto é

$$\frac{4}{x^2+3} \geq 1$$

$$4 \geq x^2+3$$

$$1 \geq x^2 \Rightarrow -1 \leq x \leq 1$$

$$\therefore \; \| x \in [-1,1] \|$$

24.

$$\sin\left(2 \operatorname{arccot} \frac{4}{3}\right) + \cos\left(2 \operatorname{arc\,cosec} \frac{5}{4}\right) = \frac{17}{25}$$

Solução

Seja

$$y = \operatorname{arc\,cotg} \frac{4}{3} \quad\Big|\quad z = \operatorname{arc\,cosec} \frac{5}{4}$$

$$\therefore \operatorname{cotg} y = \frac{4}{3} \quad\Big|\quad \therefore \operatorname{cosec} z = \frac{5}{4}$$

$$y \in (0, \pi) \quad\Big|\quad z \in \left(-\pi, -\frac{\pi}{2}\right] \cup \left(0, \frac{\pi}{2}\right]$$

Daí,

$$\sin\left(2 \operatorname{arc\,cotg} \frac{4}{3}\right) + \cos\left(2 \operatorname{arc\,cosec} \frac{5}{4}\right) =$$

$$= \sin 2y + \cos 2z$$

$$= 2 \sin y \cos y + \cos^2 z - \sin^2 z \qquad (*)$$

Mas $\left.\begin{matrix} \operatorname{cotg} y = \frac{4}{3} \\ \text{e } y \in (0, \pi) \end{matrix}\right\} \Rightarrow \underline{y \in \left(0, \frac{\pi}{2}\right)}$

Daí $\frac{4}{3} = \operatorname{cotg} y = \frac{\cos y}{\sin y} = \frac{\sqrt{1 - \sin^2 y}}{\sin y}$

$$\therefore \frac{4}{3} = \frac{\sqrt{1 - \sin^2 y}}{\sin y}$$

$\therefore \quad \frac{16}{9} = \frac{1 - \sin^2 y}{\sin^2 y}$

$\therefore \quad 16 \sin^2 y = 9 - 9 \sin^2 y$

$25 \sin^2 y = 9$

$\sin^2 y = \frac{9}{25} \xRightarrow[(y \in (0, \frac{\pi}{2}))]{} // \sin y = \frac{3}{5} //$

Daí $// \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5} //$

Mas,

$\operatorname{cosec} z = \frac{5}{4} \Rightarrow // \sin z = \frac{4}{5} > 0 //$

Sendo $z \in (-\pi, -\frac{\pi}{2}] \cup (0, \frac{\pi}{2}]$

devemos ter $\underline{\underline{z \in (0, \frac{\pi}{2}]}}$.

Daí, $// \cos z = \sqrt{1 - \sin^2 z} = \sqrt{1 - \frac{16}{25}} = \frac{3}{5} //$

Usando os valores encontrados para $\sin y$, $\cos y$, $\sin z$, $\cos z$ em (*) obtem-se:

$// \sin(2 \operatorname{arc\,cotg} \frac{4}{3}) + \cos(2 \operatorname{arc\,cosec} \frac{5}{4}) =$

$= 2 \cdot \frac{3}{5} \cdot \frac{4}{5} + \frac{9}{25} - \frac{16}{25} = \frac{17}{25} //$.