Cálculo A

Limites de funções trigonométricas

Calcule os limites das funções dadas

1. $\lim_{x\to 2} \frac{\sin x}{x}$
2. $\lim_{x\to \infty} \frac{\sin x}{x}$
3. $\lim_{x\to 0} \frac{\sin 3x}{x}$
4. $\lim_{x\to 0} \frac{\sin 5x}{\sin 2x}$
5. $\lim_{x\to 1} \frac{\sin \pi x}{\sin 3\pi x}$
6. $\lim_{n\to \infty} \left(n \sin \frac{\pi}{n}\right)$
7. $\lim_{x\to 0} \frac{1-\cos x}{x^2}$
8. $\lim_{x\to a} \frac{\sin x - \sin a}{x-a}$
9. $\lim_{x\to a} \frac{\cos x - \cos a}{x-a}$
10. $\lim_{x\to -2} \frac{\tan \pi x}{x+2}$
11. $\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$
12. $\lim_{x\to \frac{\pi}{4}} \frac{\sin x - \cos x}{1-\tan x}$
13. (a) $\lim_{x\to 0} x \sin \frac{1}{x}$
    (b)$\lim_{x\to \infty} x \sin \frac{1}{x}$
14. $\lim_{x\to 1}(1-x) \tan \frac{\pi x}{2}$
15. $\lim_{x\to 0} \cot 2x \ \cot\left(\frac{\pi}{2} - x\right)$
16. $\lim_{x\to \pi} \frac{1-\sin \frac{x}{2}}{\pi - x}$
17. $\lim_{x\to \frac{\pi}{3}} \frac{1-2\cos x}{\pi - 3x}$
18. $\lim_{x\to 0} \frac{\cos mx - \cos nx}{x^2}$
19. $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$

20. $\lim_{x\to 0} \frac{\arcsin x}{x}$

21. $\lim_{x\to 0} \frac{\arctan 2x}{\sin 3x}$

22. $\lim_{x\to 1} \frac{1-x^2}{\sin \pi x}$

23. $\lim_{x\to 0} \frac{x-\sin 2x}{x+\sin 3x}$

24. $\lim_{x\to 1} \frac{\cos \frac{\pi x}{2}}{1-\sqrt{x}}$

25. $\lim_{x\to 0} \frac{1-\sqrt{\cos x}}{x^2}$

26. $\lim_{x\to 0} \frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}$

Respostas:

1. $\frac{\sin 2}{2}$ 2. 0 3. 3 4. $\frac{5}{2}$
5. $\frac{1}{3}$ 6. π 7. $\frac{1}{2}$ 8. $\cos a$
9. $-\sin a$ 10. π 11. $\cos x$
12. $-\frac{\sqrt{2}}{2}$ 13. $(a) 0$ (b) 1 14. $\frac{2}{\pi}$ 15. $\frac{1}{2}$
16. 0 17. $-\frac{\sqrt{3}}{3}$ 18. $\frac{1}{2}(n^2-m^2)$ 19. $\frac{1}{2}$ 20. 1 21. $\frac{2}{3}$ 22. $\frac{2}{\pi}$
23. $-\frac{1}{4}$ 24. π 25. $\frac{1}{4}$ 26. 1

Limites de Funções Trigonométricas

1. $\lim\limits_{x\to 2} \dfrac{\sin x}{x} = \dfrac{\sin 2}{2}$

2. $\lim\limits_{x\to \infty} \dfrac{\sin x}{x} = \dfrac{1}{\infty} = 0$

3. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x} = \lim\limits_{x\to 0} \dfrac{\sin 3x}{3x}\, 3$

$= 3 \lim\limits_{x\to 0} \dfrac{\sin 3x}{3x} = 3$ //

4. $\lim\limits_{x\to 0} \dfrac{\sin 5x}{\sin 2x} = \lim\limits_{x\to 0} \dfrac{\frac{\sin 5x}{5x}\, 5x}{\frac{\sin 2x}{2x}\, \cancel{2x}}$

$= \dfrac{5}{2} \lim\limits_{x\to 0} \dfrac{\frac{\sin 5x}{5x}}{\frac{\sin 2x}{2x}}$

$= \dfrac{5}{2}\, \dfrac{\lim\limits_{x\to 0} \frac{\sin 5x}{5x}}{\lim\limits_{x\to 0} \frac{\sin 2x}{2x}} = \dfrac{5}{2}$ //

5. $\lim\limits_{x\to 1} \dfrac{\sin \pi x}{\sin 3\pi x} = \dfrac{0}{0}$

$\sin 3\pi x = \sin(2\pi x + \pi x) = \underline{\sin 2\pi x}\cos \pi x + \sin \pi x\, \underline{\cos 2\pi x}$

$= \underline{2 \sin \pi x \cos \pi x} \cos \pi x + \sin \pi x\,(\cos^2 \pi x - \sin^2 \pi x)$

$= 2 \sin \pi x \cos^2 \pi x + \sin \pi x \cos^2 \pi x - \sin^3 \pi x$

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$\therefore$

$$\sin 3\pi x = 3\sin \pi x \cos^2 \pi x - \sin^3 \pi x$$

$$\Rightarrow \lim_{x\to 1} \frac{\sin \pi x}{\sin 3\pi x} = \lim_{x\to 1} \frac{\sin \pi x}{3\sin \pi x \cos^2 \pi x - \sin^3 \pi x}$$

$$= \lim_{x\to 1} \frac{1}{3\cos^2 \pi x - \sin^2 \pi x}$$

$$= \frac{1}{3(-1)^2 - 0} = \frac{1}{3} //$$

6. $\lim_{n\to\infty} \left(n \sin \frac{\pi}{n} \right) = \infty \cdot 0$

Seja $x = \frac{\pi}{n}$, $n \to \infty$, $x \to 0$

Daí,

$$\lim_{n\to\infty} \left(n \sin \frac{\pi}{n} \right) = \lim_{x\to 0} \frac{\pi}{x} \sin x$$

$$= \pi \lim_{x\to 0} \frac{\sin x}{x}$$

$$= \pi //$$

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7. $$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{0}{0}$$

$$1-\cos x = 2\sin^2\frac{x}{2} \qquad \left(\sin^2 x = \frac{1-\cos 2x}{2}\right)$$

$\therefore$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2 4}$$

$$= \frac{1}{2}\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$

$$= \frac{1}{2}\left(\underbrace{\lim_{x\to 0} \frac{\sin\frac{x}{2}}{\frac{x}{2}}}\right)^2$$

$$= \frac{1}{2}(1)^2 = \frac{1}{2} \;//$$

8. $$\lim_{x\to a} \frac{\sin x - \sin a}{x-a} = \frac{0}{0}$$

Seja $y = x-a$. $x\to a$, $y\to 0$

Daí,

$$\lim_{x\to a} \frac{\sin x - \sin a}{x-a} = \lim_{y\to 0} \frac{\sin(y+a) - \sin a}{y}$$

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$$\lim_{x\to a}\frac{\sin x-\sin a}{x-a}=\lim_{y\to 0}\frac{\sin(y+a)-\sin a}{y}$$

$$=\lim_{y\to 0}\frac{\sin y\cos a+\sin a\cos y-\sin a}{y}$$

$$=\lim_{y\to 0}\left(\frac{\sin y}{y}\cos a-\sin a\frac{(1-\cos y)}{y}\right)$$

$$=\underbrace{\left(\lim_{y\to 0}\frac{\sin y}{y}\right)}_{1}\cos a-\sin a\underbrace{\left(\lim_{y\to 0}\frac{1-\cos y}{y}\right)}_{=0}$$

$$=\cos a$$

9. $\lim_{x\to a}\frac{\cos x-\cos a}{x-a}=$

$y=x-a$

$x\to a,\ y\to 0$

$$=\lim_{y\to 0}\frac{\cos(y+a)-\cos a}{y}$$

$$=\lim_{y\to 0}\frac{\cos y\cos a-\sin y\sin a-\cos a}{y}$$

$$=\lim_{y\to 0}\left(\frac{-\cos a(1-\cos y)}{y}-\frac{\sin y}{y}\sin a\right)$$

$$=-\cos a\lim_{y\to 0}\frac{1-\cos y}{y}\xrightarrow{}0-\sin a\underbrace{\lim_{y\to 0}\frac{\sin y}{y}}_{1}.$$

$$=-\sin a$$

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10. $\lim\limits_{x \to -2} \dfrac{tg\,\pi x}{x+2} = \dfrac{0}{0}$

$y = x+2 \quad ; \quad x \to -2 \quad , \; y \to 0$

Então,

$$\lim_{x \to -2} \frac{tg\,\pi x}{x+2} = \lim_{y \to 0} \frac{tg\,\pi(y-2)}{y}$$

$\sin \pi(y-2) = \sin(\pi y - 2\pi)$
$= \sin \pi y$

$\cos \pi(y-2) = \cos(\pi y - 2\pi)$
$= \cos \pi y$

$$= \lim_{y \to 0} \frac{\sin \pi(y-2)}{y} \frac{1}{\cos \pi(y-2)}$$

$$= \lim_{y \to 0} \frac{\sin \pi y}{y} \frac{1}{\cos \pi y}$$

$$= \lim_{y \to 0} \frac{\sin \pi y}{\pi y} \pi \frac{1}{\cos \pi y}$$

$$= \pi \underbrace{\lim_{y \to 0} \frac{\sin \pi y}{\pi y}}_{1} \underbrace{\lim_{y \to 0} \frac{1}{\cos \pi y}}_{1}$$

$$= \pi$$

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11. $\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \frac{0}{0}$

$= \lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$

$= \lim_{h\to 0} \frac{-\sin x (1-\cos h) + \sin h \cos x}{h}$

$= \lim_{h\to 0} \left(-\sin x \frac{(1-\cos h)}{h} + \frac{\sin h}{h} \cos x \right)$

$= -\sin x \underbrace{\lim_{h\to 0} \frac{1-\cos h}{h}}_{0} + \cos x \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{1}$

$= \cos x$ //

12. $\lim_{x\to \frac{\pi}{4}} \frac{\sin x - \cos x}{1 - \operatorname{tg} x} = \frac{0}{0}$

$= \lim_{x\to \frac{\pi}{4}} \frac{\sin x - \cos x}{1 - \frac{\sin x}{\cos x}} = \lim_{x\to \frac{\pi}{4}} \frac{\sin x - \cos x}{\frac{\cos x - \sin x}{\cos x}}$

$= \lim_{x\to \frac{\pi}{4}} \frac{\sin x - \cos x}{\cos x - \sin x} \cos x$

$= \lim_{x\to \frac{\pi}{4}} (-)\cos x = -\frac{\sqrt{2}}{2}$ //

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13. a) $\lim_{x \to 0} \left(x \sin \frac{1}{x} \right) = 0 \cdot C \qquad |C| \leq 1$

$= 0$ //

b) $\lim_{x \to +\infty} x \sin \frac{1}{x} = \infty \cdot 0$

$= \lim_{x \to +\infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} \qquad y = \frac{1}{x}\,; \quad x \to +\infty,\ y \to 0$

$= \lim_{y \to 0} \frac{\sin y}{y} = 1$ //

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14) $\lim_{x \to 1} (1-x) \operatorname{tg} \frac{\pi x}{2} = 0 \cdot \infty$

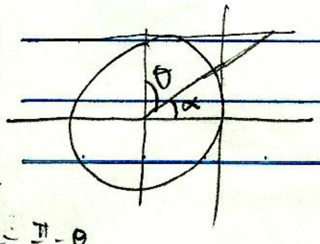
$y = 1-x\,; \quad x \to 1,\ y \to 0$

$\lim_{x \to 1} (1-x) \operatorname{tg} \frac{\pi x}{2} = \lim_{y \to 0} y \operatorname{tg} \frac{\pi}{2}(1-y)$

$= \lim_{y \to 0} y \operatorname{tg} \left(\frac{\pi}{2} - \frac{\pi}{2} y \right)$

$\operatorname{tg}\left(\frac{\pi}{2} - \frac{\pi y}{2} \right) = \qquad = \lim_{y \to 0} y \operatorname{cotg} \frac{\pi}{2} y$

$= \operatorname{tg}\left(\frac{\pi}{2} - \theta \right) = \operatorname{cotg} \theta \qquad = \lim_{y \to 0} y \frac{\cos \frac{\pi}{2} y}{\sin \frac{\pi}{2} y} =$



$\operatorname{tg}\left(\frac{\pi}{2} - \theta \right) = \frac{\sin\left(\frac{\pi}{2} - \theta \right)}{\cos\left(\frac{\pi}{2} - \theta \right)} = \frac{\cos \theta}{\sin \theta} = \operatorname{cotg} \theta$

$\alpha = \frac{\pi}{2} - \theta$

$$= \lim_{y \to 0} \frac{\cancel{y} \cos \frac{\pi}{2} y}{\sin \frac{\pi}{2} \cancel{y}}$$

$$= \lim_{y \to 0} \frac{\cos \frac{\pi}{2} y}{\frac{\sin \frac{\pi}{2} y}{y}} = \lim_{y \to 0} \frac{\cos \frac{\pi}{2} y}{\frac{\sin \frac{\pi}{2} y}{\frac{\pi}{2} y} \frac{2}{\pi}}$$

$$= \lim_{y \to 0} \frac{2}{\pi} \frac{\overset{1}{\cos \frac{\pi}{2} y}}{\frac{\sin \frac{\pi}{2} y}{\frac{\pi}{2} y}\, 1}$$

$$= \frac{2}{\pi} //$$

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15. $\lim_{x \to 0} \operatorname{cotg} 2x \operatorname{cotg} \left(\frac{\pi}{2} - x\right)$

$$= \lim_{x \to 0} \operatorname{cotg} 2x \operatorname{tg} x = \infty \cdot 0$$

$$= \lim_{x \to 0} \frac{\cos 2x}{\sin 2x} \frac{\sin x}{\cos x}$$

$$= \lim_{x \to 0} \frac{\cos^2 x - \sin^2 x}{2 \cancel{\sin x} \cos x} \frac{\cancel{\sin x}}{\cos x}$$

$$= \lim_{x \to 0} \frac{1}{2} \frac{\cos^2 x - \sin^2 x}{\cos^2 x}$$

$$= \lim_{x \to 0} \frac{1}{2} \left(1 - \overset{0}{\operatorname{tg}^2 x}\right)$$

$$= \frac{1}{2} //$$

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16. $\lim\limits_{x\to\pi} \dfrac{1-\sin\frac{x}{2}}{\pi-x} = \dfrac{0}{0}$

$y = \pi - x \quad ; \quad x \to \pi \;,\; y \to 0$

$\lim\limits_{x\to\pi} \dfrac{1-\sin\frac{x}{2}}{\pi-x} = \lim\limits_{y\to 0} \dfrac{1-\sin\frac{\pi-y}{2}}{y}$

$\sin\frac{\pi-y}{2} = \sin\left(\frac{\pi}{2}-\frac{y}{2}\right)$

$= \cos\frac{y}{2}$

$= \lim\limits_{y\to 0} \dfrac{1-\cos\frac{y}{2}}{y}$

$= \lim\limits_{y\to 0} \dfrac{1-\cos\frac{y}{2}}{\frac{y}{2}\,2}$

$= \frac{1}{2} \underbrace{\lim\limits_{y\to 0} \dfrac{1-\cos\frac{y}{2}}{\frac{y}{2}}}_{0}$

$= 0$ //

17. $\lim\limits_{x\to\frac{\pi}{3}} \dfrac{1-2\cos x}{\pi-3x} = \dfrac{0}{0}$

$y = \pi - 3x \quad ; \quad x \to \frac{\pi}{3} \;,\; y \to 0$

$\lim\limits_{x\to\frac{\pi}{3}} \dfrac{1-2\cos x}{\pi-3x} = \lim\limits_{y\to 0} \dfrac{1-2\cos\frac{\pi-y}{3}}{y} =$

$\cos\left(\frac{\pi-y}{3}\right) = \cos\left(\frac{\pi}{3}-\frac{y}{3}\right)$

$= \cos\frac{\pi}{3}\cos\frac{y}{3} + \sin\frac{\pi}{3}\sin\frac{y}{3}$

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$$= \lim_{y\to 0} \frac{1 - 2\cos\frac{\pi - y}{3}}{y}$$

$$= \lim_{y\to 0} \frac{1 - 2\left(\frac{1}{2}\cos\frac{y}{3} + \frac{\sqrt{3}}{2}\sin\frac{y}{3}\right)}{y}$$

$$= \lim_{y\to 0} \frac{1 - \cos\frac{y}{3} - \sqrt{3}\sin\frac{y}{3}}{y}$$

$$= \lim_{y\to 0} \left(\frac{1-\cos\frac{y}{3}}{y} - \sqrt{3}\,\frac{\sin\frac{y}{3}}{y}\right)$$

$$= \lim_{y\to 0} \left(\frac{1-\cos\frac{y}{3}}{\frac{y}{3}\,3} - \sqrt{3}\,\frac{\sin\frac{y}{3}}{\frac{y}{3}\,3}\right)$$

$$= \frac{1}{3}\underbrace{\lim_{y\to 0}\frac{1-\cos\frac{y}{3}}{\frac{y}{3}}}_{=0} - \frac{\sqrt{3}}{3}\underbrace{\lim_{y\to 0}\frac{\sin\frac{y}{3}}{\frac{y}{3}}}_{1}$$

$$= -\frac{\sqrt{3}}{3} \;//$$

18. $\lim\limits_{x\to 0} \dfrac{\cos mx - \cos nx}{x^2} = \dfrac{0}{0}$

Façamos

$$\begin{cases} m = a+b \\ n = a-b \end{cases} > \begin{cases} a = \frac{1}{2}(m+n) \\ b = \frac{1}{2}(m-n) \end{cases}$$

Então

$$\begin{aligned} \cos mx - \cos nx &= \cos(a+b)x - \cos(a-b)x \\ &= \cos ax \cos bx - \sin ax \sin bx \\ &\quad - (\cos ax \cos bx + \sin ax \sin bx) \\ &= -2 \sin ax \sin bx \end{aligned}$$

$\therefore$

$$\lim_{x\to 0} \frac{\cos mx - \cos nx}{x^2} =$$

$$= \lim_{x\to 0} \frac{-2 \sin ax \sin bx}{x^2}$$

$$= -2 \lim_{x\to 0} \frac{\sin ax}{x} \frac{\sin bx}{x}$$

$$= -2 \lim_{x\to 0} \frac{\sin ax}{ax \frac{1}{a}} \frac{\sin bx}{bx \frac{1}{b}}$$

$$= -2ab \underbrace{\lim_{x\to 0} \frac{\sin ax}{ax}}_{1} \underbrace{\lim_{x\to 0} \frac{\sin bx}{bx}}_{1}$$

$$= -2ab = -2 \tfrac{1}{2}(m+n) \tfrac{1}{2}(m-n) = \tfrac{1}{2}(n^2 - m^2)$$

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19. $\lim\limits_{x\to 0} \dfrac{\operatorname{tg} x - \sin x}{x^3} = \dfrac{0}{0}$

Seja $x = 2y$; $x \to 0$, $y \to 0$

$$\lim_{x\to 0} \frac{\operatorname{tg} x - \sin x}{x^3} = \lim_{y\to 0} \frac{\operatorname{tg} 2y - \sin 2y}{8y^3}$$

$$= \lim_{y\to 0} \frac{\frac{\sin 2y}{\cos 2y} - 2\sin y \cos y}{8y^3}$$

$$= \lim_{y\to 0} \frac{\frac{2\sin y\cos y}{\cos^2 y - \sin^2 y} - 2\sin y\cos y}{8y^3}$$

$$= \lim_{y\to 0} \frac{2\sin y\cos y\left(\frac{1}{\cos^2 y - \sin^2 y} - 1\right)}{8y^3}$$

$$= \lim_{y\to 0} \frac{\frac{2\sin y\cos y\,(1 - \cos^2 y + \sin^2 y)}{(\cos^2 y - \sin^2 y)}}{8y^3}$$

$$= \lim_{y\to 0} \frac{2\sin y\cos y\,(\sin^2 y + \sin^2 y)}{8y^3(\cos^2 y - \sin^2 y)}$$

$$= \lim_{y\to 0} \frac{4\sin^3 y\cos y}{8y^3(\cos^2 y - \sin^2 y)} = \frac{1}{2}\lim_{y\to 0} \underbrace{\left(\frac{\sin y}{y}\right)^3}_{1} \underbrace{\frac{\cos y}{\cos^2 y - \sin^2 y}}_{1}$$

$$= \frac{1}{2} \,//$$

20. $\lim_{x\to 0} \frac{\arcsin x}{x} = \frac{0}{0}$

$y = \arcsin x \Leftrightarrow x = \sin y$

$x \to 0 \; ; \; y \to 0$

$$\lim_{x\to 0} \frac{\arcsin x}{x} = \lim_{y\to 0} \frac{y}{\sin y}$$

$$= \lim_{y\to 0} \frac{1}{\frac{\sin y}{y}} = \frac{1}{\lim_{y\to 0} \frac{\sin y}{y}} = 1$$

21. $\lim_{x\to 0} \frac{\text{arctg}\, 2x}{\sin 3x} = \frac{0}{0}$

Seja $y = \text{arctg}\, 2x \Leftrightarrow 2x = \text{tg}\, y \; ; \; x \to 0, y \to 0$

$$\lim_{x\to 0} \frac{\text{arctg}\, 2x}{\sin 3x} = \lim_{y\to 0} \frac{y}{\sin\left(\frac{3}{2}\text{tg}\, y\right)}$$

$$= \lim_{y\to 0} \frac{\frac{y}{\frac{3}{2}\text{tg}\, y}}{\frac{\sin \frac{3}{2}\text{tg}\, y}{\frac{3}{2}\text{tg}\, y}}$$

$$= \frac{2}{3} \frac{\lim_{y\to 0} \frac{y}{\text{tg}\, y}}{\lim_{y\to 0} \frac{\sin \frac{3}{2}\text{tg}\, y}{\frac{3}{2}\text{tg}\, y}}$$

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$$\left\{\begin{array}{l}\lim\limits_{y\to 0}\dfrac{y}{tg\,y}=\lim\limits_{y\to 0}\dfrac{y}{\frac{\sin y}{\cos y}}=\lim\limits_{y\to 0}\dfrac{\cos y}{\frac{\sin y}{y}}=1\\[2ex] \lim\limits_{y\to 0}\dfrac{\sin\frac{3}{2}tg\,y}{\frac{3}{2}tg\,y}=\lim\limits_{z\to 0}\dfrac{\sin z}{z}=1\end{array}\right.$$

$$\left\{\begin{array}{l}z=\frac{3}{2}tg\,y\\ y\to 0\Rightarrow z=0\end{array}\right. \quad \therefore \quad \boxed{\lim\limits_{x\to 0}\dfrac{arc\,tg\,2x}{\sin 3x}=\dfrac{2}{3}}$$

22. $\lim\limits_{x\to 1}\dfrac{1-x^2}{\sin\pi x}=\dfrac{0}{0}$

$$\left\{\begin{array}{l}\sin\pi(x-1)=\sin\pi x\cos(\pi)-\sin\overset{0}{\pi}\cos\pi x\\ \qquad\qquad\quad=-\sin\pi x\end{array}\right.$$

$\therefore$

$$\lim_{x\to 1}\frac{1-x^2}{-\sin\pi(x-1)}=\lim_{x\to 1}\frac{(1-x)(1+x)}{-\sin\pi(x-1)}$$

$x-1=z$

$x\to 1,\ z\to 0$

$$=\lim_{z\to 0}\frac{-z\,(z+2)}{-\sin\pi z}$$

$$=\lim_{z\to 0}\frac{z+2}{\frac{\sin\pi z}{z}}=\lim_{z\to 0}\frac{z+2}{\frac{\sin\pi z}{\pi z}\pi}$$

$$=\frac{1}{\pi}\lim_{z\to 0}\frac{z+2}{\frac{\sin\pi z}{\pi z}}$$

$$=\frac{1}{\pi}\frac{\lim\limits_{z\to 0}z+2}{\lim\limits_{z\to 0}\frac{\sin\pi z}{\pi z}}=\frac{1}{\pi}\frac{2}{1}=\frac{2}{\pi}\;//$$

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23. $\lim_{x\to 0} \dfrac{x - \sin 2x}{x + \sin 3x} = \dfrac{0}{0}$

$= \lim_{x\to 0} \dfrac{1 - \frac{\sin 2x}{x}}{1 + \frac{\sin 3x}{x}}$

$= \lim_{x\to 0} \dfrac{1 - 2\frac{\sin 2x}{2x}}{1 + 3\frac{\sin 3x}{3x}}$

$= \dfrac{\lim_{x\to 0}\left(1 - 2\frac{\sin 2x}{2x}\right)}{\lim_{x\to 0}\left(1 + 3\frac{\sin 3x}{3x}\right)} = \dfrac{1-2}{1+3} = -\dfrac{1}{4}$ //

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24. $\lim_{x\to 1} \dfrac{\cos\frac{\pi x}{2}}{1 - \sqrt{x}}$

// $\sin\frac{\pi}{2}(x-1) = \sin\frac{\pi}{2}x \cancel{\cos\frac{\pi}{2}}^{0} - \sin\frac{\pi}{2}\cos\frac{\pi}{2}x$

$= -\cos\frac{\pi}{2}x$ //

$\lim_{x\to 1} \dfrac{\cos\frac{\pi x}{2}}{1-\sqrt{x}} = \lim_{x\to 1} \dfrac{-\sin\frac{\pi}{2}(x-1)}{1-\sqrt{x}}$

Mas $(1-x) = (1-\sqrt{x})(1+\sqrt{x})$

$\Leftrightarrow$

$(1-\sqrt{x}) = \dfrac{1-x}{1+\sqrt{x}}$

$= \lim_{x\to 1} \dfrac{-\sin\frac{\pi}{2}(x-1)}{\frac{1-x}{1+\sqrt{x}}}$

$= \lim_{x\to 1} (1+\sqrt{x}) \dfrac{\sin\frac{\pi}{2}(x-1)}{(x-1)} =$

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$= \lim_{x \to 1} (1+\sqrt{x}) \dfrac{\sin \frac{\pi}{2}(x-1)}{x-1}$

$= \lim_{x \to 1} (1+\sqrt{x}) \dfrac{\sin \frac{\pi}{2}(x-1)}{\frac{\pi}{2}(x-1) \frac{2}{\pi}}$

$= \frac{\pi}{2} \lim_{x \to 1} (1+\sqrt{x}) \dfrac{\sin \frac{\pi}{2}(x-1)}{\frac{\pi}{2}(x-1)}$

$= \frac{\pi}{2} \underbrace{\lim_{x \to 1} (1+\sqrt{x})}_{2} \underbrace{\lim_{x \to 1} \dfrac{\sin \frac{\pi}{2}(x-1)}{\frac{\pi}{2}(x-1)}}_{1}$

$= \frac{\pi}{2} 2 . 1 = \pi$ //

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25. $\lim_{x \to 0} \dfrac{1 - \sqrt{\cos x}}{x^2} = \frac{0}{0}$

$= \lim_{x \to 0} \dfrac{(1-\sqrt{\cos x})(1+\sqrt{\cos x})}{x^2(1+\sqrt{\cos x})}$

$= \lim_{x \to 0} \dfrac{1 - \cos x}{x^2(1+\sqrt{\cos x})} = \lim_{x \to 0} \dfrac{2\sin^2 \frac{x}{2}}{x^2(1+\sqrt{\cos x})}$

$\sin^2 x = \dfrac{1 - \cos 2x}{2}$

$\Leftrightarrow$

$2\sin^2 \frac{x}{2} = 1 - \cos x$

$= \lim_{x \to 0} 2 \dfrac{\sin^2 \frac{x}{2}}{(\frac{x}{2})^2 4} \dfrac{1}{1+\sqrt{\cos x}}$

$= \frac{1}{2} \underbrace{\lim_{x \to 0} \left(\dfrac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2}_{1} \underbrace{\lim_{x \to 0} \dfrac{1}{1+\sqrt{\cos x}}}$

$= \frac{1}{2} \frac{1}{2}$

$= \frac{1}{4}$ //

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2b. $\lim\limits_{x\to 0} \dfrac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x}$

$= \lim\limits_{x\to 0} \dfrac{\left(\sqrt{1+\sin x} - \sqrt{1-\sin x}\right)}{x} \dfrac{\left(\sqrt{1+\sin x} + \sqrt{1-\sin x}\right)}{\left(\sqrt{1+\sin x} + \sqrt{1-\sin x}\right)}$

$= \lim\limits_{x\to 0} \dfrac{\cancel{1}+\sin x - \cancel{1}+\sin x}{x\left(\sqrt{1+\sin x} + \sqrt{1-\sin x}\right)}$

$= \lim\limits_{x\to 0} \dfrac{2\sin x}{x} \dfrac{1}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}$

$= 2 \underbrace{\lim\limits_{x\to 0} \dfrac{\sin x}{x}} \underbrace{\lim\limits_{x\to 0} \dfrac{1}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}}$

$= 2 \cdot 1 \cdot \dfrac{1}{2}$

$= 1 //$